Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 405315, 17 pages
doi:10.1155/2012/405315

Research Article

Strong Convergence Theorems for a Common Fixed Point of a Finite Family of Pseudocontractive Mappings

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Received 16 May 2012; Accepted 12 July 2012

Academic Editor: Billy Rhoades

It is our purpose, in this paper, to prove strong convergence of Halpern-Ishikawa iteration method to a common fixed point of finite family of Lipschitz pseudocontractive mappings. There is no compactness assumption imposed either on C or on T. The results obtained in this paper improve most of the results that have been proved for this class of nonlinear mappings.

1. Introduction

Let C be a nonempty subset of a real Hilbert space H. The mapping $T : C \to H$ is called *Lipschitz* if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

If $L = 1$, then T is called *nonexpansive*, and if $L < 1$, then T is called a *contraction*. It follows from (1.1) that every contraction mapping is nonexpansive and every nonexpansive mapping is Lipschitz.

A mapping $T : C \to H$ is called *α-strictly pseudocontractive* [1] if for all $x, y \in C$ there exists $\alpha \in [0, 1)$ such that

$$\langle x - y, Tx - Ty \rangle \leq \|x - y\|^2 - \alpha\|(I - T)x - (I - T)y\|^2. \tag{1.2}$$

A mapping T is called *pseudocontractive* if

$$\langle x - y, Tx - Ty \rangle \leq \|x - y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

We note that (1.2) and (1.3) can be equivalently written as

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \alpha\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C, \tag{1.5}$$

respectively.

We observe from (1.4) and (1.5) that every nonexpansive mapping is α-strict pseudocontractive mapping and every α-strict pseudocontractive mapping is pseudocontractive mapping, and hence class of pseudocontractive mappings is a more general *class* of mappings. Furthermore, pseudocontractive mappings are related with the important class of nonlinear *monotone* mappings, where a mapping A with domain $D(A)$ and range $R(A)$ in H is called *monotone* if the inequality

$$\langle x - y, Ax - Ay \rangle \geq 0, \tag{1.6}$$

holds for every $x, y \in D(A)$. We note that T is pseudocontractive if and only if $A := I - T$ is monotone, and hence a fixed point of T, $F(T) := \{x \in D(T) : Tx = x\}$ is a zero of A, $N(A) := \{x \in D(A) : Ax = 0\}$. It is now well known (see, e.g., [2]) that if A is monotone, then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems. Consequently, many researchers have made efforts to obtain iterative methods for approximating fixed points of T, when T is pseudocontractive (see, e.g., [3–10] and the references contained therein).

Let C be a closed subset of a Hilbert space H, and let $T : C \to C$ be a contraction. Then the *Picard iteration method* given by

$$x_0 \in C, \quad x_{n+1} = Tx_n, \quad n \geq 1, \tag{1.7}$$

converges to the unique fixed point of T. However, this Picard iteration method may not always converge to a fixed point of T, when T is nonexpansive mapping. We can take, for example, T to be the anticlockwise rotation of the unit disk in $\mathbb{R}^2$ (with the Euclidean norm) about the origin of coordinate of an angle, say, θ.

The scheme that has been used to approximate fixed points of nonexpansive mappings is the *Mann iteration method* [5] given by

$$x_0 \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \tag{1.8}$$

where $\{\alpha_n\}$ is a real sequence in the interval $(0, 1)$ satisfying certain conditions. But it is worth mentioning that the Mann iteration process does not always converge strongly to a fixed point of nonexpansive mapping T. One has to impose compactness assumption on C (e.g., C is compact) or on T (e.g., T is *semicompact*) to get strong convergence of Mann iteration method to a fixed point of nonexpansive self-map T (see, e.g., [11, 12]).

We also note that efforts to approximate a fixed point of a Lipschitz pseudocontractive mapping defined even on a compact convex subset of a Hilbert space by Mann iteration method proved abortive. One can see an example of a Lipschitz pseudocontractive self-map of a *compact* convex subset of a Hilbert space with a unique fixed point for which no Mann sequence converges by Chidume and Mutangadura [13]. This leads now to our next concern.

Can we construct an iterative sequence for approximating fixed point of the Lipschitz pseudocontractive mappings?

In 1974, Ishikawa [14] introduced an iteration process which converges to a fixed point of Lipschitz pseudocontractive self-map T of C, when C is compact. In fact, he proved the following theorem.

Theorem I. *If C is a compact convex subset of a Hilbert space $H, T : C \mapsto C$ is a Lipschitz pseudocontractive mapping and x_0 is any point of C, then the sequence $\{x_n\}_{n\geq 0}$ converges strongly to a fixed point of T, where $\{x_n\}$ is defined iteratively for each integer $n \geq 0$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \qquad y_n = (1-\beta_n)x_n + \beta_n T x_n, \tag{1.9}$$

here $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions

$$(i)\ 0 \leq \alpha_n \leq \beta_n < 1, \quad (ii) \lim_{n\to\infty} \beta_n = 0, \quad (iii) \sum_{n\geq 0} \alpha_n \beta_n = \infty. \tag{1.10}$$

We observe that Theorem I imposes compactness assumption on C, and it is still an open problem whether or not scheme (1.9), known as *the Ishikawa iterative method*, can be used to approximate fixed points of Lipschitz pseudocontractive mappings *without compactness assumption on C or on T*.

In order to obtain a strong convergence theorem for pseudocontractive mappings without the compactness assumption, Zhou [15] established the hybrid Ishikawa algorithm for Lipschitz pseudocontractive mappings as follows:

$$\begin{aligned} y_n &= (1-\alpha_n)x_n + \alpha_n T x_n, \\ z_n &= (1-\beta_n)x_n + \beta_n T y_n, \\ C_n &= \Big\{ z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 \\ &\qquad - \alpha_n\beta_n\left(1 - 2\alpha_n - L^2{\alpha_n}^2\right)\|x_n - Tx_n\|^2 \Big\}, \\ Q_n &= \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n \geq 1. \end{aligned} \tag{1.11}$$

He proved that the sequence $\{x_n\}$ defined by (1.11) converges strongly to $P_{F(T)}x_0$, where P_C is the metric projection from H into C.

Recently, several authors (see, e.g., [16–18]) also used the hybrid Mann and hybrid Ishikawa algorithm methods to obtain strong convergence to a fixed point of Lipschitz pseudocontractive mappings. But it is worth mentioning that the *hybrid schemes are not easy to compute*. They involve computation of C_n and Q_n for each $n \geq 1$.

Another iteration scheme was introduced and studied by Chidume and Zegeye [19] with which they approximated fixed point of Lipschitz pseudocontractive mapping in a more general real Banach space.

Let K be a convex nonempty subset of real Banach space E, and let $T : K \to K$ be a mapping. From arbitrary $x_1 \in K$, define $\{x_n\}_{n\geq 1}$ by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1), \quad n \in \mathbb{N}, \tag{1.12}$$

where $\{\lambda_n\}_{n\geq 1}$ and $\{\theta_n\}_{n\geq 1}$ are real sequences in $(0,1)$ satisfying the following conditions: (i) $\lim_{n\to\infty}\theta_n = 0$; (ii) $\lambda_n = o(\theta_n)$; (iii) $\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$; (iv) $\lim_{n\to\infty}((\theta_{n-1}/\theta_n - 1)/\lambda_n\theta_n) = 0$, $\lambda_n(1+\theta_n) < 1$. Examples of real sequences which satisfy these conditions are $\lambda_n = 1/(n+1)^a$ and $\theta_n = 1/(n+1)^b$, where $0 < b < a$ and $a + b < 1$. They proved the following theorem.

Theorem CZ. *Let C be a nonempty closed convex subset of a reflexive real Banach space E with a uniformly Gâteaux differentiable norm. Let $T : C \to C$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L > 0$ and $F(T) \neq \emptyset$. Suppose every closed convex and bounded subset of K has the fixed point property for nonexpansive self-mappings. Let a sequence $\{x_n\}_{n\geq 1}$ be generated iteratively by* (1.12). *Then $\{x_n\}_{n\geq 1}$ converges strongly to a fixed point of T.*

Theorem CZ solves the open problem of approximating fixed point of Lipschitz pseudocontractive mappings that has been in the air for many years. However, it is still an open problem whether or not this scheme can be used to approximate a common fixed point of a family of Lipschitz pseudocontractive mappings. Moreover, we observe that the conditions on the real sequences $\{\theta_n\}$ and $\{\lambda_n\}$ excluded the natural choice, $\theta_n = 1/(n+1)$ and $\lambda_n = 1/(n+1)$.

Our concern now is the following: *can we construct an iterative sequence for a common fixed point of a family of Lipschitz pseudocontractive mappings?*

For a sequence $\{\alpha_n\}$ of real numbers in $[0,1]$ and an arbitrary $u \in C$, let the sequence $\{x_n\}$ in C be iteratively defined by $x_0 \in C$:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T x_n, \quad n \geq 1. \tag{1.13}$$

The recursion formula (1.13) known as *Halpern scheme* was first introduced in 1967 by Halpern [20] in the framework of Hilbert spaces. He proved that $\{x_n\}$ convergs strongly to a fixed point of nonexpansive self-mapping T of C.

Recently, considerable research efforts have been devoted to developing iterative methods for approximating a common fixed point of a family of several nonlinear mappings (see, e.g., [4, 21, 22]). In 1996, Bauschke [3] introduced the following Halpern-type iterative process for approximating a common fixed point for a finite family of N nonexpansive self-mappings. In fact, he proved the following theorem.

Theorem B. *Let C be a nonempty closed convex subset of a Hilbert space H, and let $T_1, T_2, \ldots, T_N$ be a finite family of nonexpansive mappings of C into itself with $F := F(T_1 T_N \cdots T_2) = \cdots = F(T_{N-2} \cdots T_1 T_N) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ which satisfies certain mild conditions. Given points, $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_{n+1} u + (1-\alpha_{n+1})T_{n+1} x_n, \quad n \geq 0, \tag{1.14}$$

where $T_n = T_{n(\bmod N)}$. *Then* $\{x_n\}$ *converges strongly to* $P_F u$, *where* $P_F u : H \to F$ *is the metric projection.*

But it is worth mentioning that it is still an open problem whether or not this scheme can be used to approximate a common fixed points of Lipschitz pseudocontractive mappings?

In 2008, Zhou [22] studied weak convergence of an implicit scheme to a common fixed point of finite family of pseudocontractive mappings. More precisely, he proved the following theorem.

Theorem Z. *Let* E *be a real uniformly convex Banach space with a Frêchet differentiable norm. Let* C *be a closed convex subset of* E, *and let* $\{T_i\}_{i=1}^r$ *be a finite family of Lipschitzian pseudocontractive self-mappings of* C *such that* $F := \cap_{i=1}^r F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be defined by*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geq 1, \tag{1.15}$$

where $T_n = T_{n(\bmod r)}$. *If* $\{\alpha_n\}$ *is chosen so that* $\alpha_n \in (0,1)$ *with* $\limsup_{n\to\infty} \alpha_n < 1$, *then* $\{x_n\}$ *converges weakly to a common fixed point of the family* $\{T_i\}_{i=1}^r$.

Here, we remark that the scheme in Theorem Z is *implicit*, and the convergence is *weak convergence*.

More recently, Zegeye et al. [23] proved the following strong convergence of Ishikawa iterative process for a common fixed point of finite family of Lipschitz pseudocontractive mappings.

Theorem ZSA (see [23]). *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* $T_i : C \to C$, $i = 1, 2, \ldots, N$, *be a finite family of Lipschitz pseudocontractive mappings with Lipschitzian constants* L_i, *for* $i = 1, 2, \ldots, N$, *respectively. Assume that the interior of* $F := \cap_{i=1}^n F(T_i)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_0 \in E$ *by*

$$\begin{aligned} y_n &= (1 - \beta_n) x_n + \beta_n T_n x_n, \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T_n y_n, \quad n \geq 1, \end{aligned} \tag{1.16}$$

where $T_n := T_{n(\bmod N)}$ *and* $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *satisfying certain appropriate conditions. Then,* $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_1, T_2, \ldots, T_N\}$.

From Theorem ZSA, we observe that the assumption that the interior of $F(T)$ *is nonempty is severe restriction.*

Motivated by Halpern [20] and Zegeye et al. [23], it is our purpose, in this paper, to prove strong convergence of Halpern-Ishikawa algorithm (3.3) to a common fixed point of a finite family of Lipschitz pseudocontractive mappings. No compactness assumption is imposed either on one of the mappings or on C. The assumption that interior of $F(T)$ *is nonempty* is dispensed with. Moreover, computation of closed and convex set C_n for each $n \geq 1$ is not required. The results obtained in this paper improve and extend the results of Theorems I and ZSA, Zhou [15], Yao et al. [17], and Tang et al. [16].

2. Preliminaries

In what follows we will make use of the following lemmas.

Lemma 2.1. *Let H be a real Hilbert space. Then for any given $x, y \in E$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{2.1}$$

Lemma 2.2 (see [24]). *Let C be a convex subset of a real Hilbert space H. Let $x \in H$. Then $x_0 = P_C x$ if and only if*

$$\langle z - x_0, x - x_0\rangle \leq 0, \quad \forall z \in C. \tag{2.2}$$

Lemma 2.3 (see [25]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \beta_n)a_n + \beta_n\delta_n, \quad n \geq n_0, \tag{2.3}$$

where $\{\beta_n\} \subset (0,1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions: $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$, and $\limsup_{n\to\infty}\delta_n \leq 0$. Then, $\lim_{n\to\infty}a_n = 0$.

Lemma 2.4 (see [18]). *Let H be a real Hilbert space, let C be a closed convex subset of H, and let $T : C \to C$ be a continuous pseudocontractive mapping; then*

(i) *$F(T)$ is closed convex subset of C;*

(ii) *$(I - T)$ is demiclosed at zero; that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, as $n \to \infty$, then $x = T(x)$.*

Lemma 2.5 (see [26]). *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in N$. Then there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$,and the following properties are satisfied by all (sufficiently large) numbers $k \in N$:*

$$a_{m_k} \leq a_{m_k+1}, \qquad a_k \leq a_{m_k+1}. \tag{2.4}$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.6 (see [27]). *Let H be a real Hilbert space. Then for all $x_i \in H$ and $\alpha_i \in [0,1]$ for $i = 1, 2, \ldots, n$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ the following equality holds:*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n\|^2 = \sum_{i=0}^{n} \alpha_i\|x_i\|^2 - \sum_{0\leq i,j\leq n} \alpha_i\alpha_j\|x_i - x_j\|^2. \tag{2.5}$$

3. Main Result

We now prove the following lemma and theorems.

Lemma 3.1. *Let C be a nonempty convex subset of a real Hilbert space H. Let $T_i : C \to C, i = 1, 2, \ldots, N$, be a finite family of Lipschitz pseudocontractive mappings with constants L_i, respectively. Let $S = \theta_1 T_1 + \theta_2 T_2 + \cdots + \theta_N T_N$, where $\theta_1 + \theta_2 + \cdots + \theta_N = 1$. Then S is Lipschitz pseudocontractive mapping on C.*

Proof. Let $x, y \in C$. Then

$$\begin{aligned}\langle Sx - Sy, x - y\rangle &= \theta_1\langle T_1 x - T_1 y, x - y\rangle \\ &\quad + \theta_2\langle T_2 x - T_2 y, x - y\rangle + \cdots + \theta_N\langle T_N x - T_N y, x - y\rangle \\ &\le \theta_1\|x - y\|^2 + \theta_2\|x - y\|^2 + \cdots + \theta_N\|x - y\|^2 \\ &= \|x - y\|^2.\end{aligned} \tag{3.1}$$

Hence S is pseudocontractive. Moreover, since

$$\begin{aligned}\|Sx - Sy\| &= \|(\theta_1 T_1 + \theta_2 T_2 + \cdots + \theta_N T_N)x - (\theta_1 T_1 + \theta_2 T_2 + \cdots + \theta_N T_N)y\| \\ &\le \theta_1\|T_1 x - T_1 y\| + \theta_2\|T_2 x - T_2 y\| + \cdots + \theta_N\|T_N x - T_N y\| \\ &\le L\|x - y\|,\end{aligned} \tag{3.2}$$

where $L := \max\{L_i : i = 1, 2, \ldots, N\}$, we get that S is L-Lipschitz. The proof is complete. □

Let $\{T_i : i = 1, 2, \ldots, N\}$ be a finite family of pseudocontractive mappings. The family is said to satisfy *condition* (H) if $\langle T_i x - x, T_j x - x\rangle \ge 0$, for $i, j \in \{1, 2, \ldots, N\}$.

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \to C, i = 1, 2, \ldots, N$ be a finite family of Lipschitz pseudocontractive mappings with Lipschitz constants L_i, respectively, satisfying condition (H). Assume that $F := \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let a sequence $\{x_n\}$ be a sequence generated from an arbitrary $x_1 = w \in C$ by*

$$\begin{aligned}y_n &= (1 - \beta_n)\, x_n + \beta_n S_n x_n, \\ x_{n+1} &= \alpha_n w + (1 - \alpha_n)(\gamma_n S_n y_n + (1 - \gamma_n) x_n),\end{aligned} \tag{3.3}$$

where $S_n := \theta_{n,1} T_1 + \theta_{n,2} T_2 + \cdots + \theta_{n,N} T_N$, for $\{\theta_{n,i}\} \subseteq [a, b] \subset (0, 1)$ such that $\theta_{n,1} + \theta_{n,2} + \cdots + \theta_{n,N} = 1$, for all $n \ge 1$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ satisfying the following conditions: (i) $0 \le \alpha_n \le c < 1$, for all $n \ge 1$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum \alpha_n = \infty$; (ii) $0 < \alpha \le \gamma_n \le \beta_n \le \beta < 1/[\sqrt{(1 + L^2)} + 1]$, for all $n \ge 1$, for $L := \max\{L_i : i = 1, 2, \ldots, N\}$. Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots T_N\}$ nearest to $x_1 = w$.

Proof. Let $p = P_F w$. Then from (3.3), Lemma 2.6, (1.5), and Lemma 3.1 we have the following:

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n w + (1-\alpha_n)(\gamma_n S_n y_n + (1-\gamma_n)x_n) - p\|^2 \\
&\le \alpha_n \|w - p\|^2 + (1-\alpha_n)\|\gamma_n S_n y_n + (1-\gamma_n)x_n - p\|^2 \\
&= \alpha_n \|w - p\|^2 + (1-\alpha_n)\Big[\gamma_n \|S_n y_n - p\|^2 \\
&\quad + (1-\gamma_n)\, \|x_n - p\|^2 - \gamma_n(1-\gamma_n)\|S_n y_n - x_n\|^2\Big] \\
&= \alpha_n \|w - p\|^2 + (1-\alpha_n)\gamma_n \|S_n y_n - p\|^2 \\
&\le \alpha_n \|w - p\|^2 + (1-\alpha_n)\gamma_n \\
&\quad + \left(\|y_n - p\|^2 + \|y_n - S_n y_n\|^2\right) + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 \\
&\quad - \gamma_n(1-\gamma_n)(1-\alpha_n)\|S_n y_n - x_n\|^2 \\
&= \alpha_n \|w - p\|^2 + (1-\alpha_n)\gamma_n \|y_n - p\|^2 + (1-\alpha_n)\gamma_n \|y_n - S_n y_n\|^2 \\
&\quad + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 - \gamma_n(1-\gamma_n)(1-\alpha_n)\|S_n y_n - x_n\|^2 \\
&\quad + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 - \gamma_n(1-\alpha_n)(1-\gamma_n)\|S_n y_n - x_n\|^2.
\end{aligned} \tag{3.4}$$

□

In addition, we have that

$$\begin{aligned}
\|y_n - S_n y_n\|^2 &= \|(1-\beta_n)(x_n - S_n y_n) + \beta_n(S_n x_n - S_n y_n)\|^2 \\
&= (1-\beta_n)\|x_n - S_n y_n\|^2 + \beta_n \|S_n x_n - S_n y_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - S_n x_n\|^2 \\
&\le (1-\beta_n)\|x_n - S_n y_n\|^2 + \beta_n L^2 \|x_n - y_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - S_n x_n\|^2 \\
&= (1-\beta_n)\|x_n - S_n y_n\|^2 + \beta_n^3 L^2 \|x_n - S_n x_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - S_n x_n\|^2 \\
&= (1-\beta_n)\|x_n - S_n y_n\|^2 + \beta_n\left(L^2\beta_n^2 + \beta_n - 1\right)\|x_n - S_n x_n\|^2,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(S_n x_n - p)\|^2 \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n \|S_n x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - S_n x_n\|^2 \\
&\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\Big[\|x_n - p\|^2 + \|x_n - S_n x_n\|^2\Big] \\
&\quad - \beta_n(1-\beta_n)\|x_n - S_n x_n\|^2 \\
&= \|x_n - p\|^2 + \beta_n^2 \|x_n - S_n x_n\|^2.
\end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4) we obtain that

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le{}& \alpha_n\|w-p\|^2+(1-\alpha_n)\gamma_n\Big[\|x_n-p\|^2+\beta_n^2\|x_n-S_nx_n\|^2\Big]\\
&+(1-\alpha_n)\gamma_n\Big[(1-\beta_n)\|x_n-S_ny_n\|^2+\beta_n\Big(L^2\beta_n^2+\beta_n-1\Big)\\
&\times\|x_n-S_nx_n\|^2\Big]+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2\\
&-\gamma_n(1-\gamma_n)(1-\alpha_n)\|S_ny_n-x_n\|^2\\
&+(1-\alpha_n)\gamma_n\beta_n^2\|x_n-S_nx_n\|^2+(1-\alpha_n)\gamma_n\beta_n\Big(L^2\beta_n^2+\beta_n-1\Big)\\
&\times\|x_n-S_nx_n\|^2+\big[(1-\alpha_n)(1-\beta_n)\gamma_n-(1-\alpha_n)(1-\gamma_n)\gamma_n\big]\\
&\times\|x_n-S_ny_n\|^2\\
={}& \alpha_n\|w-p\|^2+(1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n\beta_n\Big(1-2\beta_n-L^2\beta_n^2\Big)\\
&\times\|x_n-S_nx_n\|^2+(1-\alpha_n)\gamma_n(\gamma_n-\beta_n)\|x_n-S_ny_n\|^2\\
={}& \alpha_n\|w-p\|^2+\big[(1-\alpha_n)\gamma_n+(1-\alpha_n)(1-\gamma_n)\big]\|x_n-p\|^2.
\end{aligned} \tag{3.7}$$

Since from (ii), we have that $(\gamma_n-\beta_n)\le 0$ and $1-2\beta_n-L^2\beta_n^2\ge 1-2\beta-L^2\beta^2>0$ for all $n\ge 1$, (3.7) implies that

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le{}& \alpha_n\|w-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\
&-(1-\alpha_n)\gamma_n\beta_n\Big(1-2\beta-L^2\beta^2\Big)\|x_n-S_nx_n\|^2\\
\le{}& \alpha_n\|w-p\|^2+(1-\alpha_n)\|x_n-p\|^2.
\end{aligned} \tag{3.8}$$

Thus, by induction,

$$\|x_{n+1}-p\|^2\le\max\Big\{\|x_1-p\|^2,\|w-p\|^2\Big\},\quad \forall n\ge 1, \tag{3.9}$$

which implies that $\{x_n\}$ and hence $\{y_n\}$ are bounded.

Furthermore, from (3.3), Lemma 2.1, and following the methods used in (3.7) we get that

$$\begin{aligned}
\|x_{n+1}-p\|^2 ={}& \|\alpha_n(w-p)+(1-\alpha_n)[\gamma_nS_ny_n+(1-\gamma_n)x_n-p]\|^2\\
\le{}& (1-\alpha_n)\|\gamma_nS_ny_n+(1-\gamma_n)x_n-p\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle\\
={}& (1-\alpha_n)\gamma_n\|S_ny_n-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2\\
&-\gamma_n(1-\gamma_n)(1-\alpha_n)\|S_ny_n-x_n\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\gamma_n\Big[\|y_n - p\|^2 + \|y_n - S_n y_n\|^2\Big] + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 \\
&\quad - \gamma_n(1-\gamma_n)(1-\alpha_n)\|S_n y_n - x_n\|^2 + 2\alpha_n\langle w - p, x_{n+1} - p\rangle \\
&= (1-\alpha_n)\gamma_n\|y_n - p\|^2 + (1-\alpha_n)\gamma_n\|y_n - S_n y_n\|^2 \\
&\quad + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 - \gamma_n(1-\gamma_n)(1-\alpha_n)\|S_n y_n - x_n\|^2 \\
&\leq (1-\alpha_n)\gamma_n\Big[\|x_n - p\|^2 + \beta_n^2\|x_n - S_n x_n\|^2\Big] + (1-\alpha_n)\gamma_n \\
&\quad \times\Big[(1-\beta_n)\|x_n - S_n y_n\|^2 + \beta_n\Big(L^2\beta_n^2 + \beta_n - 1\Big)\|x_n - S_n x_n\|^2\Big] \\
&\quad + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 - \gamma_n(1-\gamma_n)(1-\alpha_n)\|S_n y_n - x_n\|^2 \\
&\quad + 2\alpha_n\langle w - p, x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle w - p, x_{n+1} - p\rangle \\
&\quad - (1-\alpha_n)\gamma_n\beta_n\Big(1 - 2\beta_n - \beta_n^2 L^2\Big)\|S_n y_n - x_n\|^2 \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle w - p, x_{n+1} - p\rangle \\
&\quad - (1-c)\alpha^2\Big(1 - 2\beta - \beta^2 L^2\Big)\|S_n x_n - x_n\|^2 \\
&\quad + 2\alpha_n\langle w - p, x_{n+1} - p\rangle.
\end{aligned}
\tag{3.10}
$$

On the other hand, using Lemma 2.6 and condition (H), we get that

$$
\begin{aligned}
\|x_n - S_n x_n\|^2 &= \|x_n - (\theta_{n,1}T_1 + \theta_{n,2}T_2 + \cdots + \theta_{n,N}T_N)\, x_n\|^2 \\
&= \|\theta_{n,1}(x_n - T_1 x_n) + \theta_{n,2}(x_n - T_2 x_n) + \cdots + \theta_{n,N}(x_n - T_N x_n)\|^2 \\
&= \theta_{n,1}\|x_n - T_1 x_n\|^2 + \theta_{n,2}\|x_n - T_2 x_n\|^2 + \cdots + \theta_{n,N}\|x_n - T_N x_n\|^2 \\
&\quad - \sum_{1\leq i,j\leq N}\theta_{n,i}\theta_{n,j}\|T_i x_n - T_j x_n\|^2 \\
&\quad - \sum_{1\leq i,j\leq N, i\neq j}\theta_{n,i}\theta_{n,j}\Big[\|T_i x_n - x_n\|^2 + \|x_n - T_j x_n\|^2\Big] \\
&= \theta_{n,1}[1 - \theta_{n,2} - \theta_{n,3} - \cdots - \theta_{n,N}]\|x_n - T_1 x_n\|^2 \\
&\quad + \theta_{n,2}[1 - \theta_{n,1} - \theta_{n,3} - \theta_{n,4} - \cdots - \theta_{n,N}]\|x_n - T_2 x_n\|^2 \ldots \\
&\quad + \theta_{n,N}[1 - \theta_{n,1} - \theta_{n,2} - \theta_{n,4} - \cdots - \theta_{n,N}]\|x_n - T_N x_n\|^2 \\
&\geq \theta_{n,1}\|x_n - T_1 x_n\|^2 + \theta_{n,2}\|x_n - T_2 x_n\|^2 + \cdots + \theta_{n,N}\|x_n - T_N x_n\|^2.
\end{aligned}
\tag{3.11}
$$

Thus, substituting (3.11) into (3.10) we obtain that

$$\begin{aligned}\|x_{n+1}-p\|^2 \le{}& (1-\alpha_n)\|x_n-p\|^2 + 2\alpha_n\langle w-p, x_{n+1}-p\rangle \\ &- (1-c)\alpha^2\left(1-2\beta-\beta^2L^2\right) \\ &\times \Big[\theta_{n,1}(1-\theta_{n,2}-\theta_{n,3}-\cdots-\theta_{n,N})\|x_n-T_1x_n\|^2 \\ &+ \theta_{n,2}(1-\theta_{n,1}-\theta_{n,3}-\theta_{n,4}-\cdots-\theta_{n,N})\|x_n-T_2x_n\|^2 \ldots \\ &+ \theta_{n,N}(1-\theta_{n,1}-\theta_{n,2}-\theta_{n,4}-\cdots-\theta_{n,N-1})\|x_n-T_Nx_n\|^2\Big] \qquad (3.12)\\ \le{}& (1-\alpha_n)\|x_n-p\|^2 + 2\alpha_n\langle w-p, x_{n+1}-p\rangle. \qquad (3.13)\end{aligned}$$

Now, we consider the following two cases.

Case 1. Suppose that there exists $n_0 \in N$ such that $\{\|x_n-p\|\}$ is nonincreasing. Then, we get that $\{\|x_n-p\|)\}$ is convergent. Thus, from (3.12) and the fact that $\alpha_n \to 0$, as $n \to \infty$, we have that

$$x_n - T_ix_n \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty, \qquad (3.14)$$

for each $i = 1, 2, \ldots, N$. Let $z_n = \gamma_nS_ny_n + (1-\gamma_n)x_n$. Then from (3.3) we obtain that

$$x_{n+1} - z_n = \alpha_n(w - z_n) \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty\ . \qquad (3.15)$$

Furthermore, from (3.3) and (3.14) we get that

$$\begin{aligned}\|y_n - x_n\| &= \|\beta_n(S_nx_n - x_n)\| \le \|S_nx_n - x_n\| \\ &\le \theta_{n,1}\|T_1x_n - x_n\| + \theta_{n,2}\|T_2x_n - x_n\| + \cdots + \theta_{n,N}\|T_Nx_n - x_n\| \longrightarrow 0,\end{aligned} \qquad (3.16)$$

as $n \to \infty$, and hence (3.16) and the fact that S_n is L-Lipschitz imply that

$$\begin{aligned}\|z_n - x_n\| = \|\gamma_n(S_ny_n - x_n)\| &= \|\gamma_n(S_ny_n - S_nx_n) + \gamma_n(S_nx_n - x_n)\| \\ &\le \gamma_nL\|y_n - x_n\| + \gamma_n\|S_nx_n - x_n\| \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.\end{aligned} \qquad (3.17)$$

Now, (3.15) and (3.17) imply that

$$x_{n+1} - x_n \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty\ . \qquad (3.18)$$

Moreover, since $\{x_n\}$ is bounded and E is reflexive, we choose a subsequence $\{x_{n_i+1}\}$ of $\{x_n\}$ such that $x_{n_i+1} \rightharpoonup z$ and $\limsup_{n\to\infty}\langle w-p, x_{n+1}-p\rangle = \lim_{i\to\infty}\langle w-p, x_{n_i+1}-p\rangle$. This implies from (3.18) that $x_{n_i} \rightharpoonup z$. Then, from (3.14) and Lemma 2.4 we have that $z \in F(T_i)$, for each $i = 1, 2, \ldots, N$. Hence, $z \in \cap_{i=1}^{N} F(T_i)$. Therefore, by Lemma 2.2, we immediately obtain that

$$\begin{aligned} \limsup_{n\longrightarrow\infty} \langle w - p, x_{n+1} - p \rangle &= \lim_{i\to\infty} \langle w - p, x_{n_i+1} - p \rangle \\ &= \langle w - p, z - p \rangle \le 0. \end{aligned} \tag{3.19}$$

Then, since from (3.13) we have that

$$\|x_{n+1} - p\|^2 \le (1 - \alpha_n)\|x_n - p\|^2 + 2\alpha_n \langle w - p, x_{n+1} - p \rangle. \tag{3.20}$$

It follows from (3.20), (3.19), and Lemma 2.3 that $\|x_n - p\| \to 0$, as $n \to \infty$. Consequently, $x_n \to p$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - p\| < \|x_{n_i+1} - p\|, \tag{3.21}$$

for all $i \in N$. Then, by Lemma 2.5, there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$, $\|x_{m_k} - p\| \le \|x_{m_k+1} - p\|$ and $\|x_k - p\| \le \|x_{m_k+1} - p\|$ for all $k \in N$. Now, from (3.12) and the fact that $\alpha_n \to 0$, we get that $x_{m_k} - T_i x_{m_k} \to 0$, as $k \to \infty$, for each $i = 1, 2, \ldots, N$. Thus, as in Case 1, we obtain that $x_{m_k+1} - x_{m_k} \to 0$ and that

$$\limsup_{k\to\infty} \langle w - p, x_{m_k+1} - p \rangle \le 0. \tag{3.22}$$

Now, from (3.13) we have that

$$\|x_{m_k+1} - p\|^2 \le (1 - \alpha_{m_k})\|x_{m_k} - p\|^2 + 2\alpha_{m_k} \langle w - p, x_{m_k+1} - p \rangle, \tag{3.23}$$

and hence, since $\|x_{m_k} - p\|^2 \le \|x_{m_k+1} - p\|^2$, (3.23) implies that

$$\begin{aligned} \alpha_{m_k}\|x_{m_k} - p\|^2 &\le \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + 2\alpha_{m_k} \langle w - p, x_{m_k+1} - p \rangle \\ &\le 2\alpha_{m_k} \langle w - p, x_{m_k+1} - p \rangle. \end{aligned} \tag{3.24}$$

But noting that $\alpha_{m_k} > 0$, we obtain that

$$\|x_{m_k} - p\|^2 \le 2\langle w - p, x_{m_k+1} - p \rangle. \tag{3.25}$$

Then, from (3.22) we get that $\|x_{m_k} - p\| \to 0$, as $k \to \infty$. This together with (3.23) gives that $\|x_{m_k+1} - p\| \to 0$, as $k \to \infty$. But $\|x_k - p\| \le \|x_{m_k+1} - p\|$, for all $k \in N$; thus we obtain that $x_k \to p$. Therefore, from the previous two cases, we can conclude that $\{x_n\}$ converges strongly to an element of F, and the proof is complete.

If, in Theorem 3.2, we consider single Lipschitz pseudocontractive mapping, then the assumption of condition (H) is not required. In fact, we have the following corollary.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a Lipschitz pseudocontractive mapping with Lipschitz constants L. Assume that $F(T)$ is nonempty. Let a sequence $\{x_n\}$ be a sequence generated from an arbitrary $x_1 = w \in C$ by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} &= \alpha_n w + (1-\alpha_n)(\gamma_n T y_n + (1-\gamma_n)x_n), \end{aligned} \tag{3.26}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying the following conditions: (i) $0 < \alpha_n \leq c < 1$, for all $n \geq 1$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum \alpha_n = \infty$; (ii) $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < 1/[\sqrt{(1+L^2)} + 1]$, for all $n \geq 1$. Then, $\{x_n\}$ converges strongly to a fixed point of T nearest to $x_1 = w$.

Proof. Putting $S_n := T$ in (3.3) the scheme reduces to scheme (3.26), and following the method of proof of Theorem 3.2 we get that (see, (3.10))

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1-\alpha_n)\ \|x_n - p\|^2 + 2\alpha_n\langle w - p, x_{n+1} - p\rangle \\ &\quad - (1-\alpha_n)\gamma_n\beta_n\left(1 - 2\beta - \beta^2L^2\right)\|Tx_n - x_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle w - p, x_{n+1} - p\rangle \\ &\quad - (1-c)\gamma_n\beta_n\left(1 - 2\beta - \beta^2L^2\right)\|Tx_n - x_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle w - p, x_{n+1} - p\rangle. \end{aligned} \tag{3.27}$$

Now, considering cases as in the proof of Theorem 3.2 we obtain the required result.

We now state and prove a convergence theorem for a common zero of finite family of monotone mappings. □

Corollary 3.4. *Let H be a real Hilbert space. Let $A_i : H \to H$, $i = 1,2,\ldots,N$ be a finite family of Lipschitz monotone mappings with Lipschitz constants L_i, respectively, satisfying $\langle A_i x, A_j x\rangle \geq 0$, for all $i, j \in \{1,2,\ldots,N\}$.*

Assume that $F := \bigcap_{i=1}^{N} N(A_i)$ is nonempty. Let a sequence $\{x_n\}$ be generated from an arbitrary $x_1 \in H$ by

$$\begin{aligned} y_n &= x_n - \beta_n A_n x_n, \\ x_{n+1} &= \alpha_n w + (1-\alpha_n)(x_n - \gamma_n A_n y_n), \end{aligned} \tag{3.28}$$

where $A_n := \theta_{n,1}A_1 + \theta_{n,2}A_2 + \cdots + \theta_{n,N}A_N$, for $\{\theta_{n,i}\} \subseteq [a,b] \subset (0,1)$ such that $\theta_{n,1} + \theta_{n,2} + \cdots + \theta_{n,N} = 1$, for all $n \geq 1$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying the following conditions: (i) $0 < \alpha_n \leq c < 1$, for all $n \geq 0$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum \alpha_n = \infty$; (ii) $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < 1/[\sqrt{(1+L^2)} + 1]$, for all $n \geq 1$, for $L := \max\{(1+L_i) : i = 1,2,\ldots,N\}$. Then, $\{x_n\}$ converges strongly to a common zero point of $\{A_1, A_2, \ldots A_N\}$ nearest to $x_1 = w$.

Proof. Let $T_i x := (I - A_i)x$, for $i = 1,2,\ldots,N$. Then we get that every T_i for all $i \in \{1, 2,\ldots,N\}$ is Lipschitz pseudocontractive mapping with Lipschitz constants $L'_i := (1+L_i)$ and $\cap_{i=1}^{N} F(T_i) = \cap_{i=1}^{N}(A_i) \neq \emptyset$. Moreover, when A_n is replaced with $(I - T_n)$, for each $i \in \{1,2,\ldots,N\}$, we get that scheme (3.28) reduces to scheme (3.3), and hence the conclusion follows from Theorem 3.2.

If, in Corollary 3.4 we consider a single Lipschitz monotone mapping, then we obtain the following corollary. □

Corollary 3.5. *Let H be a real Hilbert space. Let $A : H \to H$ be Lipschitz monotone mappings with Lipschitz constant L. Assume that $N(A)$ is nonempty. Let a sequence $\{x_n\}$ be generated from an arbitrary $x_1 \in H$ by*

$$\begin{aligned} y_n &= x_n - \beta_n A x_n, \\ x_{n+1} &= \alpha_n w + (1-\alpha_n)(x_n - \gamma_n A y_n), \end{aligned} \tag{3.29}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying the following conditions: (i) $0 < \alpha_n \le c < 1$, for all $n \ge 1$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum \alpha_n = \infty$; (ii) $0 < \alpha \le \gamma_n \le \beta_n \le \beta < 1/[\sqrt{(1+L^2\ )} + 1]$, for all $n \ge 1$. Then, $\{x_n\}$ converges strongly to a zero point of A nearest to $x_1 = w$.

We now give examples of Lipschitz pseudocontractive mappings satisfying condition (H). Let $X := \mathbb{R}$ and $C := [-24,3] \subset$ R. Let $T_1, T_2 := C \to C$ be defined by

$$\begin{aligned} T_1 x &:= \begin{cases} x, & x \in [-24,0), \\ x - x^3, & x \in [0,3], \end{cases} \\ T_2 x &:= \begin{cases} x, & x \in [-24,2), \\ 3x - x^2, & x \in [2,3]. \end{cases} \end{aligned} \tag{3.30}$$

Then we observe that $F(T_1) = [-24,0]$,and $F(T_2) = [-24,2]$, and hence common fixed point of T_1 and T_2 is $[-24,0]$ which is nonempty. Now, we show that T_1 and T_2 are pseudocontractive mappings. But, since

$$\begin{aligned} A_1 x &:= (I - T_1)x = \begin{cases} 0, & x \in [-24,0), \\ x^3, & x \in [0,3], \end{cases} \\ A_2 x &:= (I - T_2)x = \begin{cases} 0, & x \in [-24,2), \\ -2x + x^2, & x \in [2,3] \end{cases} \end{aligned} \tag{3.31}$$

are monotone, we have that T_1 and T_2 are pseudocontractive mappings.

Now, we show that T_1 and T_2 are Lipschitzian mappings. First, we show that T_1 is Lipschitzian with constant $L = 28$. Let $C_1 = [-24,0)$, $C_2 = [0,3]$. If $x, y \in C_1$, then we have that

$$|T_1 x - T_1 y| = |x - y| \le 28|x - y|. \tag{3.32}$$

If $x, y \in C_2$,then we have that

$$\begin{aligned} \left|T_1x - T_1y\right| &= \left|x - x^3 - \left(y - y^3\right)\right| \leq \left|x - y\right| + \left|x^3 - y^3\right| \\ &= \left|x - y\right| + \left|x - y\right|\left|x^2 + xy + y^2\right| \\ &= \left(1 + \left|x^2 + xy + y^2\right|\right)\left|x - y\right| \leq 28\left|x - y\right|. \end{aligned} \tag{3.33}$$

If $x \in C_1$ and $y \in C_{2,}$ then we get that

$$\begin{aligned} \left|T_1x - T_1y\right| &= \left|x - \left(y - y^3\right)\right| \leq \left|x - y\right| + \left|y^3\right| = \left|x - y\right| + y^2\left|y\right| \\ &\leq \left|x - y\right| + y^2\left|y - x\right| = \left(y^2 + 1\right)\left|x - y\right| \\ &\leq 10\left|x - y\right| \leq 28\left|x - y\right|. \end{aligned} \tag{3.34}$$

Therefore, from (3.32), (3.33), and (3.34), we obtain that T_1 is Lipschitz.

Next, we show that T_2 is Lipschitz with constant $L = 9$.

Let $D_1 = [-24, 2)$, $D_2 = [2, 3]$. If $x, y \in D_1$, then we have that

$$\left|T_2x - T_2y\right| = \left|x - y\right| \leq 9\left|x - y\right|. \tag{3.35}$$

If $x, y \in D_2$,then we get that

$$\begin{aligned} \left|T_2x - T_2y\right| &= \left|3x - x^2 - \left(3y - y^2\right)\right| \leq 3\left|x - y\right| + \left|x^2 - y^2\right| \\ &= 3\left|x - y\right| + \left|x - y\right|\left|x + y\right| \\ &\leq \left(3 + \left|x + y\right|\right)\left|x - y\right| \leq 9\left|x - y\right|. \end{aligned} \tag{3.36}$$

If $x \in D_1$ and $y \in D_2$ then we have that

$$\left|T_2x - T_2y\right| = \left|x - \left(3y - y^2\right)\right| \leq \left|x - y\right| + \left|y^2 - 2y\right| \tag{3.37}$$

and for $x \in [0, 2)$ (3.37) implies that

$$\begin{aligned} \left|T_2x - T_2y\right| &\leq \left|x - y\right| + \left|y^2 - 2y - \left(x^2 - 2x\right)\right| \\ &\leq \left|x - y\right| + 2\left|x - y\right| + \left|x - y\right|\left|x + y\right| \\ &\leq \left(3 + \left|x + y\right|\right)\left|x - y\right| \leq 9\left|x - y\right|. \end{aligned} \tag{3.38}$$

For $x \in [-24, 0)$ inequality (3.37) gives that

$$\begin{aligned}\left|T_2x - T_2y\right| &\leq \left|x - y\right| + \left|y - 2\right|\left|y\right|, \\ &\leq \left|x - y\right| + \left|y - 2\right|\left|y - x\right| \\ &= \left(1 + \left|y - 2\right|\right)\left|x - y\right| \\ &\leq 2\left|x - y\right| \leq 9\left|x - y\right|.\end{aligned} \tag{3.39}$$

Therefore, from (3.35), (3.36), and (3.39) we obtain that T_2 is Lipschitz. Furthermore, we show that T_1 and T_2 satisfy condition (H). If $x \in D_1$,then we have that $\langle T_1x-x, T_2x-x\rangle = 0$, and if $x \in D_2$ we get that $\langle T_1x - x, T_2x - x\rangle = \langle x - x^3 - x, (3x - x^2) - x\rangle = \langle -x^3, 2x - x^2\rangle = -x^3(2x - x^2) \geq 0$. Therefore, T_1 and T_1 satisfy property (H).

Remark 3.6. Theorem 3.2 provides convergence sequence to a common fixed point of finite family of Lipschitzian pseudocontractive mappings whereas Corollary 3.4 provides convergence sequence to a common zero of finite family of monotone mappings in Hilbert spaces. No compactness assumption is imposed either on T or C. This provides affirmative answer to the question raised.

Remark 3.7. Theorem 3.2 improves Theorem I, Theorem 3.1 of Zhou [15], Theorem 3.1 of Yao et al. [17], and Theorem 3.1 of Tang et al. [16] in the sense that either our convergence does not require compactness of T or computation of C_{n+1} from C_n for each $n \geq 1$.

Remark 3.8. Theorem 3.2 improves Theorems I and ZSA in the sense that our convergence is for a fixed point of a finite family of Lipschitz pseudocontractive mappings. The condition that interior of $F(T)$ is nonempty is dispensed with.

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