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*Research Article*

# Solution Matching for a Second Order Boundary Value Problem on Time Scales

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Let $\mathbb{T}$ be a time scale such that $a < b; a, b \in \mathbb{T}$. We will show the existence and uniqueness of solutions for the second-order boundary value problem $y^{\Delta\Delta}(t) = f(t, y(t), y^{\Delta}(t)), t \in [a, b]_{\mathbb{T}}$, $y(a) = A$, $y(b) = B$, by matching a solution of the first equation satisfying boundary conditions on $[a, c]_{\mathbb{T}}$ with a solution of the first equation satisfying boundary conditions on $[c, b]_{\mathbb{T}}$, where $c \in (a, b)_{\mathbb{T}}$.

## 1. Introduction

The result discussed in this paper was inspired by the solution matching technique that was first introduced by Bailey et al. [1]. In their work, they dealt with the existence and uniqueness of solutions for the second-order conjugate boundary value problems

$$y''(t) = f(t, y, y'), \tag{1.1}$$

$$y(a) = y_1, \qquad y(b) = y_2. \tag{1.2}$$

As shown in their work, the uniqueness and existence of the solutions of (1.1), (1.2) were obtained by matching a solution $y_1$ of (1.1) satisfying the boundary condition

$$y(a) = y_1, \qquad y'(c) = m \tag{1.3}$$

with a solution $y_2$ of (1.1) satisfying the boundary condition

$$y'(c) = m, \qquad y(b) = y_2, \tag{1.4}$$

where $m = y_1'(c) = y_2'(c)$.

Since the initial work by Bailey et al., there have been many studies utilizing the solution matching technique on boundary value problems, see for example, Rao et al. [2], Henderson [3, 4], Henderson and Taunton [5].

Existence and Uniqueness for solutions of boundary value problems have quite a history for ordinary differential equations as well for difference equations, we mentioned papers of Barr and Sherman [6], Hartman [7], Henderson [8, 9], Henderson and Yin [10], Moorti and Garner [11], Rao et al. [12] and many others.

While many of the work mentioned above considered boundary value problems for differential and difference equations, our study applies the solution matching technique to obtain a solution to a similar boundary value problem (1.1), (1.2) on a time scale. The theory of time scales was first introduced by Hilger [13] in 1990 to unify results in differential and difference equations. Since then, there has been much activity focused on dynamic equations on time scales, with a good deal of this activity devoted to boundary value problems. Efforts have been made in the context of time scales, in establishing that some results for boundary value problems for ordinary differential equations and their discrete analogues are special cases of more general results on time scales. For the context of dynamic equations on time scales, we mention the results by Bohner and Peterson [14, 15], Chyan [16], Henderson [4], and Henderson and Yin [17].

In this work, $\mathbb{T}$ is assumed to be a nonempty closed subset of $\mathbb{R}$ with $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = +\infty$. We shall also use the convention on notation that for each interval $I$ of $\mathbb{R}$,

$$I_{\mathbb{T}} = I \cap \mathbb{T}. \tag{1.5}$$

For readers' convenience, we state a few definitions which are basic to the calculus on the time scale $\mathbb{T}$. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{R}$ is defined by

$$\sigma(t) = \inf\{s > t \mid s \in \mathbb{T}\} \in \mathbb{T}. \tag{1.6}$$

If $\sigma(t) > t$, $t$ is said to be *right-scattered*, whereas, if $\sigma(t) = t$, $t$ is said to be *right-dense*. The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{R}$ is defined by

$$\rho(t) = \sup\{s < t \mid s \in \mathbb{T}\} \in \mathbb{T}. \tag{1.7}$$

If $\rho(t) < t$, $t$ is said to be *left-scattered*, and if $\rho(t) = t$, then $t$ is said to be *left-dense*. If $g : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, then the *delta derivative of $g$ at $t$*, $g^{\Delta}(t)$, is defined to be the number (provided that it exists), with the property that, given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$, such that

$$\left| [g(\sigma(t)) - g(s)] - g^{\Delta}(t)[\sigma(t) - s] \right| \le \varepsilon |\sigma(t) - s|, \tag{1.8}$$

for all $s \in U$. In this definition, $t \in \mathbb{T}^{\mathcal{K}}$, where this set is derived from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}^{\mathcal{K}} = \mathbb{T} \setminus \{m\}$. Otherwise, we define $\mathbb{T}^{\mathcal{K}} = \mathbb{T}$.

We say that the function $y$ has a generalized zero at $t$ if $y(t) = 0$ or if $y(t) \cdot y(\sigma(t)) < 0$. In the latter case, we would say the generalized zero is in the real interval $(t, \sigma(t))$.

**Theorem 1.1** (Mean Value Theorem). *If $y : \mathbb{T} \to \mathbb{R}$ is continuous and $y(t)$ has a generalized zero at $a$ and $b$, $a, b \in \mathbb{T}$, then there exists a point $r \in [a,b]_{\mathbb{T}}$ such that $y^{\Delta}$ has a generalized zero at $r$.*

Let $\mathbb{T}$ be a time scale such that $a, b \in \mathbb{T}$. In this paper, we are concerned with the existence and uniqueness of solutions of boundary value problems on the interval $[a,b]_{\mathbb{T}}$ for the second-order delta derivative equation

$$y^{\Delta\Delta} = f\left(t, y, y^{\Delta}\right), \tag{1.9}$$

satisfying the boundary conditions,

$$y(a) = y_1, \qquad y(\sigma(b)) = y_2, \tag{1.10}$$

where $a < b$ and $y_1, y_2 \in \mathbb{R}$. Throughout this paper, we will assume

(A1) $f(t, r_1, r_2)$ is a real-valued continuous function defined on $\mathbb{T} \times \mathbb{R}^2$.

We obtain solutions by matching a solution of (1.9) satisfying boundary conditions on $[a,c]_{\mathbb{T}}$ to a solution of (1.9) satisfying boundary conditions on $[c,b]_{\mathbb{T}}$. In particular, we will give sufficient conditions such that if $y_1(t)$ is a solution of (1.9) satisfying the boundary conditions $y(a) = y_1, y^{\Delta}(c) = m$ and $y_2(t)$ is a solution of (1.9) satisfying the boundary conditions $y(\sigma(b)) = y_2, y^{\Delta}(c) = m$, the solutions of (1.9) is

$$y(t) = \begin{cases} y_1(t), & t \in [a,c]_{\mathbb{T}}, \\ y_2(t), & t \in [c,b]_{\mathbb{T}}. \end{cases} \tag{1.11}$$

Moreover, we will assume the following conditions throughout this paper.

(A2) Solutions of initial value problems for (1.9) are unique and extend throughout $\mathbb{T}$.

(A3) $c \in \mathbb{T}$ is right dense and is fixed.

And the uniqueness of solutions assumptions are stated in terms of generalized zeros as follows:

(A4) For any $t_1 < t_2$ in $\mathbb{T}$, if $u$ and $v$ are solutions of (1.9) such that $(u - v)$ has a generalized zero at $t_1$ and $(u - v)^{\Delta}$ has a generalized zero at $t_2$, then $u \equiv v$ on $\mathbb{T}$.

## 2. Uniqueness of Solutions

In this section, we establish that under conditions (A1) through (A4), solutions of the conjugate boundary value problems of this paper are unique, when they exist.

**Theorem 2.1.** *Let $A, B \in \mathbb{R}$ be given and assume conditions* (A1) *through* (A4) *are satisfied. Then, given $m \in \mathbb{R}$, each of boundary value problems of* (1.9) *satisfying any of the following boundary conditions has at most one solution.*

$$y(a;m) = A, \qquad y^{\Delta}(c;m) = m, \quad a < c, \text{ where } a, c \in \mathbb{T}, \tag{2.1}$$

$$y(b;m) = B, \qquad y^{\Delta}(c;m) = m, \quad c < b, \text{ where } c, b \in \mathbb{T}. \tag{2.2}$$

*Proof.* Assume for some $m \in \mathbb{R}$, there exists distinct solutions $\alpha$ and $\beta$ of (1.9), (2.1), and set $w = \alpha - \beta$. Then, we have

$$w(a) = 0, \qquad w^{\Delta}(c) = 0. \tag{2.3}$$

Clearly, since $a < c$ and $w$ has a generalized zero at $a$ and $w^{\Delta}$ has a generalized zero at $c$, this contradicts condition (A4). Hence, the boundary value problems (1.9), (2.1) have unique solutions.

Next, we will look at a special boundary value problem of (1.9) satisfying the boundary condition

$$y(\sigma(b);m) = C, \qquad y^{\Delta}(b;m) = m. \tag{2.4}$$

We will show the uniqueness of solutions of the boundary value problems (1.9), (2.4) and use it to obtain the uniqueness of solutions of the boundary value problems (1.9), (2.2).

Assume that for some $m \in \mathbb{R}$ there are two distinct solutions, $\alpha$ and $\beta$, of (1.9), (2.4). Let $w = \alpha - \beta$. Then, we have

$$w(\sigma(b);m) = 0, \qquad w^{\Delta}(b;m) = 0. \tag{2.5}$$

By the uniqueness of solutions of initial value problems of (1.9), $w(b) \neq 0$. Without loss of generality, we may assume $w(b) > 0$. We consider the two cases of $b$.

If $b$ is right-dense, $\sigma(b) = b$, then

$$w^{\Delta}(b;m) = \lim_{t \to b} \frac{w(t;m) - w(b;m)}{t - b} = 0. \tag{2.6}$$

If $b$ is right scattered, $\sigma(b) > b$, then

$$w^{\Delta}(b;m) = \frac{w(\sigma(b);m) - w(b;m)}{\sigma(b) - b} = 0. \tag{2.7}$$

Regardless of whether $b$ is right dense or right scattered, we have $w(\sigma(b);m) = w(b;m) = 0$, which is a contradiction to condition (A2). Hence, $\alpha \equiv \beta$.

The uniqueness of solutions of boundary value problems of (1.9), (2.4) implies the uniqueness of solutions of boundary value problems of (1.9), (2.2) because the boundary conditions are defined at $c < b$. This completes the proof. □

**Theorem 2.2.** *Let $A, B \in \mathbb{R}$ be given and assume conditions* (A1) *through* (A4) *are satisfied. Then the boundary value problems* (1.9), (1.10) *has at most one solution.*

*Proof.* Again, we argue by contradiction. Assume for some values $A, B \in \mathbb{R}$, there are two distinct solutions, $\alpha$ and $\beta$, of (1.9), (1.10). Let $w = \alpha - \beta$. Then, we have $w(a) = 0$ and $w(b) = 0$. By the uniqueness of solutions of initial value problems of (1.9), $w^{\Delta}(a) \neq 0$ and $w^{\Delta}(b) \neq 0$. We may assume, without loss of generality, $w^{\Delta}(a) > 0$ and $w^{\Delta}(b) > 0$.

Then, there exists a point $c$, $a < c < b$, such that $w$ has a generalized zero at $c$. That is, either $w(c) = 0$ or $w(c) \cdot w(\sigma(c)) < 0$. But, by condition (A3), $w(c) = 0$.

Since $w(a) = 0$ and $w(c) = 0$, there exists a point $r_1 \in (a, c)_{\mathbb{T}}$ such that $w^{\Delta}$ has generalized zero at $r_1$. Since we also obtain that $w$ has a generalized zero at $a$, it implies that $\alpha \equiv \beta$, and this contradicts condition (A4).

Similarly, since $w(c) = 0$ and $w(b) = 0$, there exists a point $r_2 \in (c, b)_{\mathbb{T}}$ such that $w^{\Delta}$ has generalized zero at $r_2$. Note that $c < r_2$ and we obtain that $w$ has a generalized zero at $c$ and $w^{\Delta}$ has a generalized zero at $r_2$. This, again, implies that $\alpha \equiv \beta$, and, hence, contradicts condition (A4). □

## 3. Existence of Solutions

In this section, we establish monotonicity of the derivative as a function of $m$, of solutions of (1.9) satisfying each of the boundary conditions (2.1), (2.2). We use these monotonicity properties then to obtain solutions of (1.9), (1.10).

**Theorem 3.1.** *Suppose that conditions* (A1) *through* (A4) *are satisfied and that for each $m \in \mathbb{R}$ there exists solutions of* (1.9), (2.1) *and* (1.9), (2.2). *Then, $\alpha^{\Delta}(c; m)$ and $\beta^{\Delta}(c; m)$ are both strictly increasing function of $m$ whose range is $\mathbb{R}$.*

*Proof.* The strictness of the conclusion arises from Theorem 2.1. Let $m_1 > m_2$ and let

$$w(x) = \alpha(x; m_1) - \alpha(x; m_2). \tag{3.1}$$

Then, by Theorem 2.1,

$$\begin{aligned} w(a) &= \alpha(a; m_1) - \alpha(a; m_2) = 0, \\ w(c) &= \alpha(c; m_1) - \alpha(c; m_2) = 0, \\ w^{\Delta}(c) &= \alpha^{\Delta}(c; m_1) - \alpha^{\Delta}(c; m_2) \neq 0. \end{aligned} \tag{3.2}$$

Suppose to the contrary that $w^{\Delta}(c) < 0$. Then there exists a point $r_1 \in (a, c)_{\mathbb{T}}$ such that $w^{\Delta}$ has a generalized zero at $r_1$. This contradicts condition (A4). Thus, $w^{\Delta}(c) > 0$ and as a consequence, $w^{\Delta}(c; m)$ is a strictly increasing function of $m$.

We now show that $\{\alpha^{\Delta}(c; m) \mid m \in \mathbb{R}\} = \mathbb{R}$. Let $k \in \mathbb{R}$ and consider the solution $u(x; k)$ of (1.9), (2.1), with $u$ as defined above. Consider also the solution $\alpha(x; u^{\Delta}(c; k))$ of (1.9), (2.1). Hence, by Theorem 2.1, $\alpha(x; u^{\Delta}(c; k)) \equiv u^{\Delta}(c; k)$ and the range of $\alpha^{\Delta}(c; m)$ as a function of $m$ is the set of real numbers.

The argument for $\beta^{\Delta}$ is quite similar. This completes the proof. □

In a similar way, we also have a monotonicity result on the functions $u(t;m)$ and $v(t;m)$.

**Theorem 3.2.** *Assume the hypotheses of Theorem 3.1. Then, $u(t;m)$ and $v(t;m)$ are, respectively, strictly increasing and decreasing functions of $m$ with ranges all of $\mathbb{R}$.*

We now provide our existence result.

**Theorem 3.3.** *Assume the hypotheses of Theorem 3.1. Then, the boundary value problems* (1.9), (1.10) *has a unique solution.*

*Proof.* The existence is immediate from Theorem 3.1 or Theorem 3.2. Making use of Theorem 3.1, there exists a unique $m_0 \in \mathbb{R}$ such that $u^{\Delta}(c;m_0) = v^{\Delta}(c;m_0) = m_0$. Then,

$$y(t) = \begin{cases} u(t;m_0), & a \le t \le c, \\ v(t;m_0), & c \le t \le b \end{cases} \tag{3.3}$$

is a solution of (1.9), (1.10), and by Theorem 2.2, $y(t)$ is the unique solution. □

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