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Research Article

Generalized Transversal Lightlike Submanifolds of Indefinite Sasakian Manifolds

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We introduce and study generalized transversal lightlike submanifold of indefinite Sasakian manifolds which includes radical and transversal lightlike submanifolds of indefinite Sasakian manifolds as its trivial subcases. A characteristic theorem and a classification theorem of generalized transversal lightlike submanifolds are obtained.

1. Introduction

The theory of submanifolds in Riemannian geometry is one of the most important topics in differential geometry for years. We see from [1] that semi-Riemannian submanifolds have many similarities with the Riemannian counterparts. However, it is well known that the intersection of the normal bundle and the tangent bundle of a submanifold of a semi-Riemannian manifold may be not trivial, so it is more difficult and interesting to study the geometry of lightlike submanifolds than nondegenerate submanifolds. The two standard methods to deal with the above difficulties were developed by Kupeli [2] and Duggal-Bejancu [3, 4], respectively.

The study of CR-lightlike submanifolds of an indefinite Kaehler manifold was initiated by Duggal-Bejancu [3]. Since the book was published, many geometers investigated the lightlike submanifolds of indefinite Kaehler manifolds by generalizing the CR-lightlike submanifold [3], SCR-lightlike submanifolds [5] to GCR-lightlike submanifolds [6], and discussing the integrability and umbilication of these lightlike submanifolds. We also refer the reader to [7] for invariant lightlike submanifolds and to [8] for totally real lightlike submanifolds of indefinite Kaehler manifolds.

On the other hand, after Duggal-Sahin introduced screen real lightlike submanifolds and contact screen CR-lightlike submanifolds [9] of indefinite Sasakian manifolds by studying the integrability of distributions and the geometry of leaves of distributions as well as other properties of this submanifolds, the generalized CR-lightlike submanifold which contains contact CR and SCR-lightlike submanifolds were introduced in [4]. However, all these submanifolds of indefinite Sasakian manifolds mentioned above have the same geometric condition $\phi(\mathrm{Rad}(TM)) \subset TM$, where ϕ is the almost contact structure on indefinite Sasakian manifolds, $\mathrm{Rad}(TM)$ is the radical distribution, and TM is the tangent bundle. Until recently Yıldırım and Şahin [10] introduced radical transversal and transversal lightlike submanifold of indefinite Sasakian manifolds for which the action of the almost contact structure on radical distribution of such submanifolds does not belong to the tangent bundle, more precisely, $\phi(\mathrm{Rad}(TM)) = \mathrm{ltr}(TM)$, where $\mathrm{ltr}(TM)$ is the lightlike transversal bundle of lightlike submanifolds.

The purpose of this paper is to generalize the radical and transversal lightlike submanifolds of indefinite Sasakian manifolds by introducing generalized transversal lightlike submanifolds. The paper is arranged as follows. In Section 2, we give the preliminaries of lightlike geometry of Sasakian manifolds needed for this paper. In Section 3, we introduce the generalized transversal lightlike submanifolds and obtain a characterization theorem for such lightlike submanifolds. Section 4 is devoted to discuss the integrability and geodesic foliation of distributions of generalized transversal lightlike submanifolds. In Section 5, we investigate the geometry of totally contact umbilical generalized transversal lightlike submanifolds and obtain a classification theorem for such lightlike submanifolds.

2. Preliminaries

In this section, we follow [4, 10] developed by Duggal-Sahin and Yıldırım-Şahin, respectively, for the notations and fundamental equations for lightlike submanifolds of indefinite Sasakian manifolds.

A submanifold (M, g) of dimension m immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ of dimension $(m+n)$ is called a lightlike submanifold if the metric g induced from ambient space is degenerate and its radical distribution $\mathrm{Rad}(TM)$ is of rank r, where $m \geq 2$ and $1 \leq r \leq m$. It is well known that the radical distribution is given by $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$, where $TM^{\perp}$ is called normal bundle of M in $\overline{M}$. Thus there exist the nondegenerate complementary distribution $S(TM)$ and $S(TM^{\perp})$ of $\mathrm{Rad}(TM)$ in TM and $TM^{\perp}$, respectively, which are called the screen and screen transversal distribution on M, respectively. Thus we have

$$TM = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S(TM), \tag{2.1}$$

$$TM^{\perp} = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S\left(TM^{\perp}\right), \tag{2.2}$$

where $\oplus_{\mathrm{orth}}$ denotes the orthogonal direct sum.

Considering the orthogonal complementary distribution $S(TM)^{\perp}$ of $S(TM)$ in $T\overline{M}$, it is easy to see that $TM^{\perp}$ is a subbundle of $S(TM)^{\perp}$. As $S(TM^{\perp})$ is a nondegenerate subbundle of $S(TM)^{\perp}$, the orthogonal complementary distribution $S(TM^{\perp})^{\perp}$ of $S(TM^{\perp})$ in $S(TM)^{\perp}$ is also a nondegenerate distribution. Clearly, $\mathrm{Rad}(TM)$ is a subbundle of $S(TM^{\perp})^{\perp}$. Since for any local basis $\{\xi_i\}$ of $\mathrm{Rad}(TM)$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $S(TM)^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $\mathrm{ltr}(TM)$ locally spanned by $\{N_i\}$ (see [3]). Then we have that $S(TM^{\perp})^{\perp} = \mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)$. Let $\mathrm{tr}(TM) = S(TM^{\perp}) \oplus_{\mathrm{orth}} \mathrm{ltr}(TM)$. We call $\{N_i\}$, $\mathrm{ltr}(TM)$ and $\mathrm{tr}(TM)$ the lightlike transversal vector fields, lightlike transversal vector bundle and transversal vector bundle of M with respect to the chosen screen distribution $S(TM)$ and $S(TM^{\perp})$, respectively. Then $T\overline{M}$ is decomposed as follows:

$$\begin{aligned} T\overline{M} &= TM \oplus \mathrm{tr}(TM) = \mathrm{Rad}(TM) \oplus \mathrm{tr}(TM) \oplus_{\mathrm{orth}} S(TM) \\ &= \mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM) \oplus_{\mathrm{orth}} S(TM) \oplus_{\mathrm{orth}} S\left(TM^{\perp}\right). \end{aligned} \tag{2.3}$$

A lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\overline{g}$ is said to be as follows.

Case 1: r-lightlike if $r < \min m, n$.

Case 2: coisotropic if $r = n < m$, $S(TM^{\perp}) = 0$.

Case 3: isotropic if $r = m < n$, $S(TM) = 0$.

Case 4: totally lightlike if $r = m = n$, $S(TM^{\perp}) = S(TM) = 0$.

Let $\overline{\nabla}$, ∇, and ∇^t denote the linear connections on $\overline{M}$, M, and $\mathrm{tr}(TM)$, respectively. Then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.4}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U, \quad \forall X \in \Gamma(TM),\ \forall U \in \Gamma(\mathrm{tr}(TM)), \tag{2.5}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$, respectively, A_U is the shape operator of M with respective to U. Moreover, according to the decomposition (2.3), denoting by h^l and h^s the $\Gamma(\mathrm{ltr}(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued lightlike second fundamental form and screen second fundamental form of M, respectively, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.6}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad \forall N \in \Gamma(\mathrm{ltr}(TM)), \tag{2.7}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \quad \forall W \in \Gamma\left(S\left(TM^{\perp}\right)\right). \tag{2.8}$$

Then by using (2.4), (2.6)–(2.8), and the fact that $\overline{\nabla}$ is a metric connection, we have

$$\begin{aligned} \overline{g}(h^s(X, Y), W) + \overline{g}\left(D^l(X, W), Y\right) &= g(A_W X, Y), \\ \overline{g}(D^s(X, N), W) &= g(A_W X, N). \end{aligned} \tag{2.9}$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.1), then we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY),$$
$$\nabla_X \xi = -A^*_\xi X + \nabla^*_X \xi, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$, where h^* and A^* are the second fundamental form and shape operator of distribution $S(TM)$ and $\mathrm{Rad}(TM)$, respectively. Then we have the following:

$$\overline{g}\left(h^l(X, PY), \xi\right) = g(A^*_X, PY),$$
$$\overline{g}(h^*(X, PY), N) = g(A_N X, PY). \tag{2.11}$$
$$\overline{g}\left(h^l(X, \xi), \xi\right) = 0, \qquad A^*_\xi \xi = 0.$$

It follows from (2.6) that

$$(\nabla_X g)(Y, Z) = \overline{g}\left(h^l(X, Y), Z\right) + \overline{g}\left(h^l(X, Z), Y\right), \tag{2.12}$$

for any $X, Y, Z \in \Gamma(TM)$. Thus the induced connection ∇ on M is torsion free but is not metric, the induced connection ∇^* on $S(TM)$ is metric.

Finally, we recall some basic definitions and results of indefinite Sasakian manifolds following from [4, 11]. An odd dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ of dimension $2n+1$ is said to be with an almost contact structure (ϕ, η, V) if there exist a $(1,1)$-type tensor ϕ, a vector field V called the characteristic vector field and a 1-form η such that

$$\phi^2(X) = -X + \eta(X)V, \qquad \eta(V) = \epsilon = \pm 1. \tag{2.13}$$

It follows that $\phi(V) = 0$, $\eta \circ \phi = 0$ and $\mathrm{rank}(\phi) = 2n$. A Riemannian metric $\overline{g}$ on $(\overline{M}, \overline{g})$ is called an associated or compatible metric of an almost constant structure (ϕ, η, V) of M if

$$\overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \epsilon\eta(X)\eta(Y). \tag{2.14}$$

A semi-Riemannian manifold endowed with an almost contact structure is said to be an almost contact metric manifold if the semi-Riemannian metric is associated or compatible to almost contact structure and is denoted by $(\overline{M}, \phi, \eta, V, \overline{g})$. It is known that $\overline{g}$ is a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where N_ϕ is the Nijenhuis tensor field of ϕ. A normal contact metric manifolds is called a Sasakian manifold. We know from [12] that an almost contact metric manifold is Sasakian if and only if

$$\left(\overline{\nabla}_X \phi\right) Y = \overline{g}(X, Y)V - \epsilon\eta(Y)X. \tag{2.15}$$

It follows from (2.13) and (2.15) that

$$\overline{\nabla}_X V = -\phi X. \tag{2.16}$$

It is well known that R_q^{2m+1} can be regarded as a Sasakian manifold. Denoting by $(R_q^{2m+1}, \phi, V, \eta, \overline{g})$ the manifold R_q^{2m+1} with its usual Sasakian structure given by

$$\begin{gathered}
\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{m} y^i dx^i\right), \qquad V = 2\partial z, \\
\overline{g} = \eta \otimes \eta + \frac{1}{4}\left(-\sum_{i=1}^{q/2} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i\right), \\
\phi\left(\sum_{i=1}^{m}\left(X_i \partial x^i + Y_i \partial y^i\right) + Z\partial z\right) = \sum_{i=1}^{m}\left(Y_i \partial x^i - X_i \partial y^i\right) + \sum_{i=1}^{m} Y_i y^i \partial z,
\end{gathered} \tag{2.17}$$

where (x^i, y^i, z) are the Cartesian coordinates in R_q^{2m+1}.

3. Generalized Transversal Lightlike Submanifolds

In this section, we define a class of lightlike submanifolds of indefinite Sasakian manifolds and study the geometry of such lightlike submanifolds. Firstly, we recall the following lemma.

Lemma 3.1 (see [4]). *Let M be a lightlike submanifold of an indefinite almost contact metric manifold $\overline{M}$. If V is tangent to M, then the structure vector field V does not belong to* Rad (TM).

Definition 3.2 (see [10]). Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field V, immersed in an indefinite Sasakian manifold $(\overline{M}, \overline{g})$. We say that M is a radical transversal lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

$$\phi(\mathrm{Rad}(TM)) = \mathrm{ltr}(TM), \qquad \phi(S(TM)) = S(TM). \tag{3.1}$$

It follows from Lemma 3.1 and (2.1) that the structure vector field $V \in \Gamma(S(TM))$. Suppose that $\phi(S(TM)) = S(TM)$, then we have $X \in \Gamma(S(TM))$ such that $V = \phi(X)$. since (2.13) implies that $\phi(V) = 0$, using (2.13), we have $X = \eta(X)V$. Substituting $X = \eta(X)V$ into $V = \phi(X)$, we have $V = 0$. So it is impossible for $\phi(S(TM)) = S(TM)$ in Definition 3.2. Thus, we modify the above definition as the following one.

Definition 3.3. Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field V, immersed in an indefinite Sasakian manifold $(\overline{M}, \overline{g})$. M is said to be a radical transversal lightlike submanifold of $\overline{M}$ if there exists an invariant nondegenerate vector subbundle D such

$$\phi(\mathrm{Rad}(TM)) = \mathrm{ltr}(TM), \qquad \phi(D) = D, \tag{3.2}$$

where D is a subbundle of $S(TM)$ and $S(TM) = D \oplus \{V\}$.

Definition 3.4 (see [10]). Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field V, immersed in an indefinite Sasakian manifold $(\overline{M}, \overline{g})$. We say that M is a transversal lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

$$\phi(\mathrm{Rad}(TM)) = \mathrm{ltr}(TM), \qquad \phi(S(TM)) \subseteq S\left(TM^{\perp}\right). \tag{3.3}$$

From Definitions 3.3 and 3.4, we introduce generalized lightlike submanifold as follows.

Definition 3.5. Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field V, immersed in an indefinite Sasakian manifold $(\overline{M}, \overline{g})$. M is said to be a generalized transversal lightlike submanifold of $\overline{M}$ if there exist vector subbundle D and $D^{\perp}$ such

$$\phi(\mathrm{Rad}(TM)) = \mathrm{ltr}(TM), \qquad \phi(D) = D, \qquad \phi\left(D^{\perp}\right) \subseteq S\left(TM^{\perp}\right), \tag{3.4}$$

where D and $D^{\perp}$ are nondegenerate subbundles of $S(TM)$ and $S(TM) = D \oplus D^{\perp} \oplus \{V\}$.

Remark 3.6. It is easy to see that a generalized transversal lightlike submanifold is a radical transversal lightlike submanifold or a transversal lightlike submanifold [10] if and only if $D^{\perp} = \{0\}$ or $D = \{0\}$, respectively.

We say that M is a proper generalized transversal lightlike submanifold of $\overline{M}$ if $D \neq \{0\}$ and $D^{\perp} \neq \{0\}$. We denote by μ the orthogonal complement of $\phi(D^{\perp})$ in $S(TM^{\perp})$. The following properties of a generalized transversal lightlike submanifolds are easy to obtain.

(1) There do not exist 1-lightlike generalized transversal lightlike submanifolds of an indefinite Sasakian manifolds. The proof of the above assertion is similar to Proposition 3.1 of [10], so we omit it here. Then, $\dim(\mathrm{Rad}(TM)) \geq 2$. From (3.4), we know that there exists no a generalized transversal lightlike hypersurface of $\overline{M}$ as $\dim(\mathrm{ltr}(TM)) \geq 2$.

(2) Since (3.4) implies that $\dim(D) \geq 2$, we have $\dim(M) \geq 6$ if M is a proper generalized transversal lightlike submanifold of $\overline{M}$. It follows that any 5-dimensional generalized transversal lightlike submanifold of $\overline{M}$ must be a 2-lightlike submanifold.

(3) $\dim(\overline{M}) \geq 9$ if M is a proper generalized transversal lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$.

In this paper, we assume that the characteristic vector field is a spacelike vector filed, that is, $\epsilon = 1$. If V is a timelike vector field, then one can obtain similar results.

Proposition 3.7. *There exist no isotropic, coisotropic, and totally lightlike proper generalized transversal lightlike submanifolds of indefinite Sasakian manifolds.*

Proof. If M is isotropic or totally lightlike submanifolds of $\overline{M}$, we have $D = \{0\}$ as $S(TM) = \{0\}$, which is a contradiction with definition of proper generalized transversal lightlike submanifolds. Similarly, if M is coisotropic or totally lightlike submanifolds of $\overline{M}$, we have $D^{\perp} = \{0\}$ as $S(TM^{\perp}) = \{0\}$, there is a contradiction, which proves the assertion. □

Lemma 3.8. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifolds $\overline{M}$, then we have*

$$g(\nabla_X Y, V) = \overline{g}(Y, \phi X), \tag{3.5}$$

where $X, Y \in \Gamma(S(TM) - \{V\})$.

Proof. Noticing the fact that $\overline{\nabla}$ on $\overline{M}$ is a metric connection and Definition 3.4, we have

$$g(\nabla_X Y, V) = \overline{g}\left(\overline{\nabla}_X Y, V\right) = -\overline{g}\left(Y, \overline{\nabla}_X V\right) + X(g(Y, V)) = -\overline{g}\left(Y, \overline{\nabla}_X V\right). \tag{3.6}$$

Thus, the proof follows from (2.14), (2.16), and (3.9). □

Theorem 3.9. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifolds $\overline{M}$, then μ is an invariant distribution with respect to ϕ.*

Proof. For any $X \in \Gamma(\mu)$, $\xi \in \Gamma(\mathrm{Rad}(TM))$ and $N \in \Gamma(\mathrm{ltr}(TM))$, it follows from (2.14) that $\overline{g}(\phi X, N) = -\overline{g}(X, \phi N) = 0$, which implies that ϕX has no components in $\mathrm{ltr}(TM)$. Similarly, from (3.4) and (2.14), we have $\overline{g}(\phi X, \xi) = -\overline{g}(X, \phi\xi) = 0$, which implies that ϕX has no components in $\mathrm{Rad}(TM)$. For any $W \in \Gamma(D^{\perp})$, we have $\overline{g}(\phi X, W) = -\overline{g}(X, \phi W) = 0$ as $\phi W \in \Gamma(\phi(D^{\perp}))$. Thus, ϕX has no components in $D^{\perp}$. Finally, suppose that $\phi X = \alpha V$, where α is a smooth function on $\overline{M}$, then we get $X = 0$ by replacing X in (2.13) by ϕX. Thus, we have $\phi(\mu) \in \Gamma(\mu)$, which completes the proof. □

Next, we give a characterization theorem for generalized transversal lightlike submanifolds.

Theorem 3.10. *Let M be a lightlike submanifold of indefinite Sasakian space form $(\overline{M}(c), \overline{g})$, $c \neq 1$. Then M a generalized transversal lightlike submanifolds of $\overline{M}$ if and only if*

(1) *the maximal invariant subspaces of T_pM $(p \in M)$ define a nondegenerate distribution D with respect to ϕ;*

(2) $\overline{g}(\overline{R}(X, Y)Z, W) = 0$, *for all $X, Y \in \Gamma(D \oplus \mathrm{Rad}\,(TM))$ and for all $Z, W \in \Gamma(D^{\perp})$.*

Proof. Suppose that M is a lightlike submanifold of indefinite Sasakian space form $(\overline{M}(c), \overline{g})$. It follows from (5.1) that

$$\overline{g}\left(\overline{R}(X, Y)Z, W\right) = \frac{1-c}{2} g(X, Y)\overline{g}(\phi Z, W), \tag{3.7}$$

for any $X, Y \in \Gamma(D \oplus \mathrm{Rad}(TM))$ and $Z, W \in \Gamma(D^{\perp})$. Noticing that $\phi Z \in \Gamma(S(TM^{\perp}))$ implies $\overline{g}(\phi Z, W) = 0$, then (2) is satisfied. Also, (1) holds naturally by using the definition of generalized transversal lightlike submanifolds.

Conversely, since D is a nondegenerate distribution, we may choose $X, Y \in \Gamma(D)$ such that $g(X, Y) \neq 0$. Thus, from (3.7), we have $g(\phi Z, W) = 0$ for any $Z, W \in \Gamma(D^{\perp})$, which implies that ϕZ have components in $D^{\perp}$. For any $X \in \Gamma(D)$, $\overline{g}(\phi Z, X) = -\overline{g}(Z, \phi X) = 0$ implies that ϕZ have no components in D. Noticing condition (1), we also have $\overline{g}(\phi Z, \xi) = -\overline{g}(Z, \phi\xi) = 0$ and

$\overline{g}(\phi Z, N) = -\overline{g}(Z, \phi N) = 0$, respectively. Thus, we get $\phi(D^{\perp}) \subseteq S(TM^{\perp})$, which completes the proof. □

For any $X \in \Gamma(TM)$, we denote by P_0, P_1 and P_2 and P_3 are the projection morphisms of TM on $\mathrm{Rad}(TM)$, D and $D^{\perp}$ and $\{V\}$, respectively. Let $\phi X = TX + QX + FX$, where TX, QX, and FX are the components of ϕX on TM, $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$, respectively. Moreover, we have $TX = \phi P_1 \in \Gamma(D)$, $QX = \phi P_0 \in \Gamma(\mathrm{ltr}(TM))$ and $FX = \phi P_2 \in \Gamma(\phi(D^{\perp}))$.

Similarly, for any $U \in \Gamma(\mathrm{tr}(TM))$, we denote by U_1, U_2 and U_3 are the components of U on $\mathrm{ltr}(TM)$, $\phi(D^{\perp})$ and μ, respectively. Using Theorem 3.9, then the components of ϕU on TM and $\mathrm{tr}(TM)$ are denoted by $SU = \phi U_1 + \phi U_2 \in \Gamma(\mathrm{Rad}(TM) \oplus D^{\perp})$ and $LU = \phi U_3 \in \Gamma(\mu)$.

Let M be a lightlike submanifold of indefinite Sasakian manifolds $(\overline{M}, \overline{g})$. It follows from (2.15) that $\overline{\nabla}_X \phi Y - \phi \overline{\nabla}_X Y = g(X,Y)V - \eta(Y)X$. Substituting $X = TX + QX + FX \in \Gamma(TM)$ and $Y = TY + QY + FY \in \Gamma(TM)$ into the above equation and taking the tangential, screen transversal, and lightlike transversal parts, respectively, yield that

$$(\nabla_X T)Y = A_{QY}X + A_{FY}X + \phi h^l(X,Y) + Sh^s(X,Y) + g(X,Y)V - \eta(Y)X, \tag{3.8}$$

$$Q\nabla_X Y = h^l(X, TY) + \nabla^l_X QY + D^l(X, FY), \tag{3.9}$$

$$F\nabla_X Y = h^s(X, TY) + D^s(X, QY) + \nabla^s_X FY - Lh^s(X,Y), \tag{3.10}$$

where $(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y$.

Proposition 3.11. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifolds $\overline{M}$. Then we have*

$$\nabla_X V = -\phi X, \quad h^l(X,V) = 0, \quad h^s(X,V) = 0, \quad \forall X \in \Gamma(D), \tag{3.11}$$

$$\nabla_X V = 0, \quad h^l(X,V) = 0, \quad h^s(X,V) = -\phi X - \phi(Lh^s(X,V)), \quad \forall X \in \Gamma\left(D^{\perp}\right), \tag{3.12}$$

$$\nabla_X V = 0, \quad h^l(X,V) = -\phi X, \quad h^s(X,V) = 0, \quad \forall X \in \Gamma(\,\mathrm{Rad}\,(TM)), \tag{3.13}$$

where V is the characteristic vector filed.

Proof. Replacing Y by V in (3.8), it follows from $\phi V = 0$ and (3.8) that

$$T\nabla_X V = X - \phi h^l(X,V) - Sh^s(X,V) - \eta(X)V. \tag{3.14}$$

Similarly, replacing Y by V in (3.9) and (3.10), respectively, we get $Q\nabla_X V = 0$ and $F\nabla_X V = -Lh^s(X,V)$. Noticing that $FX \in \Gamma(\phi D^{\perp})$ for any $X \in \Gamma(TM)$ and $LW \in \Gamma(\mu)$ for any $W \in \Gamma(\mathrm{tr}(TM))$, we have $F\nabla_X V = Lh^s(X,V) = 0$.

Let $X \in \Gamma(D)$, since $\eta(X) = 0$ and $Sh^s(X,V) \in \Gamma(D^{\perp})$, we have $T\nabla_X V = X$ and $\phi h^l(X,V) = Sh^s(X,V) = 0$, which proves (3.11).

Let $X \in \Gamma(D^{\perp})$, from (3.14), we have $T\nabla_X V = \phi h^l(X,V) = 0$ and $Sh^s(X,V) = X$, which proves (3.12).

Let $X \in \Gamma(\mathrm{Rad}(TM))$, it follows from (3.14) that $T\nabla_X V = Sh^s(X,V) = 0$ and $\phi h^l(X,V) = X$, which proves (3.13). □

We know from (2.12) that the induced connection ∇ on M is not a metric connection, the following theorem gives a necessary and sufficient condition for the ∇ to be a metric connection.

Theorem 3.12. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then the following assertions are equivalent.*

(1) *The induced connection ∇ on M is a metric connection.*

(2) *$A_{\phi X}Y$ has no components in $\operatorname{Rad}(TM)$ for any $X \in \Gamma(S(TM))$ and $Y \in \Gamma(TM)$.*

(3) *$A_{\phi\xi}X$ has no components in D for any $X \in \Gamma(TM)$ and $D^s(X, N) \in \Gamma(\mu)$ for any $X \in \Gamma(TM)$.*

Proof. From (2.13) and (2.15), we have

$$g(\nabla_X \xi, V) = \overline{g}\left(\overline{\nabla}_X \xi, V\right) = \overline{g}\left(\phi\overline{\nabla}_X \xi, \phi V\right) = 0, \tag{3.15}$$

which implies that $\nabla_X \xi$ has not component in $\{V\}$ for any $X \in \Gamma(TM)$.

(1) $\Leftrightarrow$ (2). For $X_1 \in \Gamma(D)$ and $X \in \Gamma(TM)$, it follows from (2.6), (2.14), (2.15), and the fact that $\overline{\nabla}$ is metric connection, we have

$$\begin{aligned} g(\nabla_X \xi, X_1) &= \overline{g}\left(\overline{\nabla}_X \xi, X_1\right) = \overline{g}\left(\phi\overline{\nabla}_X \xi, \phi X_1\right) + \eta\left(\overline{\nabla}_X \xi\right)\eta(X_1) \\ &= \overline{g}\left(\overline{\nabla}_X \phi\xi + \eta(\xi)X - g(X, \xi)V, X_1\right) = -\overline{g}\left(\phi\xi, \overline{\nabla}_X \phi X_1\right) = \overline{g}\left(\phi\xi, A_{\phi X_1} X\right). \end{aligned} \tag{3.16}$$

For $X_2 \in \Gamma(D^{\perp})$, Together with (2.8), (2.14), and (2.15), we have

$$\begin{aligned} g(\nabla_X \xi, X_2) &= g\left(\overline{\nabla}_X \xi, X_2\right) = \overline{g}\left(\phi\overline{\nabla}_X \xi, \phi X_2\right) + \eta\left(\overline{\nabla}_X \xi\right)\eta(X_2) \\ &= \overline{g}\left(\overline{\nabla}_X \phi\xi + \eta(\xi)X - g(X, \xi)V, X_2\right) = -\overline{g}\left(\phi\xi, \overline{\nabla}_X \phi X_2\right) = \overline{g}\left(\phi\xi, A_{\phi X_2} X\right). \end{aligned} \tag{3.17}$$

Thus, we prove the equivalence between (1) and (2).

(1)$\Leftrightarrow$(3). Using the similar method shown in the above, from (3.16) and (3.17), we have

$$\begin{gathered} g(\nabla_X \xi, X_1) = \overline{g}\left(\overline{\nabla}_X \xi, X_1\right) = -g(A_{\phi}\xi X, \phi X_1), \\ g(\nabla_X \xi, X_2) = g\left(\overline{\nabla}_X \xi, X_2\right) = -\overline{g}(D^s(X, \phi\xi), \phi X_2). \end{gathered} \tag{3.18}$$

Thus, the proof follows from the above equations. □

4. Integrability and Geodesic of Distributions

Theorem 4.1. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then $D \oplus \{V\}$ is integrable if and only if $h(X, \phi Y) = h(Y, \phi X)$ for any $X, Y \in \Gamma(D)$.*

Proof. Let $X, Y \in \Gamma(D)$, we have $\phi Y = TY$ and $QY = FY = 0$. Thus, it follows from (3.9) that $Q\nabla_X Y = h^l(X, \phi Y)$. Interchanging the roles of X and Y in the above equation and subtracting, we have

$$Q[X,Y] = h^l(X,\phi Y) - h^l(Y,\phi X), \quad \forall X, Y \in \Gamma(D). \tag{4.1}$$

Similarly, it follows from (3.10) that $F\nabla_X Y = h^s(X, \phi Y) - Lh^s(X, Y)$. Noticing that h is symmetric and interchanging the roles of X and Y in the above equation and subtracting, we have

$$F[X,Y] = h^s(X,\phi Y) - h^s(Y,\phi X), \quad \forall X, Y \in \Gamma(D). \tag{4.2}$$

Thus, our assertion follows from (4.1) and (4.2). □

Corollary 4.2. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then D is not integrable.*

Proof. It follows from Lemma 3.8 that $g([X,Y], V) = 2\overline{g}(Y, \phi X)$ for any $X, Y \in \Gamma(D)$. Suppose that D is integrable, then we have $\overline{g}(Y, \phi X) = 0$, which is a contradiction to the fact that D is a nondegenerate distribution of M. □

Theorem 4.3. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then $D^\perp$ is integrable if and only if $D^l(X, FY) = D^l(Y, FX)$ and $A_{FX}Y = A_{FY}X$ for any $X, Y \in \Gamma(D^\perp)$.*

Proof. For $X, Y \in \Gamma(D^\perp)$, we have $\phi Y = FY$ and $TY = QY = 0$. Thus, it follows from (3.8) that $T\nabla_X Y = -A_{FY}X - Qh^l(X,Y) - Sh^s(X,Y) - g(X,Y)V$. As h and g are symmetric, by interchanging the roles of X and Y in the above equation and subtracting the resulting equations, we have

$$T[X,Y] = A_{FX}Y - A_{FY}X, \quad \forall X, Y \in \Gamma\left(D^\perp\right). \tag{4.3}$$

Similarly, it follows from (3.9) that $Q\nabla_X Y = D^l(X, FY)$. Interchanging the roles of X and Y in the above equation and subtracting the resulting equations, we have

$$Q[X,Y] = D^l(X,FY) - D^l(Y,FX), \quad \forall X, Y \in \Gamma\left(D^\perp\right). \tag{4.4}$$

Noticing Lemma 3.8, we have $g([X,Y], V) = 2\overline{g}(Y, \phi X) = 0$ for any $X, Y \in \Gamma(D^\perp)$. Thus, our assertion follows from (4.3) and (4.4). □

Remark 4.4. Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. If $D = \{0\}$, then M is a transversal lightlike submanifolds [9]. It follows from Theorem 4.3 that $S(TM)$ is integrable if and only if $D^l(X, FY) = D^l(Y, FX)$ for all $X, Y \in \Gamma(S(TM))$, which is just one of the conclusions shown in [10].

Theorem 4.5. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then $\mathrm{Rad}\,(TM) \oplus \{V\}$ is integrable if and only if $A_{\phi X}Y = A_{\phi Y}X$ and $D^s(X, \phi Y) - D^s(Y, \phi X)$ has no components in $\phi(D^{\perp})$ for any $X, Y \in \Gamma(\mathrm{Rad}\,(TM))$.*

Proof. Together with (2.14), (2.15), and the fact that $\overline{\nabla}$ is a metric connection, we have

$$g([X,Y],Z) = \overline{g}\Big(\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X, \phi Z\Big) = g(A_{\phi X}Y - A_{\phi Y}X, \phi Z), \tag{4.5}$$

where $Z \in \Gamma(D)$. Similarly, for $W \in \Gamma(D^{\perp})$, we have

$$g([X,Y],W) = \overline{g}\Big(\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X, \phi W\Big) = g(D^s(X,\phi Y) - D^s(Y,\phi X), \phi W), \tag{4.6}$$

Thus, our assertion follows from (4.5) and (4.6). □

Corollary 4.6. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then $\mathrm{Rad}\,(TM)$ is not integrable.*

Proof. It follows from Lemma 3.8 that $g([X,Y],V) = 2\overline{g}(Y, \phi X)$ holds for any $X, Y \in \Gamma(\mathrm{Rad}(TM))$. Suppose that $\mathrm{Rad}(TM)$ is integrable, then we have $\overline{g}(Y, \phi X) = 0$. On the other hand, for any $X \in \Gamma(\mathrm{Rad}(TM))$, there must exists $Y \in \Gamma(\mathrm{Rad}(TM))$ such that $g(X, \phi Y) \neq 0$ as $\phi(\mathrm{Rad}(TM)) = \mathrm{ltr}(TM)$, a contradiction. Then we complete the proof. □

Using the similar method in the proof of Theorems 4.1–4.5 and Corollaries 4.2 and 4.6, we have the following results.

Theorem 4.7. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then,*

(1) $\mathrm{Rad}\,(TM) \oplus D \oplus \{V\}$ *is integrable if and only if $h^s(X, \phi Y) - h^s(Y, \phi X)$ has no components in $\phi(D^{\perp})$ for any $X, Y \in \Gamma(\mathrm{Rad}\,(TM) \oplus D)$.*

(2) $\mathrm{Rad}\,(TM) \oplus D^{\perp} \oplus \{V\}$ *is integrable if and only if $A_{\phi Y}X - A_{\phi X}Y$ has no components in D for any $X, Y \in \Gamma(\mathrm{Rad}\,(TM) \oplus D^{\perp})$.*

(3) $\mathrm{Rad}\,(TM) \oplus D$ *is not integrable.*

(4) $\mathrm{Rad}\,(TM) \oplus D^{\perp}$ *is not integrable.*

We know from [3] that a distribution on M is said to define a totally geodesic foliation if any leaf of D is geodesic. we focus on this property of generalized transversal lightlike submanifolds in the following of this section.

Theorem 4.8. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then the screen distribution defines a totally geodesic foliation if and only if $g(TY, A^*_{\phi N}X) = \overline{g}(FY, h^s(X, \phi N))$ for any $X, Y \in \Gamma(S(TM))$.*

Proof. It is known that $S(TM)$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(S(TM))$ for any $X, Y \in \Gamma(S(TM))$. For $X \in \Gamma(S(TM))$, we have $QX = 0$. Also, we have

$$\begin{aligned} g(\nabla_X Y, N) &= \overline{g}\left(\overline{\nabla}_X \phi Y, \phi N\right) = -\overline{g}\left(\phi Y, \overline{\nabla}_X \phi N\right) \\ &= \overline{g}\left(\phi Y, A^*_{\phi N}\right) X - h^s(X, \phi N) = g\left(TY, A^*_{\phi N} X\right) - \overline{g}(FY, h^s(X, \phi N)). \end{aligned} \tag{4.7}$$

Thus we complete the proof. □

Remark 4.9. Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. If $D^\perp = \{0\}$, then M is radical transversal lightlike submanifolds. For $X \in \Gamma(S(TM))$, we have $FX = 0$. Thus, it follows from Theorem 4.8 that $S(TM)$ defines a totally geodesic foliation if and only if $A^*_{\phi N} X$ has no components in $S(TM)$, which is just the Theorem 3.6 proved in [10].

Theorem 4.10. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then, The following assertions are equivalent.*

(1) *$D^\perp$ defines a totally geodesic foliation.*

(2) *$D^l(X, \phi Y) = 0$ and $A_{\phi Y} X$ has no components in D.*

(3) *$h^s(X, \phi Z)$ and $h^s(X, \xi)$ has no components $\phi(D^\perp)$, where $X, Y, Z \in \Gamma(D^\perp)$ and $\xi \in \Gamma(\text{ Rad }(TM))$.*

Proof. It is easy to see that $D^\perp$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D^\perp)$ for any $X, Y \in \Gamma(D^\perp)$.

(1) $\Leftrightarrow$ (2). From (2.14) and (2.15), we have that

$$\overline{g}(\nabla_X Y, N) = \overline{g}\left(\phi \overline{\nabla}_X Y, \phi N\right) = \overline{g}\left(\overline{\nabla}_X \phi Y, \phi N\right) = \overline{g}\left(D^l(X, \phi Y), \phi N\right), \tag{4.8}$$

where $X, Y \in \Gamma(D^\perp)$ and $N \in \Gamma(\mathrm{ltr}(TM))$. Also, for $Z \in \Gamma(D)$, we have

$$\overline{g}(\nabla_X Y, Z) = \overline{g}\left(\phi \overline{\nabla}_X Y, \phi Z\right) = \overline{g}\left(\overline{\nabla}_X \phi Y, \phi Z\right) = -\overline{g}(A_{\phi Y} X, \phi Z). \tag{4.9}$$

Noticing Lemma 3.8, we have $g([X, Y], V) = 2\overline{g}(Y, \phi X) = 0$ for all $X, Y \in \Gamma(D^\perp)$. Then the equivalence between (1) and (2) follows from (4.8) and (4.9). Noticing that $\overline{\nabla}$ is a metric connection, then from (4.8) and (4.9) we have that

$$\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, h^s(X, \phi N)), \qquad \overline{g}(\nabla_X Y, Z) = -\overline{g}(h^s(X, \phi Z), \phi Y). \tag{4.10}$$

Thus, the equivalence between (1) and (3) follows from (4.10). □

From (4.9), we know that $\overline{g}(\nabla_X Y, Z) = -\overline{g}(A_{\phi Y} X, \phi Z)$ for any $X, Y \in \Gamma(D^\perp \oplus \mathrm{Rad}(TM))$ and $Z \in \Gamma(D)$, then the following corollary holds.

Corollary 4.11. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then $D^{\perp} \oplus$ Rad $(TM) \oplus \{V\}$ defines a totally geodesic foliation if and only if $A_{\phi Y}X$ has no components in D for any $X, Y \in \Gamma(D^{\perp} \oplus$ Rad $(TM)\{V\})$.*

Theorem 4.12. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then $D \oplus \{V\}$ defines a totally geodesic foliation if and only if $A^{*}_{\phi N}X$ and $A_{\phi W}X$ has no components in D for any $X \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$ and $N \in \Gamma($ ltr $(TM))$.*

Proof. Noticing that $\overline{\nabla}$ is a metric connection and (2.14) and (2.15), we have

$$\overline{g}(\nabla_X Y, N) = \overline{g}\left(\phi\overline{\nabla}_X Y, \phi N\right) = \overline{g}\left(\overline{\nabla}_X \phi Y, \phi N\right) = \overline{g}\left(A^{*}_{\phi N}X, \phi Y\right). \tag{4.11}$$

Similarly, for $W \in \Gamma(D^{\perp})$, we have

$$\overline{g}(\nabla_X Y, W) = \overline{g}\left(\phi\overline{\nabla}_X Y, \phi W\right) = \overline{g}\left(\overline{\nabla}_X \phi Y, \phi\right) = \overline{g}(A_{\phi W}X, \phi Y). \tag{4.12}$$

Then the proof follows from (4.11) and (4.12) and Lemma 3.8. □

It follows from (2.13) that $\phi V = 0$, then (4.11) and (4.12) hold for any $X, Y \in \Gamma(D\oplus\{V\})$. Thus it is easy to get the following corollary.

Corollary 4.13. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then D cannot define a totally geodesic foliation.*

Proof. It follows from Lemma 3.8 that $g(\nabla_X Y, V) = \overline{g}(Y, \phi X)$ for any $X, Y \in \Gamma(D)$. Suppose that D defines a totally geodesic foliation, then we have $\overline{g}(Y, \phi X) = 0$. Which is a contradiction to the fact that D is a nondegenerate distribution of M. □

We say that M is a contact generalized transversal lightlike product manifold if $D\oplus\{V\}$ and $D^{\perp} \oplus \mathrm{Rad}(TM)$ define totally geodesic foliations in M.

Theorem 4.14. *Let M be a generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then, M is a contact generalized transversal lightlike product manifold if and only if*

(1) *$h(X, \phi Y) = Lh^{s}(X, Y)$ for any $X, Y \in \Gamma(D \oplus \{V\})$;*

(2) *$A_{\phi X}Y$ has no components in D for any $X, Y \in \Gamma(D^{\perp} \oplus$ Rad $(TM))$.*

Proof. For $X, Y \in \Gamma(D \oplus \{V\})$, we have $\phi X = TX$ and $QX = FX = 0$ as $\phi V = 0$. It follows from (3.9) and (3.10), respectively, that

$$Q\nabla_X Y = h^{l}(X, TY), \qquad F\nabla_X Y = h^{s}(X, TY) - Lh^{s}(X, Y). \tag{4.13}$$

On the other hand, for $X, Y \in \Gamma(D^{\perp} \oplus \mathrm{Rad}(TM))$, we have $TX = 0$. Then it follows from (3.8) that

$$A_{\phi Y}X = -T\nabla_X Y - \phi h^{l}(X, Y) - Sh^{s}(X, Y) - g(X, Y)V. \tag{4.14}$$

Noticing that $\nabla_X Y$ belongs to $\Gamma(D\oplus\{V\})$ or $\Gamma(D^{\perp}\oplus\mathrm{Rad}(TM))$ if and only if $Q\nabla_X Y = F\nabla_X Y = 0$ or $T\nabla_X Y = 0$, respectively, then our assertion follows from (4.13) and (4.14). □

5. Totally Contact Umbilical Lightlike Submanifolds

In this section, we study totally contact umbilical generalized transversal lightlike submanifolds of indefinite Sasakian manifolds defined by Duggal and Sahin [9]. We mainly obtain a classification theorem for such lightlike submanifolds.

A plane section of a Sasakian manifold $(\overline{M},\phi,\eta,V,\overline{g})$ is called a ϕ-section if it is spanned by a unit vector X orthogonal to V and ϕX, where X is a non-null vector field on $\overline{M}$. The sectional curvature $K(X,\phi X)$ of a ϕ-section is called a ϕ-sectional curvature. If $\overline{M}$ has a ϕ-sectional curvature c which is not depend on the ϕ-section at each point, then c is a constant and $\overline{M}$ is called a Sasakian space form, denoted by $\overline{M}(c)$. The curvature tensor $\overline{R}$ of a Sasakian space form $\overline{M}(c)$ is given in [13] as follows:

$$\begin{aligned}\overline{R}(X,Y)Z = {} & \frac{c+3}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} \\ & + \frac{c-1}{4}\{\epsilon\eta(X)\eta(Z)Y - \epsilon\eta(Y)\eta(Z)X + \overline{g}(X,Z)\eta(Y)V \\ & \qquad -\overline{g}(Y,Z)\eta(X)V + \overline{g}(\phi Y,Z)\phi X + \overline{g}(\phi Z,X)\phi Y - 2\overline{g}(\phi X,Y)\phi Z\},\end{aligned} \tag{5.1}$$

where $X,Y,Z\in\Gamma(T\overline{M})$.

Definition 5.1 (see [9]). A lightlike submanifold of an indefinite Sasakian manifold is contact totally umbilical if

$$h^l(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_L + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V), \tag{5.2}$$

$$h^s(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_S + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V), \tag{5.3}$$

where $X,Y\in\Gamma(TM)$, $\alpha_L\in\Gamma(\mathrm{ltr}(TM))$, and $\alpha_S\in\Gamma(S(TM^{\perp}))$.

Theorem 5.2. *Let M be a totally contact umbilical proper generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M},\overline{g})$. Then $\alpha_L = 0$ if and only if D is integrable.*

Proof. Let $X,Y\in\Gamma(D)$, it follows from (2.15) and (5.2) that

$$\begin{aligned}\overline{g}([X,Y],N) &= \overline{g}\Big(\phi\overline{\nabla}_X Y - \phi\overline{\nabla}_Y X,\phi N\Big) = \overline{g}\Big(\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X,\phi N\Big) \\ &= \overline{g}\Big(h^l(X,\phi Y) - h^l(Y,\phi X),\phi N\Big) = (g(X,\phi Y) - g(Y,\phi X))\overline{g}(\alpha_L,\phi N).\end{aligned} \tag{5.4}$$

Noticing that (2.15) implies $g(X,\phi Y) + g(Y,\phi X) = 0$ for any $X, Y \in \Gamma(TM)$, then it follows from (5.4) that $\overline{g}([X,Y],N) = 2g(X,\phi Y)\overline{g}(\alpha_L,\phi N)$. On the other hand, for any $Z \in \Gamma(D^{\perp})$ it follows from (2.14), (2.15), and (5.3) that

$$\begin{aligned} \overline{g}([X,Y],Z) &= \overline{g}\left(\phi\overline{\nabla}_X Y - \phi\overline{\nabla}_Y X, \phi Z\right) = \overline{g}\left(\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X, \phi Z\right) \\ &= \overline{g}(h^s(X,\phi Y) - h^s(Y,\phi X), \phi Z) = 0, \end{aligned} \tag{5.5}$$

which means that $[X,Y]$ have components in $D^{\perp}$ for any $X, Y \in \Gamma(D)$. Noticing Lemma 3.8, we know that $[X,Y]$ has no components in $\{V\}$ for any $X, Y \in \Gamma(D)$. Thus, the proof is complete. □

Theorem 5.3. *Let M be a totally contact umbilical proper generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M},\overline{g})$. Then $\alpha_S = 0$.*

Proof. Let $X \in \Gamma(D)$, from (5.3), we have $h^s(X,Y) = g(X,Y)\alpha_S$, which implies $h^s(X,\phi Y) - h^s(Y,\phi X) = 2g(X,\phi X)\alpha_S$. Also, it follows from (3.10) that

$$h^s(X,\phi Y) = F\nabla_X Y + Lh^s(X,Y). \tag{5.6}$$

Interchanging the role of X and Y in the above equation and subtracting, we have $h^s(X,\phi Y) - h^s(Y,\phi X) = F[X,Y]$. Using the same method as shown in the proof of Theorem 5.2, we have $\alpha_S = 0$. Which proves the theorem. □

Lemma 5.4. *Let M be a totally contact umbilical proper generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M},\overline{g})$. Then, for any $X, Y \in \Gamma(D)$, we have $g(\nabla_X\phi X, V) = g(X,X)$ and $g(\nabla_{\phi X}X, V) = -g(X,X)$.*

Proof. Since $\overline{\nabla}$ is a metric connection, we have

$$X(\overline{g}(\phi X, V)) - \overline{g}\left(\overline{\nabla}_X\phi X, V\right) - \overline{g}\left(\phi X, \overline{\nabla}_X V\right) = 0. \tag{5.7}$$

Noticing (2.16) and the fact that $X \in \Gamma(D)$ implies $g(\phi X, V) = 0$ and $\phi X \in \Gamma(D)$, then from (5.7) and (2.14), we have $g(\nabla_X\phi X, V) = g(X,X)$. Interchanging the role of X and ϕX in (5.7) we get $g(\nabla_{\phi X}X, V) = -g(X,X)$. □

Lemma 5.5. *Let M be a totally contact umbilical proper generalized transversal lightlike submanifold of indefinite Sasakian manifold $(\overline{M},\overline{g})$. Then for any $X, Y \in \Gamma(D)$, one have $g(\nabla_X\xi, \phi X) = 0$ and $g(\nabla_{\phi X}\xi, X) = 0$.*

Proof. Since $\overline{\nabla}$ is a metric connection, we have

$$\phi X(\overline{g}(X,\xi)) - \overline{g}\left(\overline{\nabla}_{\phi X}X, \xi\right) - \overline{g}\left(X, \overline{\nabla}_{\phi X}\xi\right) = 0. \tag{5.8}$$

For $X \in \Gamma(D)$, it is easy to see that $g(\phi X, \xi) = 0$ and $\phi X \in \Gamma(D)$, then $g(\nabla_{\phi X}\xi, X) = -\overline{g}(h^l(\phi X, X), \xi)$. Using (5.2) we have $g(\nabla_X \xi, \phi X) = 0$. Interchanging the role of X and ϕX in (5.8) we get $g(\nabla_{\phi X}\xi, X) = 0$. □

At last, we complete this paper by a classification theorem for generalized transversal lightlike submanifolds.

Theorem 5.6. *Let M be a totally contact umbilical proper generalized transversal lightlike submanifold of indefinite Sasakian space form $(\overline{M}(c), \overline{g})$. Then $c = -3$.*

Proof. For any $X, Y, Z \in \Gamma(TM)$, it follows from (2.6)–(2.8) that

$$\begin{aligned}\overline{R}(X,Y)Z = {} & R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X \\ & + \left(\nabla_X h^l\right)(Y,Z) - \left(\nabla_Y h^l\right)(X,Z) + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) \\ & + (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) + D^s\left(X, h^l(Y,Z)\right) - D^l\left(Y, h^l(X,Z)\right),\end{aligned} \tag{5.9}$$

where $(\nabla_X h^l)(Y,Z) = \nabla^l_X h^l(Y,Z) - h^l(\nabla_X Y, Z) - h^l(\nabla_X Z, Y)$ and $(\nabla_X h^s)(Y,Z) = \nabla^s_X h^s(Y,Z) - h^s(\nabla_X Y, Z) - h^s(\nabla_X Z, Y)$.

Noticing Theorem 5.3 then we have $h^s(X,Y) = 0$ for any $X, Y \in \Gamma(TM - \{V\})$. Let $X, Y \in \Gamma(D)$ and $\xi, \xi' \in \Gamma(\mathrm{Rad}(TM))$, replacing Z by ξ in (5.9) and using (5.2), we have

$$\begin{aligned}\overline{g}\left(\overline{R}(X,Y)\xi, \xi'\right) &= \overline{g}\left(\left(\nabla_X h^l\right)(Y,\xi) - \left(\nabla_Y h^l\right)(X,\xi), \xi'\right) \\ &= \overline{g}\left(h^l(\nabla_Y X, \xi) - h^l(\nabla_X Y, \xi) + h^l(X, \nabla_X \xi) - h^l(Y, \nabla_X \xi), \xi'\right) \\ &= \overline{g}\left(\eta(\nabla_Y X) h^l(\xi, V) - \eta(\nabla_X Y) h^l(\xi, V) + g(X, \nabla_Y \xi)\alpha_L - g(Y, \nabla_X \xi)\alpha_L, \xi'\right).\end{aligned} \tag{5.10}$$

Replacing Y by ϕX in (5.10) and using Proposition 3.11 and Lemmas 5.4 and 5.5, we have

$$\overline{g}\left(\overline{R}(X,Y)\xi, \xi'\right) = 2g(X,X)g(\phi\xi, \xi'). \tag{5.11}$$

On the other hand, it follows from (5.1) that

$$\overline{g}\left(\overline{R}(X,Y)\xi, \xi'\right) = \frac{1-c}{2} g(\phi X, Y) g(\phi\xi, \xi'). \tag{5.12}$$

Replacing Y by ϕX in (5.12) and using (5.11), we have that $(c+3)g(X,X)g(\phi\xi, \xi') = 0$ for all $X \in \Gamma(D)$. As D is nondegenerate, so we may choose X such that $g(X,X) \neq 0$ in the above equation. Thus, we complete the proof. □

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