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Research Article

Deterministic Kalman Filtering on Semi-Infinite Interval

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We relate a deterministic Kalman filter on semi-infinite interval to linear-quadratic tracking control model with unfixed initial condition.

1. Introduction

In [1], Sontag considered the deterministic analogue of Kalman filtering problem on finite interval. The deterministic model allows a natural extension to semi-infinite interval. It is of a special interest because for the standard linear-quadratic stochastic control problem extension to semi-infinite interval leads to complications with the standard quadratic objective function (see, e.g., [2]). According to [1], the model which we are going to consider has the following form:

$$J(x,u,x_0) = \int_0^{+\infty} \left[u^T Ru + \left(Cx - \overline{y}\right)^T Q\left(Cx - \overline{y}\right)\right] \; dt, \tag{1.1}$$

$$\dot{x} = Ax + Bu, \tag{1.2}$$

$$x(0) = x_0. \tag{1.3}$$

Here we assume that the pair $(x,u) \in a(x_0) + Z$, where Z is a vector subspace of the Hilbert space $L_2^n[0,+\infty) \times L_2^m[0,+\infty)$ (with $L_2^n[0,+\infty)$ a Hilbert space of R^n-value square integrable functions) is defined as follows:

$$Z = \{(x, u) \in L_2^n[0, +\infty) \times L_2^m[0, +\infty) : x \text{ is absolutely continuous}, \\ \dot{x} \in L_2^n[0, +\infty), \dot{x} = Ax + Bu, x(0) = 0\}. \tag{1.4}$$

Here A is an n by n matrix; B is an n by m matrix; $R = R^T$ is an n by n and positive definite; $Q = Q^T$ is an r by r and positive definite; C is an r by n matrix; $\overline{y} \in L_2^r[0, +\infty)$. Notice that in (1.1)–(1.3) x_0 is not fixed and we minimize over all triple $(x, u, x_0) \in L_2^n[0, +\infty) \times L_2^m[0, +\infty) \times R^n$ satisfying our assumption.

Notice also that we interpret (1.1)–(1.3) as an estimation problem of the form

$$\dot{x} = Ax + Bu, \\ \overline{y} = Cx + v, \tag{1.5}$$

where we try to estimate x with the help of observation $\overline{y}$ by minimizing perturbations u, v and choosing an appropriate initial condition x_0.

2. Solution of the Deterministic Problem

Consider the algebraic Riccati equation

$$KA + A^T K + KLK - C^T QC = 0, \tag{2.1}$$

where $L = BR^{-1}B^T$. Assuming that the pair (A, B) is stabilizable and the pair (C, A) is detectable, there exists a negative definite symmetric solution K_{st} to (2.1) such that the matrix $A + LK_{st}$ is stable (see, e.g., Theorem 12.3 in [3]). According to [4], we have described a complete solution of the linear-quadratic control problem on a semi-infinite interval with the linear term in the objective function. The major motivation for this extension comes from [5] where we consider applications of primal-dual interior-point algorithms to the computational analysis of multicriteria linear-quadratic control problems in mini-max form. To compute a primal-dual direction it is required to solve linear-quadratic control problems with the same quadratic and different linear parts on each iteration. Using the results in [5], we can describe the optimal solution to (1.1)–(1.3) with fixed x_0 as follows.

There exists a unique solution $\rho_0 \in L_2^n[0, \infty)$ satisfying the differential equation

$$\dot{\rho} = -(A + LK_{st})^T \rho - C^T Q\overline{y}. \tag{2.2}$$

Moreover, ρ_0 can be explicitly described as follows:

$$\rho_0(t) = \int_0^{+\infty} \exp\left[(A + LK_{st})^T \tau\right] C^T Q\overline{y}(t + \tau) \quad d\tau. \tag{2.3}$$

The optimal solution (x, u) to (1.1)–(1.3) has the form

$$\dot{x} = (A + LK_{st})x + L\rho_0, \qquad x(0) = x_0, \tag{2.4}$$

$$u = R^{-1}B^T(K_{st}x + \rho_0). \tag{2.5}$$

For details see [5].

Notice that ρ_0 does not depend on x_0. To solve the original problem (1.1)–(1.3) we need to express the minimal value of the functional (1.1) in term of x_0.

Theorem 2.1. *Let* (x, u) *be an optimal solution of* (1.1)–(1.3) *with fixed* x_0 *given by* (2.2)–(2.5). *Then*

$$J(x, u, x_0) = \& - x_0^T K_{st} x_0 - 2\rho_0(0)^T x_0 + \int_0^{+\infty} \left[\overline{y}^T Q \overline{y} - \rho_0^T L \rho_0\right] dt. \tag{2.6}$$

Remark 2.2. Notice that $J(x, u, x_0)$ is a strictly convex function of x_0 and hence minimum of J as a function of x_0 is attained at

$$x_0^{\text{opt}} = -K_{st}^{-1}\rho_0(0). \tag{2.7}$$

Hence (2.2)–(2.5) gives a complete solution of the original problem (1.1)–(1.3).

Proof. Let $(y, w) \in a(x_0) + Z$ be feasible solution to (1.1)–(1.3), where x_0 is fixed. Consider

$$\Delta(y, w) = \left[w - R^{-1}B^T(K_{st}y + \rho_0)\right]^T \cdot R \cdot \left[w - R^{-1}B^T(K_{st}y + \rho_0)\right], \tag{2.8}$$

where we suppressed an explicit dependence on time. Notice that by (2.5)

$$\begin{aligned} \Delta(x, u) &\equiv 0, \\ \Delta(y, w) &\equiv 0, \end{aligned} \tag{2.9}$$

for any feasible solution (y, w) implies that $(y, w) = (x, u)$. Furthermore, let $\Delta(y, w) = \Delta_1 + \Delta_2 + \Delta_3$, where

$$\begin{aligned} \Delta_1 &= w^T R w, \\ \Delta_2 &= -2(K_{st}y + \rho_0)^T B w, \\ \Delta_3 &= (K_{st}y + \rho_0)^T L (K_{st}y + \rho_0). \end{aligned} \tag{2.10}$$

Now $Bw = \dot{y} - Ay$, and consequently

$$\Delta_2 = -2\left(y^T K_{st} + \rho_0^T\right)\left(\dot{y} - Ay\right) = y^T\left(K_{st}A + A^T K_{st}\right)y - 2y^T K_{st}\dot{y} - 2\rho_0^T\dot{y} + 2\rho^T Ay,$$
$$\Delta_3 = y^T K_{st} L K_{st} y + \rho_0^T L \rho_0 + 2\rho_0^T L K_{st} y. \tag{2.11}$$

Consequently,

$$\Delta(y,w) = w^T R w + \rho_0^T L \rho_0 + y^T\left(K_{st}LK_{st} + K_{st}A + A^T K_{st}\right)y - \frac{d}{dt}\left(y^T K_{st} y\right) - 2\frac{d}{dt}\left(\rho_0^T y\right)$$
$$+ 2\dot{\rho}_0^T y + 2\rho_0^T L K_{st} y + 2\rho_0^T A y. \tag{2.12}$$

Using (2.1) and (2.2), we obtain

$$\begin{aligned}\Delta(y,w) &= w^T R w + \rho_0^T L \rho_0 + y^T C^T Q C y - 2\frac{d}{dt}\left(\rho_0^T y\right) - \frac{d}{dt}\left(y^T K_{st} y\right) - 2\left(C^T Q \overline{y}\right)^T y \\ &= w^T R w + \rho_0^T L \rho_0 - 2\frac{d}{dt}\left(\rho_0^T y\right) - \frac{d}{dt}\left(y^T K_{st} y\right) + (\overline{y} - Cy)^T Q (\overline{y} - Cy) - \overline{y}^T Q \overline{y}.\end{aligned} \tag{2.13}$$

Hence, taking into account that $\rho_0(t) \to 0$, $y(t) \to 0$, $t \to +\infty$ (see, for details [5]), we obtain

$$\begin{aligned}\int_0^{+\infty} \Delta(y,w)\, dt &= \int_0^{+\infty}\left[w^T R w + (\overline{y} - Cy)^T Q (\overline{y} - Cy)\right]dt \\ &\quad + \int_0^{+\infty}\left[\rho_0^T L \rho_0 - \overline{y}^T Q \overline{y}\right]\, dt + 2\rho_0(0)^T x_0 + x_0 K_{st} x_0 \\ &= J(y,w,x_0) + 2\rho_0(0)^T x_0 + x_0 K_{st} x_0 + c,\end{aligned} \tag{2.14}$$

where $c = \int_0^{+\infty}[\rho_0^T L \rho_0 - \overline{y}^T Q \overline{y}]\, dt$.

Notice, that $\Delta(y,w) \geq 0$ and $\Delta(x,u) \equiv 0$. This shows that, indeed, (x,u) is an optimal solution to (1.1)–(1.3) (with fixed x_0) and proves (2.6). □

Remark 2.3. By (2.14) and $\Delta(x,u) \equiv 0$, we have $J(y,w,x_0) \geq J(x,u,x_0)$ and the equality occurs if and only if $(y,w) \equiv (x,u)$ (see also (2.9)). Hence (x,u) is a unique solution to the problem (1.1)–(1.3). Similary reasoning works in discrete time case.

3. Steady-State Deterministic Kalman Filtering

In light of (2.7), it is natural to consider the process

$$z(t) = -K_{st}^{-1}\rho_0(t), \quad t \in [0,+\infty) \tag{3.1}$$

as a natural estimate for the optimal solution to problem (1.1)–(1.3). Let us find the differential equation for z.

Proposition 3.1. *One has*

$$\dot{z} = Az + K_{st}^{-1}C^T Q(\overline{y} - Cz). \tag{3.2}$$

Remark 3.2. Notice that K_{st}^{-1} is a solution to the algebraic equation

$$L - PC^T QCP + AP + PA^T = 0. \tag{3.3}$$

In other words, the differential equation (3.2) is a precise deterministic analogue for the stochastic differential equation describing the optimal (steady-state) estimation in Kalman filtering problem. See, for example, [2].

Proof. Using (2.2) and (3.1), we obtain

$$\begin{aligned}\dot{z} &= K_{st}^{-1}(A + LK_{st})^T\rho_0 + K_{st}^{-1}C^T Q\overline{y} \\ &= -\left(K_{st}^{-1}A^T + L\right)(K_{st}z) + K_{st}^{-1}C^T Q\overline{y}.\end{aligned} \tag{3.4}$$

Since K_{st} is a solution to (2.1), we have

$$-K_{st}^{-1}A^T K_{st} - LK_{st} = A - K_{st}^{-1}C^T QC. \tag{3.5}$$

Hence,

$$\dot{z} = Az - K_{st}^{-1}C^T QCz + K_{st}^{-1}C^T Q\overline{y}. \tag{3.6}$$

Hence, we obtain (3.2). □

Remark 3.3. Notice that due to (3.1) $\Delta(z,0) \equiv 0$ and consequently $(z,0)$ would be an optimal solution to (1.1)–(1.3) if it were feasible for this problem.

4. The Solution of the Discrete Deterministic Problem

It is natural to consider the discrete version for the problem (1.1)–(1.3). In this case, the problem can be reformulated as follows:

$$J(x,u,x_0) = \frac{1}{2}\sum_{k=1}^{\infty}\left[u_k^T R u_k + \left(Cx_k - \overline{y}_k\right)^T Q\left(Cx_k - \overline{y}_k\right)\right] \longrightarrow \min, \tag{4.1}$$

$$x_{k+1} = Ax_k + Bu_k, \tag{4.2}$$

$$x_0 = x_o. \tag{4.3}$$

Here we let x denote a sequence $\{x_k\} \subset \mathbb{R}^n$ for $k = 0,\ldots,\infty$. We say that $x \in l_2^n(\mathbb{N})$ if $\sum_{i=1}^{\infty}\|x_i\|^2 < \infty$, where $\|\cdot\|$ is a norm induced by an inner product $\langle , \rangle$ in $\mathbb{R}^n$. Let $(x,u) \in l_2^n(\mathbb{N}) \times l_2^m(\mathbb{N})$.

Like in the continuous case, we assume that the pair $(x,u) \in a(x_0) + Z$, where Z is a vector subspace of the Hilbert space $l_2^n(\mathbb{N}) \times l_2^m(\mathbb{N})$.

Observe now the inner product in H has the following form:

$$\langle (x,y),(u,v)\rangle_H = \sum_{k=0}^{\infty}\{\langle x_k,u_k\rangle + \langle y_k,v_k\rangle\}. \tag{4.4}$$

The vector subspace Z now takes the following form:

$$Z = \{(x,u) \in H : \quad x_{k+1} = Ax_k + Bu_k,\ k = 0,1,\ldots, \quad x_0 = 0\}. \tag{4.5}$$

Here A is an n by n matrix. B is an n by m matrix. $R = R^T$ is an n by n and positive definite. $Q = Q^T$ is an r by r and positive definite. C is an r by n matrix and $\overline{y} \in l_2^r(\mathbb{N})$.

As in the continuous case, we interpret (4.1)–(4.3) as an estimation problem of the form

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ \overline{y}_k &= Cx_k + v_k, \end{aligned} \tag{4.6}$$

where we try to estimate x with the help of observation $\overline{y}$ by minimizing perturbations u, v and choosing an appropriate initial condition x_0.

According to [4], a general cost function for a discrete linear-quadratic control problem with linear term on the cost function has the following form:

$$J(x,u,x_0) = \sum_{k=1}^{\infty}\frac{1}{2}\left[x_k^T Q x_k + u_k^T R u_k\right] + x_k^T\psi_k + u_k^T\phi_k \longrightarrow \min, \tag{4.7}$$

where $\psi \in l_2^n(\mathbb{N})$ and $\phi \in l_2^m(\mathbb{N})$. The solution to the particular class of problems can be completely described by solving several system of recurrence relations and the following discrete algebraic Riccati equation (DARE):

$$K = A^T KA - \left(A^T KB\right)\left(R + B^T KB\right)^{-1}\left(A^T KB\right)^T + Q. \tag{4.8}$$

We assume that this equation has a positive definite stabilizing solution K_{st}. For sufficient conditions, see [6].

In our situation, we have

$$J(x, u, x_0) = \frac{1}{2}\sum_{k=1}^{\infty}\left[u_k^T Ru_k + (Cx_k - \overline{y}_k)^T Q(Cx_k - \overline{y}_k)\right] \longrightarrow \min. \tag{4.9}$$

It is easy to see that $\psi_k = -C^T Q\overline{y}_k$ and $\phi_k = 0, k = 0, 1, \ldots$. By [4], there is a unique solution $\overline{\rho} = \{\overline{\rho}_k\} \in l_2^n(\mathbb{N})$ of the following recurrence relations

$$\rho_k = \left[A^T - \left(A^T K_{st}B\right)\left(R + B^T K_{st}B\right)^{-1} B^T\right]\rho_{k+1} + C^T Q\overline{y}_k. \tag{4.10}$$

For details on an explicit solution of the above recurrence relation, see [4]. For simplicity, we let

$$\overline{R} = \left(R + B^T K_{st}B\right), \tag{4.11}$$

and we also let

$$L = B\overline{R}^{-1}B^T. \tag{4.12}$$

So our recurrence relation for ρ now takes the form

$$\rho_k = \left[A^T - A^T K_{st}L\right]\rho_{k+1} + C^T Q\overline{y}_k \tag{4.13}$$

with the corresponding DARE

$$\begin{aligned} K = \;& A^T KA - (A^T KB)\overline{R}^{-1}(A^T KB)^T + C^T QC, \\ K =\;& A^T KA - A^T KLKA + C^T QC. \end{aligned} \tag{4.14}$$

The optimal solution to (4.1)–(4.3) has the following form:

$$x_{k+1} = \left(A^T - A^T K_{st}L\right)^T x_k + L\overline{\rho}_{k+1}, \tag{4.15}$$

$$u_k = -\overline{R}^{-1}B^T K_{st}Ax_k + \overline{R}^{-1}B^T\overline{\rho}_{k+1}. \tag{4.16}$$

For details, see [4]. To solve the original problem (4.1)–(4.3) we need to express the minimal value of the functional (4.1) in terms of x_0.

Theorem 4.1. *Let* (x,u) *be an optimal solution of* (4.1)–(4.3) *with fixed* x_0 *given by* (4.15)-(4.16). *Then*

$$J(x,u,x_0) = \frac{1}{2}x_0^T K_{st} x_0 - \overline{\rho}_0^T x_0 + \frac{1}{2}\sum_{k=0}^{\infty}\left[2\overline{y}_k^T Q \overline{y}_k - \overline{\rho}_{k+1}^T L \overline{\rho}_{k+1}\right]. \tag{4.17}$$

Proof. For simplicity of notation, we use K for K_{st}. Let

$$\begin{aligned}\Delta(y_k,w_k) &= \left[w_k + \overline{R}^{-1}B^T(KAy_k - \overline{\rho}_{k+1})\right]^T \cdot \overline{R}\cdot \left[w_k + \overline{R}^{-1}B^T(KAy_k - \overline{\rho}_{k+1})\right] \\ &= \Delta_1 + \Delta_2 + \Delta_3,\end{aligned} \tag{4.18}$$

where

$$\begin{aligned}\Delta_1 &= w_k^T \overline{R} w_k, \\ \Delta_2 &= 2(KAy_k - \overline{\rho}_{k+1})^T B w_k \\ &= 2(KAy_k - \overline{\rho}_{k+1})^T (y_{k+1} - Ay_k) \\ &= 2y_k^T A^T K y_{k+1} - 2y_k^T A^T KAy_k - 2\overline{\rho}_{k+1}^T y_{k+1} + 2\overline{\rho}_{k+1}^T A y_k, \\ \Delta_3 &= (KAy_k - \overline{\rho}_{k+1})^T L (KAy_k - \overline{\rho}_{k+1}) \\ &= y_k^T A^T KLKAy_k + \overline{\rho}_{k+1}^T L \overline{\rho}_{k+1} - 2y_k^T A^T KL\overline{\rho}_{k+1}.\end{aligned} \tag{4.19}$$

We assume that $(y,w) \in a(x_0) + Z$. Since $A^TKA - A^TKLKA = K - C^TQC$ and $[A^T - A^TKL]\overline{\rho}_{k+1} = \overline{\rho}_k - C^TQ\overline{y}_k$, we have

$$\begin{aligned}\Delta(y_k,w_k) &= w_k^T\overline{R}w_k - 2y_k^T\left[A^TKA - A^TKLKA\right]y_k - y_k^TA^TKLKAy_k \\ &\quad - 2\overline{\rho}_{k+1}^T y_{k+1} + 2\overline{\rho}_{k+1}^T A y_k + \overline{\rho}_{k+1}^T L\overline{\rho}_{k+1} - 2y_k^TA^TKL\overline{\rho}_{k+1} + 2y_k^TA^TKy_{k+1} \\ &= w_k^T\overline{R}w_k - 2y_k^T\left[K - C^TQC\right]y_k - y_k^TA^TKLKAy_k - 2\overline{\rho}_{k+1}^T y_{k+1} \\ &\quad + 2y_k^TA^T\overline{\rho}_{k+1} + \overline{\rho}_{k+1}^T L\overline{\rho}_{k+1} - 2y_k^TA^TKL\overline{\rho}_{k+1} + 2y_k^TA^TKy_{k+1} \\ &= w_k^T\overline{R}w_k - 2y_k^T\left[K - C^TQC\right]y_k - y_k^TA^TKLKAy_k - 2\overline{\rho}_{k+1}^T y_{k+1} \\ &\quad + 2y_k^T\left[A^T - A^TKL\right]\overline{\rho}_{k+1} + \overline{\rho}_{k+1}^T L\overline{\rho}_{k+1} + 2y_k^TA^TKy_{k+1}\end{aligned}$$

$$\begin{aligned}
&= w_k^T \overline{R} w_k - 2y_k^T \left[K - C^T Q C\right] y_k - y_k^T A^T K L K A y_k - 2\rho_{k+1}^T y_{k+1} \\
&\quad + 2y_k^T \left[\overline{\rho}_k - C^T Q \overline{y}_k\right] + \overline{\rho}_{k+1}^T L \overline{\rho}_{k+1} + 2y_k^T A^T K y_{k+1} \\
&= w_k^T \overline{R} w_k - 2y_k^T \left[K - C^T Q C\right] y_k - y_k^T A^T K L K A y_k - 2\overline{\rho}_{k+1}^T y_{k+1} \\
&\quad + 2y_k^T \overline{\rho}_k - 2y_k^T C^T Q \overline{y}_k + \overline{\rho}_{k+1}^T L \overline{\rho}_{k+1} + 2y_k^T A^T K y_{k+1}.
\end{aligned} \tag{4.20}$$

By recalling now the definition of $\overline{R}$, we have

$$\begin{aligned}
w_k^T \overline{R} w_k &= w_k^T \left(R + B^T K B\right) w_k \\
&= w_k^T R w_k + w_k^T B^T K B w_k \\
&= w_k^T R w_k + \left(y_{k+1} - A y_k\right)^T K \left(y_{k+1} - A y_k\right) \\
&= w_k^T R w_k + y_{k+1}^T K y_{k+1} + y_k^T A^T K A y_k - 2y_k^T A^T K y_{k+1}.
\end{aligned} \tag{4.21}$$

Therefore,

$$\begin{aligned}
\Delta\left(y_k, w_k\right) = {} & w_k^T R w_k - y_k^T K y_k - y_k^T C^T Q C y_k - 2\overline{\rho}_{k+1}^T y_{k+1} + 2y_k^T \overline{\rho}_k \\
& - 2y_k^T C^T Q \overline{y}_k + \overline{\rho}_{k+1}^T L \overline{\rho}_{k+1} + y_{k+1}^T K y_{k+1}.
\end{aligned} \tag{4.22}$$

We then rearrange the terms and complete the square to obtain a useful expression for Δ:

$$\begin{aligned}
\Delta\left(y_k, \mathrm{w}_k\right) &= w_k^T R w_k - y_k^T K y_k + y_{k+1}^T K y_{k+1} - 2\overline{\rho}_{k+1}^T y_{k+1} + 2\overline{\rho}_k^T y_k \\
&\quad + \overline{\rho}_{k+1}^T L \overline{\rho}_{k+1} + y_k^T C^T Q C y_k - 2y_k^T C^T Q \overline{y}_k \\
&= w_k^T R w_k - y_k^T K y_k + y_{k+1}^T K y_{k+1} - 2\overline{\rho}_{k+1}^T y_{k+1} + 2\overline{\rho}_k^T y_k \\
&\quad + \overline{\rho}_{k+1}^T L \overline{\rho}_{k+1} + \left(C y_k - \overline{y}_k\right)^T Q \left(C y_k - \overline{y}_k\right) - 2\overline{y}_k^T Q \overline{y}_k.
\end{aligned} \tag{4.23}$$

Notice, since we fixed x_0, we let $y_0 = x_0$ and take summation of both sides:

$$\begin{aligned}
\sum_{k=0}^{\infty} \Delta\left(y_k, w_k\right) = {} & -x_0^T K x_0 + 2\overline{\rho}_0^T x_0 + \sum_{k=0}^{\infty} \left[w_k^T R w_k + \left(C y_k - \overline{y}_k\right)^T Q \left(C y_k - \overline{y}_k\right)\right] \\
& + \sum_{k=0}^{\infty} \left[\overline{\rho}_{k+1}^T L \overline{\rho}_{k+1} - 2\overline{y}_k^T Q \overline{y}_k\right].
\end{aligned} \tag{4.24}$$

By the definition of $\Delta(y_k, w_k)$, $\Delta(x_k, u_k) = 0$. Therefore,

$$0 = -x_0^T K x_0 + 2\overline{\rho}_0^T x_0 + 2J(x, u, x_0) + \sum_{k=0}^{\infty} \left[\overline{\rho}_{k+1}^T L \overline{\rho}_{k+1} - 2\overline{y}_k^T Q \overline{y}_k\right]. \tag{4.25}$$

As a result,

$$J(x, u, x_0) = \frac{1}{2} x_0^T K x_0 - \overline{\rho}_0^T x_0 + \frac{1}{2} \sum_{k=0}^{\infty} \left[2\overline{y}_k^T Q \overline{y}_k - \overline{\rho}_{k+1}^T L \overline{\rho}_{k+1}\right]. \tag{4.26}$$

Then the proof is completed. □

As in continuous case, for the discrete case, $J(x, u, x_0)$ is a strictly convex function of x_0 and hence minimum of J as a function of x_0 is attained at

$$x_0^{\text{opt}} = K_{st}^{-1} \overline{\rho}_0, \tag{4.27}$$

where $\overline{\rho}$ is the unique l_2 solution to (4.13).

Since we have (4.27), it is natural to consider the process

$$z_k = K_{st}^{-1} \overline{\rho}_k, \tag{4.28}$$

as an estimate for the optimal solution to problem (4.1)–(4.3). Let us find the recurrence relation for z_k.

Proposition 4.2. *Assuming that the closed loop matrix A-LKA is invertible, one has*

$$z_{k+1} = Az_k - K_{st}^{-1} \left[A^T - A^T K_{st} L\right]^{-1} \left(\overline{y}_k - Cz_k\right). \tag{4.29}$$

Proof. We can rewrite (4.13) in the form

$$K_{st} z_k = \left[A^T - A^T K_{st} L\right] K_{st} z_{k+1} + C^T Q \overline{y}_k. \tag{4.30}$$

Using the algebraic Riccati equation

$$K_{st} = A^T K_{st} A - A^T K_{st} L K_{st} A + C^T Q C, \tag{4.31}$$

we can rewrite (4.30) in the form

$$C^T Q C z_k + \left(A^T K_{st} - A^T K_{st} L K_{st}\right) A z_k = \left(A^T - A^T K_{st} L\right) K_{st} z_{k+1} + C^T Q \overline{y}_k, \tag{4.32}$$

which is equivalent to

$$\left(A^T - A^T K_{st} L\right) K_{st} z_{k+1} = \left(A^T - A^T K_{st} L\right) K_{st} A z_k - C^T Q\left(\overline{y}_k - C z_k\right). \tag{4.33}$$

The result follows. □

Remark 4.3. Notice that (4.29) is the analogue of the "limiting" discrete Kalman filter [6, Page 384, (17.6.1)].

5. Concluding Remarks

In this paper, we relate a deterministic Kalman filter on semi-infinite interval to linear-quadratic tracking control model with unfixed initial condition. Solutions of the deterministic problems both continuous and discrete cases are described. This extends the result of Sontag to semi-infinite interval.

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