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Research Article

Sharp Integral Inequalities Based on a General Four-Point Quadrature Formula via a Generalization of the Montgomery Identity

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We consider families of general four-point quadrature formulae using a generalization of the Montgomery identity via Taylor's formula. The results are applied to obtain some sharp inequalities for functions whose derivatives belong to L_p spaces. Generalizations of Simpson's 3/8 formula and the Lobatto four-point formula with related inequalities are considered as special cases.

1. Introduction

The most elementary quadrature rules in four nodes are Simpson's 3/8 rule based on the following four point formula

$$\int_a^b f(t)dt = \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}(\xi), \tag{1.1}$$

where $\xi \in [a, b]$, and Lobatto rule based on the following four point formula

$$\int_{-1}^1 f(t)dt = \frac{1}{6}\left[f(-1) + 5f\left(-\frac{\sqrt{5}}{5}\right) + 5f\left(\frac{\sqrt{5}}{5}\right) + f(1)\right] - \frac{2}{23625} f^{(6)}(\eta), \tag{1.2}$$

where $\eta \in [-1,1]$. Formula (1.1) is valid for any function f with a continuous fourth derivative $f^{(4)}$ on $[a,b]$ and formula (1.2) is valid for any function f with a continuous sixth derivative $f^{(6)}$ on $[-1,1]$.

Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and $f' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Then the Montgomery identity holds (see [1])

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \int_a^b P(x,t)f'(t)dt, \tag{1.3}$$

where the Peano kernel is

$$P(x,t) = \begin{cases} \dfrac{t-a}{b-a}, & a \le t \le x, \\[2mm] \dfrac{t-b}{b-a}, & x < t \le b. \end{cases} \tag{1.4}$$

In [2], Pečarić proved the following weighted Montgomery identity

$$f(x) = \int_a^b w(t)f(t)dt + \int_a^b P_w(x,t)f'(t)dt, \tag{1.5}$$

where $w : [a,b] \to [0,\infty\rangle$ is some probability density function, that is, integrable function, satisfying $\int_a^b w(t)dt = 1$, and $W(t) = \int_a^t w(x)dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$ and $P_w(x,t)$ is the weighted Peano kernel defined by

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x < t \le b. \end{cases} \tag{1.6}$$

Now, let us suppose that I is an open interval in $\mathbb{R}$, $[a,b] \subset I$, $f : I \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 2$, $w : [a,b] \to [0,\infty\rangle$ is a probability density function. Then the following generalization of the weighted Montgomery identity via Taylor's formula states (given by Aglić Aljinović and Pečarić in [3])

$$\begin{aligned} f(x) = &\int_a^b w(t)f(t)dt - \sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds \\ &+ \frac{1}{(n-1)!}\int_a^b T_{w,n}(x,s)f^{(n)}(s)ds, \end{aligned} \tag{1.7}$$

where $x \in [a,b]$ and

$$T_{w,n}(x,s) = \begin{cases} \displaystyle\int_a^s w(u)(u-s)^{n-1}du, & a \le s \le x, \\ \displaystyle -\int_s^b w(u)(u-s)^{n-1}du, & x < s \le b. \end{cases} \tag{1.8}$$

If we take $w(t) = 1/(b-a)$, $t \in [a,b]$, equality (1.7) reduces to

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt - \sum_{i=0}^{n-2} f^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)} + \frac{1}{(n-1)!}\int_a^b T_n(x,s)f^{(n)}(s)ds, \tag{1.9}$$

where $x \in [a,b]$ and

$$T_n(x,s) = \begin{cases} -\dfrac{(a-s)^n}{n(b-a)}, & a \le s \le x, \\ -\dfrac{(b-s)^n}{n(b-a)}, & x < s \le b. \end{cases} \tag{1.10}$$

For $n = 1$, (1.9) reduces to the Montgomery identity (1.3).

In this paper, we generalize the results from [4]. Namely, we use identities (1.7) and (1.9) to establish for each number $x \in (a,(a+b)/2]$ a general four-point quadrature formula of the type

$$\int_a^b w(t)f(t)dt = \left(\frac{1}{2} - A(x)\right)[f(a)+f(b)] + A(x)[f(x)+f(a+b-x)] + R(f,w;x), \tag{1.11}$$

where $R(f,w;x)$ is the remainder and $A : (a,(a+b)/2] \to \mathbb{R}$ is a real function. The obtained formula is used to prove a number of inequalities which give error estimates for the general four-point formula for functions whose derivatives are from L_p-spaces. These inequalities are generally sharp. As special cases of the general non-weighted four-point quadrature formula, we obtain generalizations of the well-known Simpson's 3/8 formula and Lobatto four-point formula with related inequalities.

2. General Weighted Four-Point Formula

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n \ge 2$. We introduce the following notation for each $x \in (a,(a+b)/2]$:

$$D(x) = \left(\frac{1}{2} - A(x)\right)[f(a)+f(b)] + A(x)[f(x)+f(a+b-x)],$$

$$
\begin{aligned}
t_{w,n}(x) = {} & A(x)\left[\sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\right. \\
& \left.+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a+b-x)}{(i+1)!}\int_a^b w(s)(s-a-b+x)^{i+1}ds\right] \\
& +\left(\frac{1}{2}-A(x)\right)\left[\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{(i+1)!}\int_a^b w(s)(s-a)^{i+1}ds\right. \\
& \left.+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{(i+1)!}\int_a^b w(s)(s-b)^{i+1}ds\right],
\end{aligned}
$$

$$
\begin{aligned}
\widehat{T}_{w,n}(x,s) = {} & -\left(\frac{1}{2}-A(x)\right)[T_{w,n}(a,s)+T_{w,n}(b,s)] - A(x)[T_{w,n}(x,s)+T_{w,n}(a+b-x,s)] \\
= {} & \begin{cases}
-\left(\dfrac{1}{2}+A(x)\right)\displaystyle\int_a^s w(u)(u-s)^{n-1}du \\
\quad +\left(\dfrac{1}{2}-A(x)\right)\displaystyle\int_s^b w(u)(u-s)^{n-1}du, & a \le s \le x, \\
-\dfrac{1}{2}\left[\displaystyle\int_a^s w(u)(u-s)^{n-1}du - \int_s^b w(u)(u-s)^{n-1}du\right], & x < s \le a+b-x, \\
-\left(\dfrac{1}{2}-A(x)\right)\displaystyle\int_a^s w(u)(u-s)^{n-1}du \\
\quad +\left(\dfrac{1}{2}+A(x)\right)\displaystyle\int_s^b w(u)(u-s)^{n-1}du, & a+b-x < s \le b.
\end{cases}
\end{aligned}
\tag{2.1}
$$

In the next theorem we establish the general weighted four-point formula.

Theorem 2.1. *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $w : [a,b] \to [0,\infty\rangle$ be some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 2$. Then for each $x \in (a,(a+b)/2]$ the following identity holds*

$$
\int_a^b w(t)f(t)dt = D(x) + t_{w,n}(x) + \frac{1}{(n-1)!}\int_a^b \widehat{T}_{w,n}(x,s)f^{(n)}(s)ds. \tag{2.2}
$$

Proof. We put $x \equiv a$, $x \equiv x$, $x \equiv a+b-x$ and $x \equiv b$ in (1.7) to obtain four new formulae. After multiplying these four formulae by $1/2 - A(x)$, $A(x)$, $A(x)$ and $1/2 - A(x)$, respectively, and adding, we get (2.2). □

Remark 2.2. Identity (2.2) holds true in the case $n = 1$. It can also be obtained by taking $x \equiv a$, $x \equiv x$, $x \equiv a+b-x$ and $x \equiv b$ in (1.5), multiplying these four formulae by $1/2 - A(x)$, $A(x)$, $A(x)$ and $1/2 - A(x)$, respectively, and adding. In this special case we have

$$
\int_a^b w(t)f(t)dt = D(x) + \int_a^b \widehat{T}_{w,1}(x,s)f'(s)ds, \tag{2.3}
$$

where

$$\widehat{T}_{w,1}(x,s) = -\left(\frac{1}{2} - A(x)\right)[T_{w,1}(a,s) + T_{w,1}(b,s)] - A(x)[T_{w,1}(x,s) + T_{w,1}(a+b-x,s)]$$
$$= -\left(\frac{1}{2} - A(x)\right)[P_w(a,s) + P_w(b,s)] - A(x)[P_w(x,s) + P_w(a+b-x,s)]$$
$$= \begin{cases} \frac{1}{2} - A(x) - W(s), & a \le s \le x, \\ \frac{1}{2} - W(s), & x < s \le a+b-x, \\ \frac{1}{2} + A(x) - W(s), & a+b-x < s \le b. \end{cases} \tag{2.4}$$

Theorem 2.3. *Suppose that all assumptions of Theorem 2.1 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is, $1 \le p,\ q \le \infty$, $1/p + 1/q = 1$, let $f^{(n)} \in L^p[a,b]$ for some $n \ge 1$. Then for each $x \in (a,(a+b)/2]$ we have*

$$\left|\int_a^b w(t)f(t)dt - D(x) - t_{w,n}(x)\right| \le \frac{1}{(n-1)!}\left\|\widehat{T}_{w,n}(x,\cdot)\right\|_q \left\|f^{(n)}\right\|_p. \tag{2.5}$$

Inequality (2.5) *is sharp for $1 < p \le \infty$.*

Proof. By applying the Hölder inequality we have

$$\left|\frac{1}{(n-1)!}\int_a^b \widehat{T}_{w,n}(x,s)f^{(n)}(s)ds\right| \le \frac{1}{(n-1)!}\left\|\widehat{T}_{w,n}(x,\cdot)\right\|_q \left\|f^{(n)}\right\|_p. \tag{2.6}$$

By using the above inequality from (2.2) we obtain estimate (2.5). Let us denote $U_n^x(s) = \widehat{T}_{w,n}(x,s)$. For the proof of sharpness, we will find a function f such that

$$\left|\int_a^b U_n^x(s)f^{(n)}(s)ds\right| = \|U_n^x\|_q \left\|f^{(n)}\right\|_p. \tag{2.7}$$

For $1 < p < \infty$, take f to be such that

$$f^{(n)}(s) = \operatorname{sign} U_n^x(s) \cdot |U_n^x(s)|^{1/(p-1)}, \tag{2.8}$$

where for $p = \infty$ we put

$$f^{(n)}(s) = \operatorname{sign} U_n^x(s). \tag{2.9}$$

□

Remark 2.4. Inequality (2.5) for $A(x) = 1/4$ was proved by Aglić Aljinović et al. in [4].

3. Non-Weighted Four-Point Formula and Applications

Here we define

$$\begin{aligned}\widehat{t}_n(x) = {} & A(x)\sum_{i=0}^{n-2}\left[f^{(i+1)}(x) + (-1)^{i+1}f^{(i+1)}(a+b-x)\right]\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)} \\ & + \left(\frac{1}{2} - A(x)\right)\sum_{i=0}^{n-2}\left[f^{(i+1)}(a) + (-1)^{i+1}f^{(i+1)}(b)\right]\frac{(b-a)^{i+1}}{(i+2)!},\end{aligned} \tag{3.1}$$

$$\begin{aligned}\widehat{T}_n(x,s) = {} & -n\left\{\left(\frac{1}{2} - A(x)\right)[T_n(a,s) + T_n(b,s)] + A(x)[T_n(x,s) + T_n(a+b-x,s)]\right\} \\ = {} & \begin{cases}\left(\frac{1}{2} + A(x)\right)\frac{(a-s)^n}{(b-a)} + \left(\frac{1}{2} - A(x)\right)\frac{(b-s)^n}{(b-a)}, & a \le s \le x, \\ \frac{(a-s)^n + (b-s)^n}{2(b-a)}, & x < s \le a+b-x, \\ \left(\frac{1}{2} - A(x)\right)\frac{(a-s)^n}{(b-a)} + \left(\frac{1}{2} + A(x)\right)\frac{(b-s)^n}{(b-a)}, & a+b-x < s \le b.\end{cases}\end{aligned} \tag{3.2}$$

Theorem 3.1. *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 1$. Then for each $x \in (a,(a+b)/2]$ the following identity holds*

$$\frac{1}{b-a}\int_a^b f(t)dt = D(x) + \widehat{t}_n(x) + \frac{1}{n!}\int_a^b \widehat{T}_n(x,s)f^{(n)}(s)ds. \tag{3.3}$$

Proof. We take $w(t) = 1/(b-a)$, $t \in [a,b]$ in (2.2). □

Theorem 3.2. *Suppose that all assumptions of Theorem 3.1 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is, $1 \le p,\ q \le \infty$, $1/p + 1/q = 1$ and $f^{(n)} \in L^p[a,b]$ for some $n \ge 1$. Then for each $x \in (a,(a+b)/2]$ we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - D(x) - \widehat{t}_n(x)\right| \le \frac{1}{n!}\left\|\widehat{T}_n(x,\cdot)\right\|_q\left\|f^{(n)}\right\|_p. \tag{3.4}$$

Inequality (3.4) *is sharp for* $1 < p \le \infty$.

Proof. We take $w(t) = 1/(b-a)$, $t \in [a,b]$ in (2.5). □

Now, we set

$$A(x) = \frac{(b-a)^2}{12(x-a)(b-x)}, \quad x \in \left(a, \frac{a+b}{2}\right]. \tag{3.5}$$

This special choice of the function A enables us to consider generalizations of the well-known Simpson's 3/8 formula (1.1) and Lobatto formula (1.2)

3.1. $x = (2a+b)/3$

Suppose that all assumptions of Theorem 3.1 hold. Then the following generalization of Simpson's 3/8 formula reads

$$\frac{1}{b-a}\int_a^b f(t)dt = D\left(\frac{2a+b}{3}\right) + \hat{t}_n\left(\frac{2a+b}{3}\right) + \frac{1}{n!}\int_a^b \hat{T}_n\left(\frac{2a+b}{3}, s\right) f^{(n)}(s)ds, \tag{3.6}$$

where

$$\begin{aligned}
D\left(\frac{2a+b}{3}\right) &= \frac{1}{8}\left(f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right),\\
\hat{t}_n\left(\frac{2a+b}{3}\right) &= \frac{1}{8}\sum_{i=0}^{n-2}\left[f^{(i+1)}\left(\frac{2a+b}{3}\right) + (-1)^{i+1} f^{(i+1)}\left(\frac{a+2b}{3}\right)\right]\frac{\left[2^{i+2} + (-1)^{i+1}\right](b-a)^{i+1}}{3^{i+1}(i+2)!}\\
&\quad + \frac{1}{8}\sum_{i=0}^{n-2}\left[f^{(i+1)}(a) + (-1)^{i+1} f^{(i+1)}(b)\right]\frac{(b-a)^{i+1}}{(i+2)!},\\
\hat{T}_n\left(\frac{2a+b}{3}, s\right) &= -\frac{n}{8}\left[T_n(a,s) + 3T_n\left(\frac{2a+b}{3}, s\right) + 3T_n\left(\frac{a+2b}{3}, s\right) + T_n(b,s)\right]\\
&= \begin{cases}
\dfrac{7(a-s)^n + (b-s)^n}{8(b-a)} & a \le s \le \dfrac{2a+b}{3},\\[2mm]
\dfrac{(a-s)^n + (b-s)^n}{2(b-a)}, & \dfrac{2a+b}{3} < s \le \dfrac{a+2b}{3},\\[2mm]
\dfrac{(a-s)^n + 7(b-s)^n}{8(b-a)} & \dfrac{a+2b}{3} < s \le b.
\end{cases}
\end{aligned} \tag{3.7}$$

In the next corollaries we will use the beta function and the incomplete beta function of Euler type defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad B_r(x,y) = \int_0^r t^{x-1}(1-t)^{y-1}dt, \quad x,y > 0. \tag{3.8}$$

Corollary 3.3. *Suppose that all assumptions of Theorem 3.1 hold. Additionally, assume that (p,q) is a pair of conjugate exponents and $n \in \mathbb{N}$.*

(a) *If $f^{(n)} \in L^\infty[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - D\left(\frac{2a+b}{3}\right)\right| \le \frac{25}{288}(b-a)\|f'\|_\infty,$$

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - D\left(\frac{2a+b}{3}\right) - \hat{t}_n\left(\frac{2a+b}{3}\right)\right|$$
$$\leq \frac{1}{(n+1)!}\left(\frac{\left[3^{n+1}+3\cdot 2^{n+1}+3(-1)^n\right](b-a)^n}{4\cdot 3^{n+1}} - \left(\frac{b-a}{2}\right)^n\left[\frac{(-1)^{n+1}+1}{2}\right]\right)\left\|f^{(n)}\right\|_\infty, \quad n\geq 2. \tag{3.9}$$

(b) *If* $f^{(n)} \in L^2[a,b]$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - D\left(\frac{2a+b}{3}\right) - \hat{t}_n\left(\frac{2a+b}{3}\right)\right|$$
$$\leq \frac{1}{n!}\left(\frac{\left[3^{2n}+5\cdot 2^{2n+1}+11\right](b-a)^{2n-1}}{32\cdot 3^{2n}(2n+1)} + \frac{(-1)^n(b-a)^{2n-1}}{32} \times\left[7B(n+1,n+1)+9B_{2/3}(n+1,n+1)-9B_{1/3}(n+1,n+1)\right]\right)^{1/2}\left\|f^{(n)}\right\|_2. \tag{3.10}$$

(c) *If* $f^{(n)} \in L^1[a,b]$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - D\left(\frac{2a+b}{3}\right) - \hat{t}_n\left(\frac{2a+b}{3}\right)\right| \leq \frac{1}{n!}K_n\left(\frac{2a+b}{3}\right)\left\|f^{(n)}\right\|_1, \tag{3.11}$$

where $K_1((2a+b)/3) = 5/24$, $K_2((2a+b)/3) = (5/18)\ (b-a)$, $K_3((2a+b)/3) = (7/54)(b-a)^2$ *and* $K_n((2a+b)/3) = (1/8)(b-a)^{n-1}$, *for* $n \geq 4$.

The first and the second inequality are sharp.

Proof. We apply (3.4) with $x = (2a+b)/3$ and $p = \infty$

$$\int_a^b\left|\hat{T}_n\left(\frac{2a+b}{3},s\right)\right|ds = \int_a^{(2a+b)/3}\left|\frac{7(a-s)^n+(b-s)^n}{8(b-a)}\right|ds + \int_{(2a+b)/3}^{(a+2b)/3}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right|ds + \int_{(a+2b)/3}^{b}\left|\frac{(a-s)^n+7(b-s)^n}{8(b-a)}\right|ds$$
$$= 2\frac{\left[3^{n+1}-2^{n+1}+7\cdot(-1)^n\right](b-a)^n}{8\cdot 3^{n+1}(n+1)}$$

$$+\frac{\left(2^{n+1}+(-1)^{n+1}\right)(b-a)^n}{3^{n+1}(n+1)}-\frac{\left(1+(-1)^{n+1}\right)(b-a)^n}{2^{n+1}(n+1)}$$
$$=\frac{\left[3^{n+1}+3\cdot 2^{n+1}+3(-1)^n\right](b-a)^n}{4\cdot 3^{n+1}(n+1)}-\left(\frac{b-a}{2}\right)^n\left[\frac{(-1)^{n+1}+1}{2(n+1)}\right], \tag{3.12}$$

for $n \geq 2$ and

$$\int_a^b \left|\widehat{T}_1\left(\frac{2a+b}{3},s\right)\right| ds = \frac{25}{288}(b-a). \tag{3.13}$$

To obtain the second inequality we take $p = 2$

$$\int_a^b \left|\widehat{T}_n\left(\frac{2a+b}{3},s\right)\right|^2 ds = \int_a^{(2a+b)/3}\left|\frac{7(a-s)^n+(b-s)^n}{8(b-a)}\right|^2 ds$$
$$+\int_{(2a+b)/3}^{(a+2b)/3}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right|^2 ds+\int_{(a+2b)/3}^{b}\left|\frac{(a-s)^n+7(b-s)^n}{8(b-a)}\right|^2 ds$$
$$=\frac{\left[3^{2n}+5\cdot 2^{2n+1}+11\right](b-a)^{2n-1}}{32\cdot 3^{2n}(2n+1)}+\frac{(-1)^n(b-a)^{2n-1}}{32}$$
$$\times\left[7B(n+1,n+1)+9B_{2/3}(n+1,n+1)-9B_{1/3}(n+1,n+1)\right]. \tag{3.14}$$

If $p = 1$, we have

$$\sup_{s\in[a,b]}\left|\widehat{T}_n\left(\frac{2a+b}{3},s\right)\right| = \max\left\{\sup_{s\in[a,(2a+b)/3]}\left|\frac{7(a-s)^n+(b-s)^n}{8(b-a)}\right|,\right.$$
$$\sup_{s\in[(2a+b)/3,(a+2b)/3]}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right| \tag{3.15}$$
$$\left.\sup_{s\in[(a+2b)/3,b]}\left|\frac{(a-s)^n+7(b-s)^n}{8(b-a)}\right|\right\}.$$

By an elementary calculation we get

$$\sup_{s\in[a,(2a+b)/3]}\left|\frac{7(a-s)+(b-s)}{8(b-a)}\right| = \sup_{s\in[(a+2b)/3,b]}\left|\frac{(a-s)+7(b-s)}{8(b-a)}\right| = \frac{5}{24}(b-a),$$

$$\sup_{s\in[a,(2a+b)/3]}\left|\frac{7(a-s)^2+(b-s)^2}{8(b-a)}\right| = \sup_{s\in[(a+2b)/3,b]}\left|\frac{(a-s)^2+7(b-s)^2}{8(b-a)}\right| = \frac{11}{72}(b-a),$$
$$\sup_{s\in[a,(2a+b)/3]}\left|\frac{7(a-s)^n+(b-s)^n}{8(b-a)}\right| = \sup_{s\in[(a+2b)/3,b]}\left|\frac{(a-s)^n+7(b-s)^n}{8(b-a)}\right| = \frac{(b-a)^{n-1}}{8}, \tag{3.16}$$

for $n \geq 3$. The function $y : [a,b] \to \mathbb{R}$, $y(x) = (a-x)^n + (b-x)^n$, is decreasing on $\langle a,(a+b)/2\rangle$ and increasing on $\langle (a+b)/2, b\rangle$ if n is even, and decreasing on $\langle a,b\rangle$ if n is odd. Thus

$$\sup_{s\in[(2a+b)/3,(a+2b)/3]}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right| = \frac{\left((-1)^n+2^n\right)(b-a)^{n-1}}{2\cdot 3^n}. \tag{3.17}$$

Finally,

$$\sup_{s\in[a,b]}\left|\widehat{T}_1\left(\frac{2a+b}{3},s\right)\right| = \frac{5}{24} \tag{3.18}$$

and for $n \geq 2$

$$\sup_{s\in[a,b]}\left|\widehat{T}_n\left(\frac{2a+b}{3},s\right)\right| = (b-a)^{n-1}\max\left\{\frac{1}{8},\frac{2^n+(-1)^n}{2\cdot 3^n}\right\}. \tag{3.19}$$

□

3.2. $[a,b] = [-1,1]$, $x = -\sqrt{5}/5$

Suppose that all assumptions of Theorem 3.1 hold. Then the following generalization of Lobatto formula reads

$$\frac{1}{2}\int_{-1}^{1} f(t)dt = D\left(-\frac{\sqrt{5}}{5}\right) + \hat{t}_n\left(-\frac{\sqrt{5}}{5}\right) + \frac{1}{n!}\int_{-1}^{1}\widehat{T}_n\left(-\frac{\sqrt{5}}{5},s\right)f^{(n)}(s)ds, \tag{3.20}$$

where

$$D\left(-\frac{\sqrt{5}}{5}\right) = \frac{1}{12}\left(f(-1)+5f\left(-\frac{\sqrt{5}}{5}\right)+5f\left(\frac{\sqrt{5}}{5}\right)+f(1)\right),$$
$$\hat{t}_n\left(-\frac{\sqrt{5}}{5}\right) = \frac{5}{12}\sum_{i=0}^{n-2}\left[f^{(i+1)}\left(-\frac{\sqrt{5}}{5}\right)+(-1)^{i+1}f^{(i+1)}\left(\frac{\sqrt{5}}{5}\right)\right]$$
$$\times\frac{\left(5+\sqrt{5}\right)^{i+2}+(-1)^{i+1}\left(5-\sqrt{5}\right)^{i+2}}{2\cdot 5^{i+2}(i+2)!}$$
$$+\frac{1}{12}\sum_{i=0}^{n-2}\left[f^{(i+1)}(-1)+(-1)^{i+1}f^{(i+1)}(1)\right]\frac{2^{i+1}}{(i+2)!},$$

$$\widehat{T}_n\left(-\frac{\sqrt{5}}{5},s\right) = -\frac{n}{12}\left[T_n(-1,s)+5T_n\left(-\frac{\sqrt{5}}{5},s\right)+5T_n\left(\frac{\sqrt{5}}{5},s\right)+T_n(1,s)\right]$$
$$= \begin{cases} \dfrac{11(-1-s)^n+(1-s)^n}{24} & -1\le s\le -\dfrac{\sqrt{5}}{5}, \\ \dfrac{(-1-s)^n+(1-s)^n}{4} & -\dfrac{\sqrt{5}}{5}< s\le \dfrac{\sqrt{5}}{5}, \\ \dfrac{(-1-s)^n+11(1-s)^n}{24} & \dfrac{\sqrt{5}}{5}< s\le 1. \end{cases} \tag{3.21}$$

Corollary 3.4. *Suppose that all assumptions of Theorem 3.1 hold. Additionally, assume that (p,q) is a pair of conjugate exponents and $n\in\mathbb{N}$.*

(a) *if $f^{(n)}\in L^{\infty}[-1,1]$, then*

$$\left|\frac{1}{2}\int_{-1}^{1}f(t)dt-D\left(-\frac{\sqrt{5}}{5}\right)\right|\le\left(\frac{101}{180}-\frac{\sqrt{5}}{6}\right)\|f'\|_{\infty},$$
$$\left|\frac{1}{2}\int_{-1}^{1}f(t)dt-D\left(-\frac{\sqrt{5}}{5}\right)-\widehat{t}_n\left(-\frac{\sqrt{5}}{5}\right)\right|$$
$$\le\frac{1}{(n+1)!}\left(\frac{2^{n+1}\cdot 5^n+\left(5+\sqrt{5}\right)^{n+1}-\left(-5+\sqrt{5}\right)^{n+1}}{12\cdot 5^n}-\frac{1+(-1)^{n+1}}{2}\right)\left\|f^{(n)}\right\|_{\infty},\quad n\ge 2. \tag{3.22}$$

(b) *if $f^{(n)}\in L^{2}[-1,1]$, then*

$$\left|\frac{1}{2}\int_{-1}^{1}f(t)dt-D\left(-\frac{\sqrt{5}}{5}\right)-\widehat{t}_n\left(-\frac{\sqrt{5}}{5}\right)\right|$$
$$\le\frac{1}{n!}\cdot\frac{2^{n-2}}{3}\left(\frac{35\left(5+\sqrt{5}\right)^{2n+1}+85\left(5-\sqrt{5}\right)^{2n+1}+10^{2n+1}}{10^{2n+1}(2n+1)}\right.$$
$$+(-1)^n\left[11B(n+1,n+1)+25B_{(5+\sqrt{5})/10}(n+1,n+1)\right.$$
$$\left.\left.-25B_{(5-\sqrt{5})/10}(n+1,n+1)\right]\right)^{1/2}\left\|f^{(n)}\right\|_{2}. \tag{3.23}$$

(c) *if* $f^{(n)} \in L^1[-1,1]$, *then*

$$\left| \frac{1}{2}\int_{-1}^{1} f(t)dt - D\left(-\frac{\sqrt{5}}{5}\right) - \widehat{t}_n\left(-\frac{\sqrt{5}}{5}\right) \right| \leq \frac{1}{n!} K_n\left(-\frac{\sqrt{5}}{5}\right) \left\| f^{(n)} \right\|_1, \tag{3.24}$$

where $K_1(-\sqrt{5}/5) = 1/(2\sqrt{5})$, $K_2(-\sqrt{5}/5) = 3/5$, $K_3(-\sqrt{5}/5) = 8/(5\sqrt{5})$, $K_4(-\sqrt{5}/5) = 28/25$, $K_5(-\sqrt{5}/5) = 88/(25\sqrt{5})$, $K_n(-\sqrt{5}/5) = 2^{n-3}/3$, *for* $n \geq 6$.
The first and the second inequality are sharp.

Proof. Applying (3.4) with $[a,b] = [-1,1]$, $x = -\sqrt{5}/5$ and $p = \infty, p = 2, p = 1$ and carrying out the same analysis as in Corollay 3.3 we obtain the above inequalities. □

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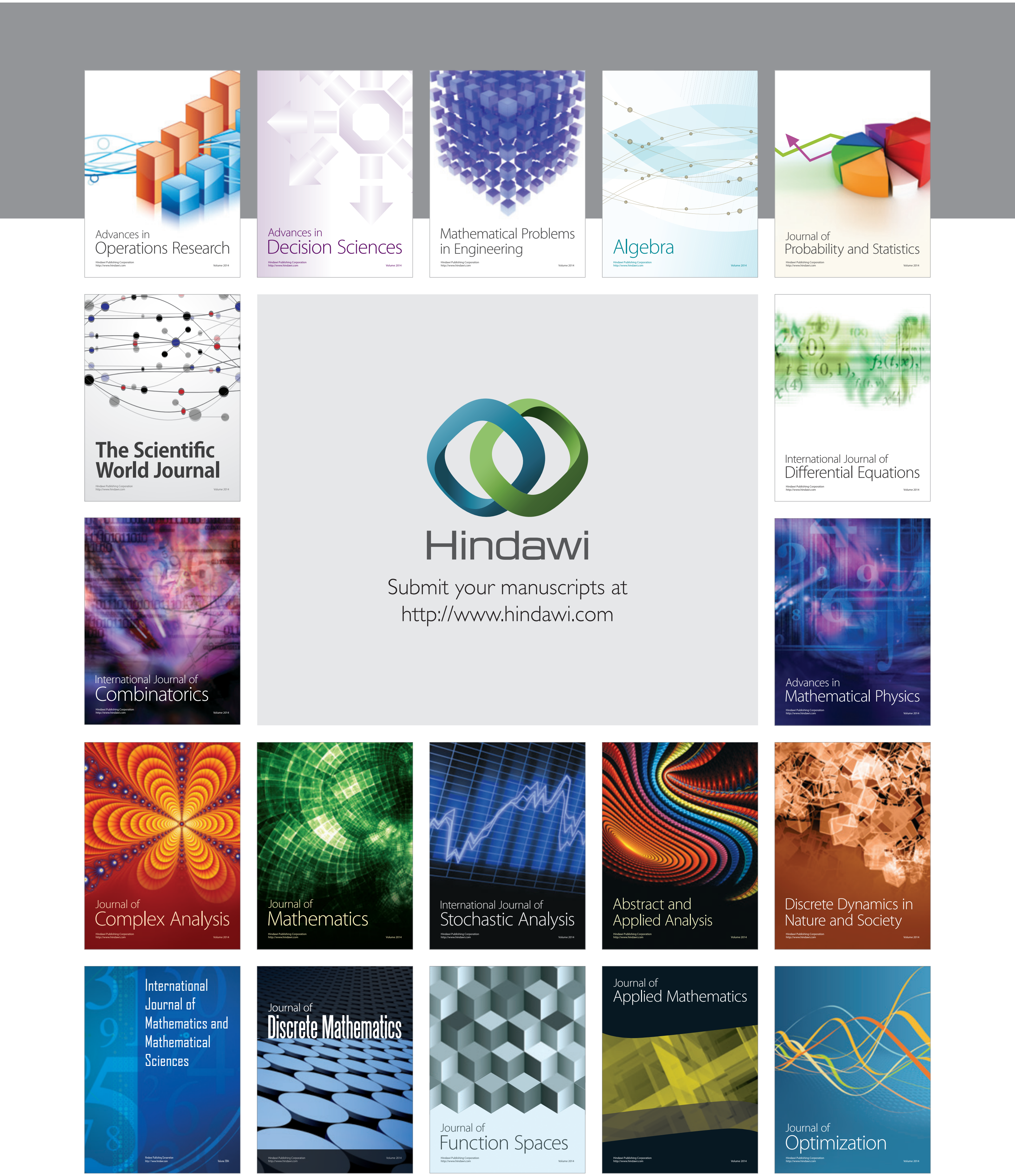
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