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Research Article

A New Subclass of Harmonic Univalent Functions Associated with Dziok-Srivastava Operator

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The purpose of the present paper is to study a certain subclass of harmonic univalent functions associated with Dziok-Srivastava operator. We obtain coefficient conditions, distortion bounds, and extreme points for the above class of harmonic univalent functions belonging to this class and discuss a class preserving integral operator. We also show that class studied in this paper is closed under convolution and convex combination. The results obtained for the class reduced to the corresponding results for several known classes in the literature are briefly indicated.

1. Introduction

A continuous complex-valued function $f = u+iv$ defined in a simply connected domain D, is said to be harmonic in D if both u and v are harmonic in D. In any simply connected domain D we can write $f = h+\overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$. See Clunie and Sheil-Small [1].

Denote by S_H the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h+\overline{g} \in S_H$, we may express the analytic function h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

Note that S_H, is reduced to S the class of normalized analytic univalent functions if the coanalytic part of $f = h + \overline{g}$ is identically zero.

For more basic results on harmonic univalent functions, one may refer to the following introductory text book by Duren [2] (see also [3–5]).

For $\alpha_j \in C$, $(j = 1,2,3,\dots,q)$ and $\beta_j \in C - \{0,-1,-2,-3,\dots\}$, $(j = 1,2,3,\dots,s)$, the generalized hypergeometric functions are defined by

$$ {}_qF_s(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\beta_2,\dots,\beta_s;z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k\cdots(\alpha_q)_k z^k}{(\beta_1)_k(\beta_2)_k\cdots(\beta_s)_k k!}, \quad (q \le s+1;\ q,s \in N_0 = \{0,1,2,\dots\}), \tag{1.2}$$

where $(\alpha)_k$ is the Pochhammer symbol defined by

$$(\alpha)_k = \begin{cases} \dfrac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+k-1), & \text{if } k = 1,2,3,\dots, \\ 1, & \text{if } k = 0. \end{cases} \tag{1.3}$$

Corresponding to the function

$$h(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\beta_2,\dots,\beta_s;z) = z\,{}_qF_s(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\beta_2,\dots,\beta_s;z). \tag{1.4}$$

The Dziok-Srivastava operator [6, 7] $H_{q,s}(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\beta_2,\dots,\beta_s)$ is defined for $f \in S$ by $H_{q,s}(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\beta_2,\dots,\beta_s)f(z) = h(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\beta_2,\dots,\beta_s;z) * f(z)$

$$= z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1}(\alpha_2)_{k-1}\dots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}(\beta_2)_{k-1}\dots(\beta_s)_{k-1}} a_k \frac{z^k}{(k-1)!}, \tag{1.5}$$

where $*$ stands for convolution of two power series.

To make the notation simple, we write

$$H_{q,s}[\alpha_1]f(z) = H_{q,s}(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\beta_2,\dots,\beta_s)f(z). \tag{1.6}$$

Special cases of the Dziok-Srivastava operator includes the Hohlov operator [8], the Carlson-Shaffer operator $L(a,c)$ [9], the Ruscheweyh derivative operator D^n [10], and the Srivastava-Owa fractional derivative operators ([11–13]).

We define the Dziok-Srivastava operator of the harmonic functions $f = h + \overline{g}$ given by (1.1) as

$$H_{q,s}[\alpha_1]f(z) = H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}. \tag{1.7}$$

Recently, Porwal [14, Chapter 5] defined the subclass $M_H(\beta) \subset S_H$ consisting of harmonic univalent functions $f(z)$ satisfying the following condition:

$$M_H(\beta) = \left\{ f \in S_H : \operatorname{Re}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right) < \beta \right\}, \quad \left(1 < \beta \le \frac{4}{3}\right),\ z \in U. \tag{1.8}$$

He proved that if $f = h + \overline{g}$ is given by (1.1) and if

$$\sum_{k=2}^{\infty} \frac{(k-\beta)}{(\beta-1)}|a_k| + \sum_{k=1}^{\infty} \frac{(k+\beta)}{(\beta-1)}|b_k| \leq 1, \quad \left(1 < \beta \leq \frac{4}{3}\right). \tag{1.9}$$

For $g \equiv 0$ the class of $M_H(\beta)$ is reduced to the class $M(\beta)$ studied by Uralegaddi et al. [15].

Generalizing the class $M_H(\beta)$, we let $M_H(\alpha_1, \beta)$ denote the family of functions $f = h + \overline{g}$ of form (1.1) which satisfy the condition

$$\operatorname{Re}\left\{ \frac{z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}}{H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}} \right\} < \beta, \tag{1.10}$$

where $(1 < \beta \leq 4/3)$ and $\Gamma(\alpha_1, k) = |(\alpha_1)_{k-1}(\alpha_2)_{k-1} \cdots (\alpha_q)_{k-1} / (\beta_1)_{k-1}(\beta_2)_{k-1} \cdots (\beta_s)_{k-1}|$.

Further, let $\overline{M}_H(\alpha_1, \beta)$ be the subclass of $M_H(\alpha_1, \beta)$ consisting of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \, \overline{z}^k. \tag{1.11}$$

In this paper, we give a sufficient condition for $f = h + \overline{g}$, given by (1.1) to be in $M_H(\alpha_1, \beta)$, and it is shown that this condition is also necessary for functions in $\overline{M}_H(\alpha_1, \beta)$. We then obtain distortion theorem, extreme points, convolution conditions, and convex combinations and discuss a class preserving integral operator for functions in $\overline{M}_H(\alpha_1, \beta)$.

2. Main Results

First, we give a sufficient coefficient bound for the class $M_H(\alpha_1, \beta)$.

Theorem 2.1. *If $f = h + \overline{g} \in S_H$ is given by* (1.1) *and if*

$$\sum_{k=2}^{\infty} \frac{(k-\beta)}{(\beta-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!}|a_k| + \sum_{k=1}^{\infty} \frac{(k+\beta)}{(\beta-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!}|b_k| \leq 1, \tag{2.1}$$

then $f \in M_H(\alpha_1, \beta)$.

Proof. Let

$$\sum_{k=2}^{\infty} \frac{(k-\beta)}{(\beta-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!}|a_k| + \sum_{k=1}^{\infty} \frac{(k+\beta)}{(\beta-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!}|b_k| \leq 1. \tag{2.2}$$

It suffices to show that

$$\left|\frac{\left(z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}\right)/\left(H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}\right) - 1}{\left(z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}\right)/\left(H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}\right) - (2\beta - 1)}\right| < 1, \quad z \in U. \tag{2.3}$$

We have

$$\begin{aligned}
&\left|\frac{\left(z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}\right)/\left(H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}\right) - 1}{\left(z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}\right)/\left(H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}\right) - (2\beta - 1)}\right| \\
&\quad \le \frac{\sum_{k=2}^{\infty}(k-1)\mathcal{A}|a_k||z|^{k-1} + \sum_{k=1}^{\infty}(k+1)\mathcal{A}|b_k||z|^{k-1}}{2(\beta-1) - \sum_{k=2}^{\infty}(k-2\beta+1)\mathcal{A}|a_k||z|^{k-1} - \sum_{k=1}^{\infty}(k+2\beta-1)\mathcal{A}|b_k||z|^{k-1}} \\
&\quad \le \frac{\sum_{k=2}^{\infty}(k-1)\mathcal{A}|a_k| + \sum_{k=1}^{\infty}(k+1)\mathcal{A}|b_k|}{2(\beta-1) - \sum_{k=2}^{\infty}(k-2\beta+1)\mathcal{A}|a_k| - \sum_{k=1}^{\infty}(k+2\beta-1)\mathcal{A}|b_k|},
\end{aligned} \tag{2.4}$$

where $\mathcal{A}$ denotes $(\Gamma(\alpha_1, k)/(k-1)!)$.

The last expression is bounded above by 1, if

$$\begin{aligned}
\sum_{k=2}^{\infty}(k-1)\frac{\Gamma(\alpha_1,k)}{(k-1)!}|a_k| + \sum_{k=1}^{\infty}(k+1)\frac{\Gamma(\alpha_1,k)}{(k-1)!}|b_k| \le 2(\beta-1) - \sum_{k=2}^{\infty}(k-2\beta+1)\frac{\Gamma(\alpha_1,k)}{(k-1)!}|a_k| \\
- \sum_{k=1}^{\infty}(k+2\beta-1)\frac{\Gamma(\alpha_1,k)}{(k-1)!}|b_k|,
\end{aligned} \tag{2.5}$$

which is equivalent to

$$\sum_{k=2}^{\infty}\frac{(k-\beta)}{(\beta-1)}\frac{\Gamma(\alpha_1,k)}{(k-1)!}|a_k| + \sum_{k=1}^{\infty}\frac{(k+\beta)}{(\beta-1)}\frac{\Gamma(\alpha_1,k)}{(k-1)!}|b_k| \le 1. \tag{2.6}$$

But (2.6) is true by hypothesis.

Hence

$$\left|\frac{\left(z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}\right)/\left(H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}\right) - 1}{\left(z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}\right)/\left(H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}\right) - (2\beta - 1)}\right| < 1, \quad z \in U, \tag{2.7}$$

and the theorem is proved.

In the following theorem, it is proved that the condition (2.1) is also necessary for functions $f = h + \overline{g} \in \overline{M}_H(\alpha_1, \beta)$ that are given by (1.11). □

Theorem 2.2. *A function* $f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \overline{z}^k$ *is in* $\overline{M}_H(\alpha_1, \beta)$*, if and only if*

$$\sum_{k=2}^{\infty} (k-\beta) \frac{\Gamma(\alpha_1, k)}{(k-1)!} |a_k| + \sum_{k=1}^{\infty} (k+\beta) \frac{\Gamma(\alpha_1, k)}{(k-1)!} |b_k| \leq (\beta - 1). \tag{2.8}$$

Proof. Since $\overline{M}_H(\alpha_1, \beta) \subset M_H(\alpha_1, \beta)$, we only need to prove the "only if" part of the theorem. For this we show that $f \notin \overline{M}_H(\alpha_1, \beta)$ if the condition (2.8) does not hold.

Note that a necessary and sufficient condition for $f = h + \overline{g}$ given by (1.9) is in $\overline{M}_H(\alpha_1, \beta)$ if

$$\operatorname{Re}\left\{ \frac{z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}}{H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}} \right\} < \beta \tag{2.9}$$

is equivalent to

$$\operatorname{Re}\left\{ \frac{(\beta-1)z - \sum_{k=2}^{\infty}(k-\beta)(\Gamma(\alpha_1,k)/(k-1)!)|a_k| z^k - \sum_{k=1}^{\infty}(k+\beta)(\Gamma(\alpha_1,k)/(k-1)!)|b_k| \overline{z}^k}{z + \sum_{k=2}^{\infty}(\Gamma(\alpha_1,k)/(k-1)!)|a_k| z^k - \sum_{k=1}^{\infty}(\Gamma(\alpha_1,k)/(k-1)!)|b_k| \overline{z}^k} \right\} \geq 0. \tag{2.10}$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\frac{(\beta-1) - \sum_{k=2}^{\infty}(k-\beta)(\Gamma(\alpha_1,k)/(k-1)!)|a_k| r^{k-1} - \sum_{k=1}^{\infty}(k+\beta)(\Gamma(\alpha_1,k)/(k-1)!)|b_k| r^{k-1}}{1 + \sum_{k=2}^{\infty}(\Gamma(\alpha_1,k)/(k-1)!)|a_k| r^{k-1} - \sum_{k=1}^{\infty}(\Gamma(\alpha_1,k)/(k-1)!)|b_k| r^{k-1}} \geq 0. \tag{2.11}$$

If the condition (2.8) does not hold then the numerator of (2.11) is negative for r and sufficiently close to 1. Thus there exists a $z_0 = r_0$ in (0,1) for which the quotient in (2.11) is negative. This contradicts the required condition for $f \in \overline{M}_H(\alpha_1, \beta)$ and so the proof is complete.

Next, we determine the extreme points of the closed convex hulls of $\overline{M}_H(\alpha_1, \beta)$, denoted by clco $\overline{M}_H(\alpha_1, \beta)$. □

Theorem 2.3. $f \in \operatorname{clco} \overline{M}_H(\alpha_1, \beta)$*, if and only if*

$$f(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z)), \tag{2.12}$$

where

$$
\begin{aligned}
h_1(z) &= z, \\
h_k(z) &= z + \frac{(\beta - 1)}{(k - \beta)} \frac{(k-1)!}{\Gamma(\alpha_1, k)} z^k, \quad k = (2, 3, \ldots), \\
g_k(z) &= z - \frac{(\beta - 1)}{(k + \beta)} \frac{(k-1)!}{\Gamma(\alpha_1, k)} \overline{z}^k, \quad k = (1, 2, 3, \ldots), \quad \sum_{k=1}^{\infty} (x_k + y_k) = 1,
\end{aligned}
\tag{2.13}
$$

$x_k \geq 0$, *and* $y_k \geq 0$. *In particular the extreme points of* $\overline{M}_H(\alpha_1, \beta)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

Proof. For functions f of the form (2.12), we have

$$
\begin{aligned}
f(z) &= \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z)) \\
&= z + \sum_{k=2}^{\infty} \frac{(\beta - 1)}{(k - \beta)} \frac{(k-1)!}{\Gamma(\alpha_1, k)} x_k z^k - \sum_{k=1}^{\infty} \frac{(\beta - 1)}{(k + \beta)} \frac{(k-1)!}{\Gamma(\alpha_1, k)} y_k \overline{z}^k.
\end{aligned}
\tag{2.14}
$$

Then

$$
\begin{aligned}
&\sum_{k=2}^{\infty} \frac{(k - \beta)}{(\beta - 1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!} \left(\frac{(\beta - 1)}{(k - \beta)} \frac{(k-1)!}{\Gamma(\alpha_1, k)} x_k \right) + \sum_{k=1}^{\infty} \frac{(k + \beta)}{(\beta - 1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!} \left(\frac{(\beta - 1)}{(k + \beta)} \frac{(k-1)!}{\Gamma(\alpha_1, k)} y_k \right) \\
&\quad = \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k \\
&\quad = 1 - x_1 \leq 1,
\end{aligned}
\tag{2.15}
$$

and so $f \in \operatorname{clco} \overline{M}_H(\alpha_1, \beta)$.

Conversely, suppose that $f \in \operatorname{clco} \overline{M}_H(\alpha_1, \beta)$. Set $x_k = ((k - \beta)/(\beta - 1))((k - 1)!/\Gamma(\alpha_1, k))|a_k|$, $(k = 2, 3, 4, \ldots)$ and $y_k = ((k + \beta)/(\beta - 1))((k - 1)!/\Gamma(\alpha_1, k))|b_k|$, $(k = 1, 2, 3, \ldots)$.

Then note that by Theorem 2.2, $0 \leq x_k \leq 1$, $(k = 2, 3, 4, \ldots)$ and $0 \leq y_k \leq 1$, $(k = 1, 2, 3, \ldots)$. We define $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$, and by Theorem 2.2, $x_1 \geq 0$.

Consequently, we obtain $f(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z))$ as required. □

Theorem 2.4. *If* $f \in \overline{M}_H(\alpha_1, \beta)$, *then*

$$
\begin{aligned}
|f(z)| &\leq (1 + |b_1|) r + \frac{1}{\Gamma(\alpha_1, 2)} \left(\frac{(\beta - 1)}{(2 - \beta)} - \frac{(\beta + 1)}{(2 - \beta)} |b_1| \right) r^2, \quad |z| = r < 1, \\
|f(z)| &\geq (1 - |b_1|) r - \frac{1}{\Gamma(\alpha_1, 2)} \left(\frac{(\beta - 1)}{(2 - \beta)} - \frac{(\beta + 1)}{(2 - \beta)} |b_1| \right) r^2, \quad |z| = r < 1.
\end{aligned}
\tag{2.16}
$$

Proof. We only prove the right hand inequality. The proof for left hand inequality is similar and will be omitted. Let $f \in \overline{M}_H(\alpha_1, \beta)$. Taking the absolute value of f, we have

$$\begin{aligned}
\left|f(z)\right| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\
&\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \\
&= (1+|b_1|)r + \frac{(\beta-1)}{(2-\beta)}\frac{1}{\Gamma(\alpha_1,2)}\sum_{k=2}^{\infty}\left(\frac{(2-\beta)\Gamma(\alpha_1,2)}{(\beta-1)}|a_k| + \frac{(2-\beta)\Gamma(\alpha_1,2)}{(\beta-1)}|b_k|\right)r^2 \\
&\leq (1+|b_1|)r + \frac{(\beta-1)}{(2-\beta)}\frac{1}{\Gamma(\alpha_1,2)}\sum_{k=2}^{\infty}\left(\frac{(k-\beta)}{(\beta-1)}\frac{\Gamma(\alpha_1,k)}{(k-1)!}|a_k| + \frac{(k+\beta)}{(\beta-1)}\frac{\Gamma(\alpha_1,k)}{(k-1)!}|b_k|\right)r^2 \\
&\leq (1+|b_1|)r + \frac{(\beta-1)}{(2-\beta)}\frac{1}{\Gamma(\alpha_1,2)}\left(1-\frac{(\beta+1)}{(\beta-1)}|b_1|\right)r^2 \\
&= (1+|b_1|)r + \frac{1}{\Gamma(\alpha_1,2)}\left(\frac{(\beta-1)}{(2-\beta)} - \frac{(\beta+1)}{(2-\beta)}|b_1|\right)r^2.
\end{aligned} \tag{2.17}$$

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$\begin{aligned}
f(z) &= z + \sum_{k=2}^{\infty}|a_k|z^k - \sum_{k=1}^{\infty}|b_k|\overline{z}^k, \\
F(z) &= z + \sum_{k=2}^{\infty}|A_k|z^k - \sum_{k=1}^{\infty}|B_k|\overline{z}^k.
\end{aligned} \tag{2.18}$$

We define the convolution of two harmonic functions f and F as

$$(f * F)(z) = f(z) * F(z) = z + \sum_{k=2}^{\infty}|a_k A_k|z^k - \sum_{k=1}^{\infty}|b_k B_k|\overline{z}^k. \tag{2.19}$$

Using this definition, we show that the class $\overline{M}_H(\alpha_1, \beta)$ is closed under convolution. □

Theorem 2.5. *For $1 < \gamma \leq \beta \leq 4/3$, let $f \in \overline{M}_H(\alpha_1, \gamma)$ and $F \in \overline{M}_H(\alpha_1, \beta)$. Then*

$$f * F \in \overline{M}_H(\alpha_1, \gamma) \subseteq \overline{M}_H(\alpha_1, \beta). \tag{2.20}$$

Proof. Let $f(z) = z + \sum_{k=2}^{\infty}|a_k|z^k - \sum_{k=1}^{\infty}|b_k|\overline{z}^k$ be in $\overline{M}_H(\alpha_1, \gamma)$, and $F(z) = z + \sum_{k=2}^{\infty}|A_k|z^k - \sum_{k=1}^{\infty}|B_k|\overline{z}^k$ be in $\overline{M}_H(\alpha_1, \beta)$.

Then the convolution $f * F$ is given by (2.19). We wish to show that the coefficients of $f * F$ satisfy the required condition given in Theorem 2.2. For $F(z) \in \overline{M}_H(\alpha_1, \beta)$ we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now, for the convolution function $f * F$, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{(k-\alpha)}{(\alpha-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!} |a_k A_k| + \sum_{k=1}^{\infty} \frac{(k+\alpha)}{(\alpha-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!} |b_k B_k| \\
&\quad \leq \sum_{k=2}^{\infty} \frac{(k-\alpha)}{(\alpha-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!} |a_k| + \sum_{k=1}^{\infty} \frac{(k+\alpha)}{(\alpha-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!} |b_k| \\
&\quad \leq 1, \quad \left(\text{since } f \in \overline{M}_H(\alpha_1, \gamma)\right).
\end{aligned} \tag{2.21}$$

Therefore $f * F \in \overline{M}_H(\alpha_1, \gamma) \subseteq \overline{M}_H(\alpha_1, \beta)$. □

Theorem 2.6. *The class $\overline{M}_H(\alpha_1, \beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$, let $f_i(z) \in \overline{M}_H(\alpha_1, \beta)$, where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{k=2}^{\infty} |a_{k_i}| z^k - \sum_{k=1}^{\infty} |b_{k_i}| \overline{z}^k. \tag{2.22}$$

Then by Theorem 2.2,

$$\sum_{k=2}^{\infty} \frac{(k-\beta)}{(\beta-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!} |a_{k_i}| + \sum_{k=1}^{\infty} \frac{(k+\beta)}{(\beta-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!} |b_{k_i}| \leq 1. \tag{2.23}$$

For $\sum_{i-1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i-1}^{\infty} t_i f_i(z) = z + \sum_{k=2}^{\infty} \left(\sum_{i-1}^{\infty} t_i |a_{k_i}| \right) z^k - \sum_{k=1}^{\infty} \left(\sum_{i-1}^{\infty} t_i |b_{k_i}| \right) \overline{z}^k. \tag{2.24}$$

Then by (2.23),

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{(k-\beta)}{(\beta-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!} \left(\sum_{i-1}^{\infty} t_i |a_{k_i}| \right) + \sum_{k=1}^{\infty} \frac{(k+\beta)}{(\beta-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!} \left(\sum_{i-1}^{\infty} t_i |b_{k_i}| \right) \\
&\quad = \sum_{i-1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \frac{(k-\beta)}{(\beta-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!} |a_{k_i}| + \sum_{k=1}^{\infty} \frac{(k+\beta)}{(\beta-1)} \frac{\Gamma(\alpha_1, k)}{(k-1)!} |b_{k_i}| \right) \\
&\quad \leq \sum_{i-1}^{\infty} t_i = 1.
\end{aligned} \tag{2.25}$$

This is the condition required by Theorem 2.2 and so $\sum_{i-1}^{\infty} t_i f_i(z) \in \overline{M}_H(\alpha_1, \beta)$. □

3. A Family of Class Preserving Integral Operator

Let $f(z) = h(z) + \overline{g(z)}$ be defined by (1.1); then $F(z)$ defined by the relation

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt + \overline{\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt}, \quad (c > -1). \tag{3.1}$$

Theorem 3.1. *Let $f(z) = h(z) + \overline{g(z)} \in S_H$ be given by* (1.11) *and $f \in \overline{M}_H(\alpha_1, \beta)$, then $F(z)$ is defined by* (3.1) *also belong to $\overline{M}_H(\alpha_1, \beta)$.*

Proof. Let $f(z) = z + \sum_{k=2}^{\infty}|a_k|z^k - \sum_{k=1}^{\infty}|b_k|\overline{z}^k$ be in $\overline{M}_H(\alpha_1, \beta)$, then by Theorem 2.2, we have

$$\sum_{k=2}^{\infty}\frac{(k-\beta)}{(\beta-1)}\frac{\Gamma(\alpha_1,k)}{(k-1)!}|a_k| + \sum_{k=1}^{\infty}\frac{(k+\beta)}{(\beta-1)}\frac{\Gamma(\alpha_1,k)}{(k-1)!}|b_k| \le 1. \tag{3.2}$$

By definition of $F(z)$, we have

$$F(z) = z + \sum_{k=2}^{\infty}\frac{(c+1)}{(c+k)}|a_k|z^k - \sum_{k=1}^{\infty}\frac{(c+1)}{(c+k)}|b_k|\overline{z}^k. \tag{3.3}$$

Now we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{(k-\beta)}{(\beta-1)}\frac{\Gamma(\alpha_1,k)}{(k-1)!}\left(\frac{(c+1)}{(c+k)}|a_k|\right) + \sum_{k=1}^{\infty}\frac{(k+\beta)}{(\beta-1)}\frac{\Gamma(\alpha_1,k)}{(k-1)!}\left(\frac{(c+1)}{(c+k)}|b_k|\right) \\
&\quad \le \sum_{k=2}^{\infty}\frac{(k-\beta)}{(\beta-1)}\frac{\Gamma(\alpha_1,k)}{(k-1)!}|a_k| + \sum_{k=1}^{\infty}\frac{(k+\beta)}{(\beta-1)}\frac{\Gamma(\alpha_1,k)}{(k-1)!}|b_k| \\
&\quad \le 1.
\end{aligned} \tag{3.4}$$

Thus $F(z) \in \overline{M}_H(\alpha_1, \beta)$. □

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