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Research Article

Index of Quasiconformally Symmetric Semi-Riemannian Manifolds

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We find the index of $\tilde{\nabla}$-quasiconformally symmetric and $\tilde{\nabla}$-concircularly symmetric semi-Riemannian manifolds, where $\tilde{\nabla}$ is metric connection.

1. Introduction

In 1923, Eisenhart [1] gave the condition for the existence of a second-order parallel symmetric tensor in a Riemannian manifold. In 1925, Levy [2] proved that a second-order parallel symmetric nonsingular tensor in a real-space form is always proportional to the Riemannian metric. As an improvement of the result of Levy, Sharma [3] proved that any second-order parallel tensor (not necessarily symmetric) in a real-space form of dimension greater than 2 is proportional to the Riemannian metric. In 1939, Thomas [4] defined and studied the index of a Riemannian manifold. A set of metric tensors (a metric tensor on a differentiable manifold is a symmetric nondegenerate parallel $(0,2)$ tensor field on the differentiable manifold) $\{H_1,\ldots,H_\ell\}$ is said to be *linearly independent* if

$$c_1H_1+\cdots+c_\ell H_\ell=0,\qquad c_1,\ldots,c_\ell\in\mathbf{R},\tag{1.1}$$

implies that

$$c_1=\cdots=c_\ell=0.\tag{1.2}$$

The set $\{H_1, \ldots, H_\ell\}$ is said to be a complete set if any metric tensor H can be written as

$$H = c_1 H_1 + \cdots + c_\ell H_\ell, \qquad c_1, \ldots, c_\ell \in \mathbf{R}. \tag{1.3}$$

More precisely, the number of linearly independent metric tensors in a complete set of metric tensors of a Riemannian manifold is called the index of the Riemannian manifold [4, page 413]. Thus, the problem of existence of a second-order parallel symmetric tensor is closely related with the index of Riemannian manifolds. Later, in 1968, Levine and Katzin [5] studied the index of conformally flat Riemannian manifolds. They proved that the index of an n-dimensional conformally flat manifold is $n(n+1)/2$ or 1 according as it is a flat manifold or a manifold of nonzero constant curvature. In 1981, Stavre [6] proved that if the index of an n-dimensional conformally symmetric Riemannian manifold (except the four cases of being conformally flat, of constant curvature, an Einstein manifold or with covariant constant Einstein tensor) is greater than one, then it must be between 2 and $n+1$. In 1982, Starve and Smaranda [7] found the index of a conformally symmetric Riemannian manifolds with respect to a semisymmetric metric connection of Yano [8]. More precisely, they proved the following result: "Let a Riemannian manifold be conformally symmetric with respect to a semisymmetric metric connection $\overline{\nabla}$. Then (a) the index $i_{\overline{\nabla}}$ is 1 if there is a vector field U such that $\overline{\nabla}_U \overline{E} = 0$ and $\overline{\nabla}_U \overline{r} \neq 0$, where $\overline{E}$ and $\overline{r}$ are the Einstein tensor field and the scalar curvature with respect to the connection $\overline{\nabla}$, respectively; and (b) the index $i_{\overline{\nabla}}$ satisfies $1 < i_{\overline{\nabla}} \leq n+1$ if $\overline{\nabla}\ \overline{E} \neq 0$."

A real-space form is always conformally flat, and a conformally flat manifold is always conformally symmetric. But the converse is not true in both the cases. On the other hand, the quasiconformal curvature tensor [9] is a generalization of the Weyl conformal curvature tensor and the concircular curvature tensor. The Levi-Civita connection and semisymmetric metric connection are the particular cases of a metric connection. Also, a metric connection is Levi-Civita connection when its torsion is zero and it becomes the Hayden connection [10] when it has nonzero torsion. Thus, metric connections include both the Levi-Civita connections and the Hayden connections (in particular, semisymmetric metric connections).

Motivated by these circumstances, it becomes necessary to study the index of quasiconformally symmetric semi-Riemannian manifolds with respect to any metric connection. The paper is organized as follows. In Section 2, we give the definition of the index of a semi-Riemannian manifold and give the definition and some examples of the Ricci symmetric metric connections $\tilde{\nabla}$. In Section 3, we give the definition of the quasiconformal curvature tensor with respect to a metric connection $\tilde{\nabla}$. We also obtain a complete classification of $\tilde{\nabla}$-quasiconformally flat (and in particular, quasiconformally flat) manifolds. In Section 4, we find out the index of $\tilde{\nabla}$-quasiconformally symmetric manifolds and $\tilde{\nabla}$-concircularly symmetric manifolds. In the last section, we discuss some of applications in theory of relativity.

2. Index of a Semi-Riemannian Manifold

Let M be an n-dimensional differentiable manifold. Let $\tilde{\nabla}$ be a linear connection in M. Then torsion tensor $\tilde{T}$ and curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ are given by

$$\begin{aligned}\tilde{T}(X,Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y],\\ \tilde{R}(X,Y)Z &= \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z\end{aligned} \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of vector fields in M. By a semi-Riemannian metric [11] on M, we understand a nondegenerate symmetric $(0,2)$ tensor field g. In [4], a semi-Riemannian metric is called simply a metric tensor. A positive definite symmetric $(0,2)$ tensor field is well known as a Riemannian metric, which, in [4], is called a fundamental metric tensor. A symmetric $(0,2)$ tensor field g of rank less than n is called a degenerate metric tensor [4].

Let (M, g) be an n-dimensional semi-Riemannian manifold. A linear connection $\tilde{\nabla}$ in M is called a metric connection with respect to the semi-Riemannian metric g if $\tilde{\nabla} g = 0$. If the torsion tensor of the metric connection $\tilde{\nabla}$ is zero, then it becomes Levi-Civita connection ∇, which is unique by the fundamental theorem of Riemannian geometry. If the torsion tensor of the metric connection $\tilde{\nabla}$ is not zero, then it is called a Hayden connection [10, 12]. Semisymmetric metric connections [8] and quarter symmetric metric connections [13] are some well-known examples of Hayden connections.

Let (M, g) be an n-dimensional semi-Riemannian manifold. For a metric connection $\tilde{\nabla}$ in M, the curvature tensor $\tilde{R}$ with respect to the $\tilde{\nabla}$ satisfies the following condition:

$$\begin{aligned}\tilde{R}(X,Y,Z,V) + \tilde{R}(Y,X,Z,V) &= 0,\\ \tilde{R}(X,Y,Z,V) + \tilde{R}(X,Y,V,Z) &= 0\end{aligned} \tag{2.2}$$

for all $X, Y, Z, V \in \mathfrak{X}(M)$, where

$$\tilde{R}(X,Y,Z,V) = g\left(\tilde{R}(X,Y)Z, V\right). \tag{2.3}$$

The Ricci tensor $\tilde{S}$ and the scalar curvature $\tilde{r}$ of the semi-Riemannian manifold with respect to the metric connection $\tilde{\nabla}$ is defined by

$$\begin{aligned}\tilde{S}(X,Y) &= \sum_{i=1}^{n} \varepsilon_i \tilde{R}(e_i, X, Y, e_i),\\ \tilde{r} &= \sum_{i=1}^{n} \varepsilon_i \tilde{S}(e_i, e_i),\end{aligned} \tag{2.4}$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of vector fields in the manifold M and $\varepsilon_i = g(e_i, e_i)$. The Ricci operator $\tilde{Q}$ with respect to the metric connection $\tilde{\nabla}$ is defined by

$$\tilde{S}(X,Y) = g\left(\tilde{Q}X, Y\right), \quad X, Y \in \mathfrak{X}(M). \tag{2.5}$$

Define

$$\tilde{e}X = \tilde{Q}X - \frac{\tilde{r}}{n}X, \quad X \in \mathfrak{X}(M),$$
$$\tilde{E}(X,Y) = g(\tilde{e}\,X,Y), \quad X,Y \in \mathfrak{X}(M). \tag{2.6}$$

Then, consider

$$\tilde{E} = \tilde{S} - \frac{\tilde{r}}{n}\,g. \tag{2.7}$$

The $(0,2)$ tensor $\tilde{E}$ is called tensor of Einstein [14] with respect to the metric connection $\tilde{\nabla}$. If $\tilde{S}$ is symmetric, then $\tilde{E}$ is also symmetric.

Definition 2.1. A metric connection $\tilde{\nabla}$ with symmetric Ricci tensor $\tilde{S}$ will be called a "*Ricci-symmetric metric connection*."

Example 2.2. In a semi-Riemannian manifold (M,g), a semisymmetric metric connection $\overline{\nabla}$ of Yano [8] is given by

$$\overline{\nabla}_X Y = \nabla_X Y + u(Y)X - g(X,Y)U, \quad X,Y \in \mathfrak{X}(M), \tag{2.8}$$

where ∇ is Levi-Civita connection, U is a vector field, and u is its associated 1 form given by $u(X) = g(X,U)$. The Ricci tensor $\overline{S}$ with respect to $\overline{\nabla}$ is given by

$$\overline{S} = S - (n-2)\alpha - \operatorname{trace}(\alpha)g, \tag{2.9}$$

where S is the Ricci tensor, and α is a $(0,2)$ tensor field defined by

$$\alpha(X,Y) = (\nabla_X u)(Y) - u(X)u(Y) + \frac{1}{2}u(U)g(X,Y), \quad X,Y \in \mathfrak{X}(M). \tag{2.10}$$

The Ricci tensor $\overline{S}$ is symmetric if 1 form, u is closed.

Example 2.3. An (ε)-almost para contact metric manifold $(M,\varphi,\xi,\eta,g,\varepsilon)$ is given by

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X,\varphi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \tag{2.11}$$

where φ is a tensor field of type $(1,1)$, η is 1 form, ξ is a vector field and $\varepsilon = \pm 1$. An (ε)-almost para contact metric manifold satisfying

$$(\nabla_X \varphi)Y = -\,g(\varphi X,\varphi Y)\xi - \varepsilon\eta(Y)\varphi^2 X \tag{2.12}$$

is called an (ε)-para Sasakian manifold [15]. In an (ε)-para Sasakian manifold, the semisymmetric metric connection $\overline{\nabla}$ given by

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi \tag{2.13}$$

is a Ricci symmetric metric connection.

Example 2.4. An almost contact metric manifold (M,φ,ξ,η,g) is given by

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.14}$$

where φ is a tensor field of type $(1,1)$, η is 1-form and ξ is a vector field. An almost contact metric manifold is a *Kenmotsu manifold* [16] if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{2.15}$$

and is a Sasakian manifold [17] if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.16}$$

In an almost contact metric manifold M, the semisymmetric metric connection $\overline{\nabla}$ given by

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi \tag{2.17}$$

is a Ricci symmetric metric connection if M is Kenmotsu, but the connection fails to be Ricci symmetric if M is Sasakian.

Let (M,g) be an n-dimensional semi-Riemannian manifold equipped with a metric connection $\tilde{\nabla}$. A symmetric $(0,2)$ tensor field H, which is covariantly constant with respect to $\tilde{\nabla}$, is called a *special quadratic first integral* (for brevity SQFI) [18] with respect to $\tilde{\nabla}$. The semi-Riemannian metric g is always an SQFI. A set of SQFI tensors $\{H_1,\ldots,H_\ell\}$ with respect to $\tilde{\nabla}$ is said to be *linearly independent* if

$$c_1H_1 + \cdots + c_\ell H_\ell = 0, \quad c_1,\ldots,c_\ell \in \mathbf{R} \tag{2.18}$$

implies that

$$c_1 = \cdots = c_\ell = 0. \tag{2.19}$$

The set $\{H_1,\ldots,H_\ell\}$ is said to be a complete set if any SQFI tensor H with respect to $\tilde{\nabla}$ can be written as

$$H = c_1H_1 + \cdots + c_\ell H_\ell, \quad c_1,\ldots,c_\ell \in \mathbf{R}. \tag{2.20}$$

The "*index*" [4] of the manifold M with respect to $\tilde{\nabla}$, denoted by $i_{\tilde{\nabla}}$, is defined to be the number ℓ of members in a complete set $\{H_1, \ldots, H_\ell\}$.

We will need the following Lemma.

Lemma 2.5. *Let (M, g) be an n-dimensional semi-Riemannian manifold equipped with a Ricci symmetric metric connection $\tilde{\nabla}$. Then the following statements are true.*

(a) *If $\tilde{\nabla}_X \tilde{S} = 0$, then $\tilde{\nabla}_X \tilde{E} = 0$. Conversely, if $\tilde{r}$ is constant and $\tilde{\nabla}_X \tilde{E} = 0$ then $\tilde{\nabla}_X \tilde{S} = 0$.*

(b) *If $\tilde{\nabla}_X \tilde{S} \neq 0$ and ψ is a nonvanishing differentiable function such that $\psi \tilde{\nabla}_X \tilde{S}$ and g are linearly dependent, then $\tilde{\nabla}_X \tilde{E} = 0$.*

The proof is similar to Lemmas 1.2 and 1.3 in [7] for a semisymmetric metric connection and is therefore omitted.

3. Quasiconformal Curvature Tensor

Let (M, g) be an n-dimensional $(n > 3)$ semi-Riemannian manifold equipped with a metric connection $\tilde{\nabla}$. The conformal curvature tensor $\tilde{\mathcal{C}}$ with respect to the $\tilde{\nabla}$ is defined by [19, page 90] as follow:

$$\begin{aligned}\tilde{\mathcal{C}}(X, Y, Z, V) = \tilde{R}(X, Y, Z, V) &- \frac{1}{n-2}\Big(\tilde{S}(Y, Z) g(X, V) - \tilde{S}(X, Z) g(Y, V) \\ &+ g(Y, Z)\tilde{S}(X, V) - g(X, Z)\tilde{S}(Y, V)\Big) \\ &+ \frac{\tilde{r}}{(n-1)(n-2)}\big(g(Y, Z) g(X, V) - g(X, Z) g(Y, V)\big),\end{aligned} \tag{3.1}$$

and the concircular curvature tensor $\tilde{\mathcal{Z}}$ with respect to $\tilde{\nabla}$ is defined by ([20], [21, page 87]) as follows:

$$\tilde{\mathcal{Z}}(X, Y, Z, V) = \tilde{R}(X, Y, Z, V) - \frac{\tilde{r}}{n(n-1)}\big(g(Y, Z) g(X, V) - g(X, Z) g(Y, V)\big). \tag{3.2}$$

As a generalization of the notion of conformal curvature tensor and concircular curvature tensor, the quasiconformal curvature tensor $\tilde{\mathcal{C}}_*$ with respect to $\tilde{\nabla}$ is defined by [9] as follows:

$$\begin{aligned}\tilde{\mathcal{C}}_*(X, Y, Z, V) = a\tilde{R}(X, Y, Z, V) &+ b\Big(\tilde{S}(Y, Z) g(X, V) - \tilde{S}(X, Z) g(Y, V) \\ &+ g(Y, Z)\tilde{S}(X, V) - g(X, Z)\tilde{S}(Y, V)\Big) \\ &- \frac{\tilde{r}}{n}\left\{\frac{a}{n-1} + 2b\right\}\big(g(Y, Z) g(X, V) - g(X, Z) g(Y, V)\big),\end{aligned} \tag{3.3}$$

where a and b are constants. In fact, we have

$$\tilde{\mathcal{C}}_*(X, Y, Z, V) = -(n-2)b\, \tilde{\mathcal{C}}(X, Y, Z, V) + (a + (n-2)b)\tilde{\mathcal{Z}}(X, Y, Z, V). \tag{3.4}$$

Since there is no restrictions for manifolds if $a = 0$ and $b = 0$, therefore it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$. From (3.4) it is clear that if $a = 1$ and $b = -1/(n-2)$, then $\widetilde{\mathcal{C}}_* = \widetilde{\mathcal{C}}$; if $a = 1$ and $b = 0$, then $\widetilde{\mathcal{C}}_* = \widetilde{\mathcal{Z}}$.

Now, we need the following.

Definition 3.1. A semi-Riemannian manifold (M, g) equipped with a metric connection $\widetilde{\nabla}$ is said to be

(a) $\widetilde{\nabla}$-quasiconformally flat if $\widetilde{\mathcal{C}}_* = 0$,

(b) $\widetilde{\nabla}$-conformally flat if $\widetilde{\mathcal{C}} = 0$, and

(c) $\widetilde{\nabla}$-concircularly flat if $\widetilde{\mathcal{Z}} = 0$.

In particular, with respect to the Levi-Civita connection ∇, $\widetilde{\nabla}$-quasiconformally flat, $\widetilde{\nabla}$ conformally flat, and $\widetilde{\nabla}$-concircularly flat become simply quasiconformally flat, conformally flat, and concircularly flat, respectively.

Definition 3.2. A semi-Riemannian manifold (M, g) equipped with a metric connection $\widetilde{\nabla}$ is said to be

(a) $\widetilde{\nabla}$-quasiconformally symmetric if $\widetilde{\nabla}\ \widetilde{\mathcal{C}}_* = 0$,

(b) $\widetilde{\nabla}$-conformally symmetric if $\widetilde{\nabla}\widetilde{\mathcal{C}} = 0$, and

(c) $\widetilde{\nabla}$-concircularly symmetric if $\widetilde{\nabla}\widetilde{\mathcal{Z}} = 0$.

In particular, with respect to the Levi-Civita connection ∇, $\widetilde{\nabla}$-quasiconformally symmetric, $\widetilde{\nabla}$-conformally symmetric, and $\widetilde{\nabla}$-concircularly symmetric become simply quasiconformally symmetric, conformally symmetric, and concircularly symmetric, respectively.

Theorem 3.3. *Let M be a semi-Riemannian manifold of dimension $n > 2$. Then M is $\widetilde{\nabla}$-quasiconformally flat if and only if one of the following statements is true:*

(i) *$a + (n-2)b = 0$, $a \neq 0 \neq b$, and M is $\widetilde{\nabla}$-conformally flat,*

(ii) *$a + (n-2)b \neq 0$, $a \neq 0$, M is $\widetilde{\nabla}$-conformally flat, and $\widetilde{\nabla}$-concircularly flat,*

(iii) *$a + (n-2)b \neq 0$, $a = 0$ and Ricci tensor $\widetilde{S}$ with respect to $\widetilde{\nabla}$ satisfies*

$$\widetilde{S} - \frac{\widetilde{r}}{n} g = 0, \tag{3.5}$$

where $\widetilde{r}$ is the scalar curvature with respect to $\widetilde{\nabla}$.

Proof. Using $\widetilde{\mathcal{C}}_* = 0$ in (3.3), we get

$$\begin{aligned} 0 = a\widetilde{R}(X, Y, Z, V) &+ b\Big(\widetilde{S}(Y, Z)g(X, V) - \widetilde{S}(X, Z)g(Y, V) \\ &+ g(Y, Z)\widetilde{S}(X, V) - g(X, Z)\widetilde{S}(Y, V)\Big) \\ &- \frac{\widetilde{r}}{n}\left(\frac{a}{n-1} + 2b\right)\big(g(Y, Z)g(X, V) - g(X, Z)g(Y, V)\big), \end{aligned} \tag{3.6}$$

from which we obtain the following:

$$(a + (n-2)b)\left(\tilde{S} - \frac{\tilde{r}}{n}\, g\right) = 0. \tag{3.7}$$

Case 1 ($a + (n-2)b = 0$ and $a \neq 0 \neq b$). Then from (3.3) and (3.1), it follows that $(n-2)b\,\tilde{\mathcal{C}} = 0$, which gives $\tilde{\mathcal{C}} = 0$. This gives the statement (i).

Case 2 ($a + (n-2)b \neq 0$ and $a \neq 0$). Then from (3.7)

$$\tilde{S}(Y, Z) = \frac{\tilde{r}}{n} g(Y, Z). \tag{3.8}$$

Using (3.8) in (3.6), we get

$$a\left(\tilde{R}(X, Y, Z, V) - \frac{\tilde{r}}{n(n-1)}(g(Y, Z)g(X, V) - g(X, Z)g(Y, V))\right) = 0. \tag{3.9}$$

Since $a \neq 0$, then by (3.2) $\tilde{\mathcal{Z}} = 0$ and by using (3.9), (3.8) in (3.1), we get $\tilde{\mathcal{C}} = 0$. This gives the statement (ii).

Case 3 ($+(n-2)b \neq 0$ and $a = 0$, we get (3.5)). This gives the statement (iii). Converse is true in all cases. □

Corollary 3.4 (see [22], Theorem 5.1). *Let M be a semi-Riemannian manifold of dimension $n > 2$. Then M is quasiconformally flat if and only if one of the following statements is true:*

(i) $a + (n-2)b = 0$, $a \neq 0 \neq b$, *and M is conformally flat,*

(ii) $a + (n-2)b \neq 0$, $a \neq 0$,*and M is of constant curvature, and*

(iii) $a + (n-2)b \neq 0$, $a = 0$, *and M is Einstein manifold.*

Remark 3.5. In [23], the following three results are known.

(a) [23, Proposition 1.1]. A quasiconformally flat manifold is either conformally flat or Einstein.

(b) [23, Corollary 1.1]. A quasiconformally flat manifold is conformally flat if the constant $a \neq 0$.

(c) [23, Corollary 1.2]. A quasiconformally flat manifold is Einstein if the constants $a = 0$ and $b \neq 0$.

However, the converses need not be true in these three results. But, in Corollary 3.4 we get a complete classification of quasiconformally flat manifolds.

4. $\tilde{\nabla}$-Quasiconformally Symmetric Manifolds

Let (M, g) be an n-dimensional semi-Riemannian manifold equipped with the metric connection $\tilde{\nabla}$. Let $\tilde{R}$ be the curvature tensor of M with respect to the metric connection $\tilde{\nabla}$.

If H is a parallel symmetric $(0,2)$ tensor with respect to the metric connection $\tilde{\nabla}$, then we easily obtain that

$$H\left(\left(\tilde{\nabla}_U\tilde{R}\right)(X,Y)Z,V\right)+H\left(Z,\left(\tilde{\nabla}_U\tilde{R}\right)(X,Y)V\right)=0,\qquad X,Y,Z,V,U\in\mathfrak{X}(M). \tag{4.1}$$

The solutions H of (4.1) is closely related to the index of quasiconformally symmetric and concircularly symmetric manifold with respect to the $\tilde{\nabla}$.

Lemma 4.1. *Let (M,g) be an n-dimensional semi-Riemannian $\tilde{\nabla}$-quasiconformally symmetric manifold, $n>2$ and $b\neq 0$. Then*

$$\operatorname{trace}\left(\tilde{\nabla}_U\tilde{E}\right)=0. \tag{4.2}$$

Proof. Using (2.7) in (3.3), we get the following:

$$\begin{aligned}\tilde{\mathcal{C}}_*(X,Y,Z,V) = {}& a\tilde{R}(X,Y,Z,V)+b\Big(\tilde{E}(Y,Z)g(X,V)-\tilde{E}(X,Z)g(Y,V)\\ &+g(Y,Z)\tilde{E}(X,V)-g(X,Z)\tilde{E}(Y,V)\Big)\\ &-\frac{a\,\tilde{r}}{n(n-1)}\left(g(Y,Z)g(X,V)-g(X,Z)g(Y,V)\right).\end{aligned} \tag{4.3}$$

Taking covariant derivative of (4.3) and using $\tilde{\nabla}_U\tilde{\mathcal{C}}_*=0$, we get

$$\begin{aligned}a\left(\tilde{\nabla}_U\tilde{R}\right)(X,Y,Z,V) = {}& b\Big(\left(\tilde{\nabla}_U\tilde{E}\right)(X,Z)g(Y,V)-\left(\tilde{\nabla}_U\tilde{E}\right)(Y,Z)g(X,V)\\ &-g(Y,Z)\left(\tilde{\nabla}_U\tilde{E}\right)(X,V)+g(X,Z)\left(\tilde{\nabla}_U\tilde{E}\right)(Y,V)\Big)\\ &+\frac{a\left(\tilde{\nabla}_U\tilde{r}\right)}{n(n-1)}\left(g(Y,Z)g(X,V)-g(X,Z)g(Y,V)\right).\end{aligned} \tag{4.4}$$

Contracting (4.4) with respect to Y and Z and using (2.2), we get

$$\begin{aligned}a\left(\tilde{\nabla}_U\tilde{S}\right)(X,V) = {}& -\ b\operatorname{trace}\left(\tilde{\nabla}_U\tilde{E}\right)g(X,V)\\ &-\ (n-2)b\left(\tilde{\nabla}_U\tilde{E}\right)(X,V)+\frac{a\left(\tilde{\nabla}_U\tilde{r}\right)}{n}g(X,V).\end{aligned} \tag{4.5}$$

Using (4.5), we get (4.2). □

Theorem 4.2. *If (M, g) is an n-dimensional semi-Riemannian $\tilde{\nabla}$-quasiconformally symmetric manifold, $n > 2$ and $b \neq 0$, then* (4.1) *takes the form*

$$\det\begin{pmatrix} H(X,Z) - \frac{1}{n}\operatorname{trace}(H)g(X,Z) & H(Y,V) - \frac{1}{n}\operatorname{trace}(H)g(Y,V) \\ \left(\tilde{\nabla}_U \tilde{E}\right)(X,Z) & \left(\tilde{\nabla}_U \tilde{E}\right)(Y,V) \end{pmatrix} = 0. \tag{4.6}$$

If $\tilde{\nabla}_U \tilde{E} \neq 0$, then (4.6) *has the general solution*

$$H_U(X,Y) = f\left(\tilde{\nabla}_U \tilde{S}\right)(X,Y) + \frac{1}{n}\left(\operatorname{trace}(H_U) - f\left(\tilde{\nabla}_U \tilde{r}\right)\right)g(X,Y), \tag{4.7}$$

where f is an arbitrary nonvanishing differentiable function.

Proof. Using (4.4) in (4.1), we get

$$\begin{aligned} 0 = b\Big(&\left(\tilde{\nabla}_U \tilde{E}\right)(X,Z)H(Y,V) - \left(\tilde{\nabla}_U \tilde{E}\right)(Y,Z)H(X,V) \\ &- g(Y,Z)H\left(\left(\tilde{\nabla}_U \tilde{e}\right)X,V\right) + g(X,Z)H\left(\left(\tilde{\nabla}_U \tilde{e}\right)Y,V\right) \\ &+ \left(\tilde{\nabla}_U \tilde{E}\right)(X,V)H(Y,Z) - \left(\tilde{\nabla}_U \tilde{E}\right)(Y,V)H(X,Z) \\ &-g(Y,V)H\left(\left(\tilde{\nabla}_U \tilde{e}\right)X,Z\right) + g(X,V)H\left(\left(\tilde{\nabla}_U \tilde{e}\right)Y,Z\right)\Big) \\ &+ \frac{a\left(\tilde{\nabla}_U \tilde{r}\right)}{n(n-1)}\big(g(Y,Z)H(X,V) - g(X,Z)H(Y,V) \\ &\qquad +g(Y,V)H(X,Z) - g(X,V)H(Y,Z)\big). \end{aligned} \tag{4.8}$$

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of vector fields in M. Taking $X = Z = e_i$ in (4.8) and summing up to n terms, then, using (4.2), we have

$$\begin{aligned} 0 = b\Big(&(n-1)H\left(\left(\tilde{\nabla}_U \tilde{e}\right)Y,V\right) + H\left(\left(\tilde{\nabla}_U \tilde{e}\right)V,Y\right) \\ &-\operatorname{trace}(H)\left(\tilde{\nabla}_U \tilde{E}\right)(Y,V) - g(Y,V)\sum_{i=1}^{n} H\left(\left(\tilde{\nabla}_U \tilde{e}\right)e_i, e_i\right)\Big) \\ &+ \frac{a\left(\tilde{\nabla}_U \tilde{r}\right)}{n(n-1)}\left(\operatorname{trace}(H)g(Y,V) - nH(Y,V)\right). \end{aligned} \tag{4.9}$$

Interchanging Y and V in (4.9) and subtracting the so-obtained formula from (4.9), we deduce that

$$H\left(\left(\tilde{\nabla}_U \tilde{e}\right)Y,V\right) = H\left(\left(\tilde{\nabla}_U \tilde{e}\right)V,Y\right). \tag{4.10}$$

Now, interchanging X and Z, Y, and V in (4.8) and taking the sum of the resulting equation and (4.8) and using (4.9) and (4.10), we get (4.6). If $\tilde{\nabla}_U \tilde{E} \neq 0$, then using (2.7) leads to (4.7). □

Theorem 4.3. *If (M, g) is an n-dimensional semi-Riemannian $\tilde{\nabla}$-quasiconformally symmetric manifold, $n > 2$ and $b \neq 0$, and if there is a vector field U so that*

$$\tilde{\nabla}_U \tilde{E} = 0, \qquad \tilde{\nabla}_U \tilde{r} \neq 0, \tag{4.11}$$

then the solution of (4.1) is $H = f\ g$, where f is a differentiable nonvanishing function.

Proof. Using (4.11), (4.8) becomes

$$g(Y, Z)H(X, V) - g(X, Z)H(Y, V) + g(Y, V)H(X, Z) - g(X, V)H(Y, Z) = 0, \tag{4.12}$$

Interchanging X and Z, Y and V in (4.12) and taking the sum of the resulting equation and (4.12), we get

$$g(X, Z)H(Y, V) - g(Y, V)H(X, Z) = 0. \tag{4.13}$$

Therefore, the tensor fields H and g are proportional. □

Theorem 4.4. *Let (M, g) be an n-dimensional semi-Riemannian $\tilde{\nabla}$-quasiconformally symmetric manifold, $n > 2$ and $b \neq 0$. If there is a vector field U satisfying the condition (4.11), then $i_{\tilde{\nabla}} = 1$.*

Proof. By Theorem 4.3 and from the fact that $\tilde{\nabla}_U g = 0$ and $\tilde{\nabla}_U H = 0$, it follows that f is constant. Thus, $i_{\tilde{\nabla}} = 1$. □

Theorem 4.5. *Let (M, g) be an n-dimensional semi-Riemannian $\tilde{\nabla}$-quasiconformally symmetric manifold, $n > 2$ and $b \neq 0$, for which the tensor field $\tilde{E}$ is not covariantly constant with respect to the Ricci symmetric metric connection $\tilde{\nabla}$. If $i_{\tilde{\nabla}} > 1$, then there is a vector field U, so that the equation*

$$\tilde{\nabla}_U H = 0 \tag{4.14}$$

has the fundamental solutions

$$H_1 = g, \qquad H_2 = \varphi \tilde{\nabla}_U \tilde{S}, \tag{4.15}$$

where φ is a differentiable nonvanishing function.

Proof. Given that $\tilde{\nabla}_U E \neq 0$, there is U so that the tensorial equation (4.1) has general solution which depends on U. g is obviously a solution of (4.14) because $\tilde{\nabla}_U g = 0$, g also satisfies the tensorial equation (4.1), and H_U given by (4.7) is also a solution of (4.14). Equation (4.14) has at least two solution as $i_{\tilde{\nabla}} > 1$. These two solutions are independent. By Lemma 2.5(b) $\varphi \tilde{\nabla}_U \tilde{S}$ and g are independent and we get two fundamental solution of $\tilde{\nabla}_U \widetilde{H} = 0$ which is $H_1 = g, H_2 = \varphi \tilde{\nabla}_U \tilde{S}$, where φ is a differentiable nonvanishing function. □

Theorem 4.6. *Let (M, g) be an n-dimensional semi-Riemannian $\tilde{\nabla}$-quasiconformally symmetric manifold, $n > 2$ and $b \neq 0$, for which the tensor field $\tilde{E}$ is not covariantly constant with respect to the metric connection $\tilde{\nabla}$. Then $1 \leq i_{\tilde{\nabla}} \leq n+1$.*

Proof. Let U_i, $i = 1, \ldots, p$ be independent vector fields, for which

$$\tilde{\nabla}_{U_i} \tilde{E} \neq 0, \tag{4.16}$$

and let $\varphi_i \tilde{\nabla}_{U_i} \tilde{S}$ and g be the fundamental solutions of $\tilde{\nabla}_{U_i} \widetilde{H} = 0$. Obviously $p < n$, as U_i are independent. Therefore, we have $p+1$ solutions. This completes the proof. □

Remark 4.7. The previous results of this section will be true for $\tilde{\nabla}$-conformally symmetric semi-Riemannian manifold, where $\tilde{\nabla}$ is any Ricci symmetric metric connection.

Theorem 4.8. *If (M, g) be an n-dimensional semi-Riemannian $\tilde{\nabla}$-concircularly symmetric manifold, then the (4.1) takes the form*

$$\det\begin{pmatrix} H(X,Z) & H(Y,V) \\ g(X,Z) & g(Y,V) \end{pmatrix} = 0. \tag{4.17}$$

Proof. Taking covariant derivative of (3.2) and using $\tilde{\nabla}_U \tilde{\mathcal{Z}} = 0$, we get

$$\left(\tilde{\nabla}_U \tilde{R}\right)(X,Y,Z,V) = \frac{\tilde{\nabla}_U \tilde{r}}{n(n-1)} \left(g(Y,Z)g(X,V) - g(X,Z)g(Y,V)\right), \tag{4.18}$$

which, when used in (4.1), yields

$$\begin{aligned} 0 = \frac{\tilde{\nabla}_U \tilde{r}}{n(n-1)} &\left(g(Y,Z)H(X,V) - g(X,Z)H(Y,V)\right. \\ &\left.+ g(Y,V)H(X,Z) - g(X,V)H(Y,Z)\right). \end{aligned} \tag{4.19}$$

Now, we interchange X with Z and Y with V in (4.19) and take the sum of the resulting equation and (4.19), and we get (4.17). □

Theorem 4.9. *Let (M, g) be an n-dimensional semi-Riemannian $\tilde{\nabla}$-concircularly symmetric manifold. Then $i_{\tilde{\nabla}} = 1$.*

Proof. By Theorem 4.8 and from the fact that $\tilde{\nabla}_U g = 0$ and $\tilde{\nabla}_U H = 0$, we get $i_{\tilde{\nabla}} = 1$. □

5. Discussion

A semi-Riemannian manifold is said to be *decomposable* [4] (or locally reducible) if there always exists a local coordinate system (x^i) so that its metric takes the form

$$ds^2 = \sum_{a,b=1}^{r} g_{ab} dx^a dx^b + \sum_{\alpha,\beta=r+1}^{n} g_{\alpha\beta} dx^\alpha dx^\beta, \tag{5.1}$$

where g_{ab} are functions of $x^1, \ldots, x^r$ and $g_{\alpha\beta}$ are functions of $x^{r+1}, \ldots, x^n$. A semi-Riemannian manifold is said to be reducible if it is isometric to the product of two or more semi-Riemannian manifolds; otherwise, it is said to be *irreducible* [4]. A reducible semi-Riemannian manifold is always decomposable but the converse needs not to be true.

The concept of the index of a (semi-)Riemannian manifold gives a striking tool to decide the reducibility and decomposability of (semi-)Riemannian manifolds. For example, a Riemannian manifold is decomposable if and only if its index is greater than one [4]. Moreover, a complete Riemannian manifold is reducible if and only if its index is greater than one [4]. A second-order $(0,2)$-symmetric parallel tensor is also known as a special Killing tensor of order two. Thus, a Riemannian manifold admits a special Killing tensor other than the Riemannian metric g if and only if the manifold is reducible [1], that is the index of the manifold is greater than 1. In 1951, Patterson [24] found a similar result for semi-Riemannian manifolds. In fact, he proved that a semi-Riemannian manifold (M, g) admitting a special Killing tensor K_{ij}, other than g, is reducible if the matrix (K_{ij}) has at least two distinct characteristic roots at every point of the manifold. In this case, the index of the manifold is again greater than 1.

By Theorem 4.6, we conclude that a $\widetilde{\nabla}$-quasiconformally symmetric Riemannian manifold (where $\widetilde{\nabla}$ is any Ricci symmetric metric connection, not necessarily Levi-Civita connection) is decomposable, and it is reducible if the manifold is complete.

It is known that the maximum number of linearly independent Killing tensors of order 2 in a semi-Riemannian manifold (M^n, g) is $(1/12)n(n+1)^2(n+2)$, which is attained if and only if M is of constant curvature. The maximum number of linearly independent Killing tensors in a four-dimensional spacetime is 50, and this number is attained if and only if the spacetime is of constant curvature [25]. But, from Theorem 4.6, we also conclude that the maximum number of linearly independent special Killing tensors in a 4-dimensional Robertson-Walker spacetime [11, page 341] is 5.

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