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Research Article

Toeplitz Operators with Essentially Radial Symbols

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For Topelitz operators with radial symbols on the disk, there are important results that characterize boundedness, compactness, and its relation to the Berezin transform. The notion of essentially radial symbol is a natural extension, in the context of multiply-connected domains, of the notion of radial symbol on the disk. In this paper we analyze the relationship between the boundary behavior of the Berezin transform and the compactness of T_ϕ when $\phi \in L^2(\Omega)$ is essentially radial and Ω is multiply-connected domains.

1. Introduction

Toeplitz operators are object of intense study. Many papers have been dedicated to the study of these concrete class of operators generating many interesting results. A very important tool to study the behavior of these operators is the Berezin transform. This tool is particularly relevant with its connections with quantum mechanics, especially in the case of the Toeplitz operators on the Segal-Bargmann space. In this case, they arises naturally as anti-Wick quantization operators, and there is a natural equivalence between Toeplitz operators and a generalization of pseudodifferential operators, the so-called Weyl's quantization.

In a fundamental paper, Axler and Zheng proved that, if $S \in \mathcal{B}(L^2(\mathbf{D}))$ can be written as a finite sum of finite products of Toeplitz operators with L^∞-symbols, then S is compact if and only if S has a Berezin transform which vanishes at the boundary of the disk $\mathbf{D}$. As they expected, this result has been extended into several directions, and it has been proved even for operators which are not of the Toeplitz type. Therefore it has been an important open problem to characterize the class of operators for which the compactness is equivalent to the vanishing of the Berezin transform. Since there are operators which are not compact but have a Berezin transform which vanishes at the boundary, it is now clear that the two notions are not equivalent. Moreover, it is possible to show that in the context of Toeplitz operators there are examples of unbounded symbols whose corresponding operators are bounded and even compact.

Recently, many papers have been written in the case when the operator has an unbounded radial symbol $\varphi \in L^2(\mathbf{D})$. Of course, for a square-integrable symbol, the Toeplitz operator is densely defined but is not necessarily bounded. However, it is possible (see [1] of Grudsky and Vasilevski, [2] of Zorboska, and [3] of Korenblum and Zhu) to show that operators with unbounded radial symbols can have a very rich structure. Moreover, there is a very neat and elegant way to characterize boundedness and compactness. The reason being that the operators with radial symbols on the disk are diagonal operators. In this context the relation between compactness and the Berezin transform has been studied in depth, and interesting results have been established.

In a previous paper (see [4]), the author showed that it is possible to extend the notion of radial symbol when Ω is a bounded multiply-connected domain in the complex plane $\mathbb{C}$, whose boundary $\partial\Omega$ consists of finitely many simple closed smooth analytic curves γ_j $(j = 1, 2, \dots, n)$ where γ_j are positively oriented with respect to Ω and $\gamma_j \cap \gamma_i = \emptyset$ if $i \neq j$. The key ingredient for this extension is to observe two facts. The first fact is that the structure of the Bergman kernel suggests that there is in any planar domain an internal region that we can neglect when we are interested in boundedness and compactness of the Toeplitz operators with square integrable symbols. The second observation consists in exploiting the geometry of the domain and conformal equivalence in order to partially recover the notion of radial symbol. For this class of essentially radial symbols, the compactness and boundedness have been studied and necessary and sufficient conditions established. In this paper we carry forward our analysis by investigating the relationship between the compactness and the vanishing of the Berezin transform. It is important to observe that in the case of the disk the analysis uses the fact that the Berezin transform can be easily written in a simple way since we can write explicitly an orthonormal basis, namely the collection of functions $\{\sqrt{k+1}z^k\}_{k=0}^{\infty}$. In the case of a planar domain, this is not possible because it is very hard to construct explicitly an orthonormal basis for the Bergman space. However, it is possible to reach interesting results that fully extend what it is known in the case of the disk.

The paper is organized as follows. In Section 2 we describe the setting where we work, give the relevant definitions, and state our main result. In Section 3 we prove the main result and we study several important consequences.

2. Preliminaries

Let Ω be a bounded multiply-connected domain in the complex plane $\mathbb{C}$, whose boundary $\partial\Omega$ consists of finitely many simple closed smooth analytic curves γ_j $(j = 1, 2, \dots, n)$ where γ_j are positively oriented with respect to Ω and $\gamma_j \cap \gamma_i = \emptyset$ if $i \neq j$. We also assume that γ_1 is the boundary of the unbounded component of $\mathbb{C} \setminus \Omega$. Let Ω_1 be the bounded component of $\mathbb{C} \setminus \gamma_1$, and Ω_j $(j = 2, \dots, n)$ the unbounded component of $\mathbb{C} \setminus \gamma_j$, respectively, so that $\Omega = \bigcap_{j=1}^{n} \Omega_j$.

For $d\nu = (1/\pi)dx\,dy$ we consider the usual L^2-space $L^2(\Omega) = L^2(\Omega, d\nu)$. The Bergman space $L_a^2(\Omega, d\nu)$, consisting of all holomorphic functions which are L^2-integrable, is a closed subspace of $L^2(\Omega, d\nu)$ with the inner product given by

$$\langle f, g \rangle = \int_{\Omega} f(z)\overline{g(z)}d\nu(z), \tag{2.1}$$

for $f, g \in L^2(\Omega, d\nu)$. The Bergman projection is the orthogonal projection

$$P : L^2(\Omega, d\nu) \longrightarrow L^2_a(\Omega, d\nu); \tag{2.2}$$

it is well-known that for any $f \in L^2(\Omega, d\nu)$ we have

$$Pf(w) = \int_\Omega f(z) K^\Omega(z, w) d\nu(z), \tag{2.3}$$

where K^Ω is the Bergman reproducing kernel of Ω. For $\varphi \in L^\infty(\Omega, d\nu)$ the Toeplitz operator $T_\varphi : L^2_a(\Omega, d\nu) \to L^2_a(\Omega, d\nu)$ is defined by $T_\varphi = PM_\varphi$ where M_φ is the standard multiplication operator. A simple calculation shows that

$$T_\varphi f(z) = \int_\Omega \varphi(w) f(w) K^\Omega(w, z) d\nu(w). \tag{2.4}$$

We use the symbol Δ to indicate the punctured disk $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Let Γ be any one of the domains Ω, Δ, Ω_j $(j = 2, \ldots, n)$.

We call $K^\Gamma(z, w)$ the reproducing kernel of Γ, and we use the symbol $k^\Gamma(z, w)$ to indicate the normalized reproducing kernel; that is, $k^\Gamma(z, w) = K^\Gamma(z, w)/K^\Gamma(w, w)^{1/2}$.

For any $A \in \mathcal{B}(L^2_a(\Gamma, d\nu))$, the space of bounded operators on $L^2_a(\Gamma, d\nu)$, we define $\widetilde{A}$, the Berezin transform of A, by

$$\widetilde{A}(w) = \left\langle A k^\Gamma_w, k^\Gamma_w \right\rangle = \int_\Gamma A k^\Gamma_w(z) \overline{k^\Gamma_w(z)} d\nu(z), \tag{2.5}$$

where $k^\Gamma_w(\cdot) = K^\Gamma(\cdot, w) K^\Gamma(w, w)^{-1/2}$.

If $\varphi \in L^\infty(\Gamma)$, then we indicate with the symbol $\widetilde{\varphi}$ the Berezin transform of the associated Toeplitz operator T_φ, and we have

$$\widetilde{\varphi}(w) = \int_\Gamma \varphi(z) \left| k^\Gamma_w(z) \right|^2 d\nu(z). \tag{2.6}$$

We remind the reader that it is well known that $\widetilde{A} \in \mathcal{C}^\infty_b(\Gamma)$ and we have $\|\widetilde{A}\|_\infty \le \|A\|_{\mathcal{B}(L^2(\Omega))}$. It is possible, in the case of bounded symbols, to give a characterization of compactness using the Berezin transform (see [5, 6]).

We remind the reader that any Ω bounded multiply-connected domain in the complex plane $\mathbb{C}$, whose boundary $\partial\Omega$ consists of finitely many simple closed smooth analytic curves γ_j $(j = 1, 2, \ldots, n)$, is conformally equivalent to a canonical bounded multiply-connected domain whose boundary consists of finitely many circles (see [7]). This means that it is possible to find a conformally equivalent domain $D = \bigcap_{i=1}^n D_i$ where $D_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $D_j = \{z \in \mathbb{C} : |z - a_j| > r_j\}$ for $j = 2, \ldots, n$. Here $a_j \in D_1$ and $0 < r_j < 1$ with $|a_j - a_k| > r_j + r_k$ if $j \ne k$ and $1 - |a_j| > r_j$. Before we state the main result of this paper, we need to give a few definitions.

Definition 2.1. Let $\Omega = \bigcap_{i=1}^{n} \Omega_i$ be a canonical bounded multiply-connected domain. One says that the set of $n+1$ functions $\mathfrak{P} = \{p_0, p_1, \ldots, p_n\}$ is a ∂-partition for Ω if

(1) for every $j = 0, 1, \ldots, n$, $p_j : \Omega \to [0,1]$ is a Lipschitz, C^∞-function;

(2) for every $j = 2, \ldots, n$ there exists an open set $W_j \subset \Omega$ and an $\epsilon_j > 0$ such that $U_{\epsilon_j} = \{\zeta \in \Omega : r_j < |\zeta - a_j| < r_j + \epsilon_j\}$ and the support of p_j are contained in W_j and

$$p_j(\zeta) = 1 \quad \forall \zeta \in U_{\epsilon_j}; \tag{2.7}$$

(3) for $j = 1$ there exists an open set $W_1 \subset \Omega$ and an $\epsilon_1 > 0$ such that $U_{\epsilon_1} = \{\zeta \in \Omega : 1 - \epsilon_1 < |\zeta| < 1\}$ and the support of p_1 are contained in W_1 and

$$p_1(\zeta) = 1 \quad \forall \zeta \in U_{\epsilon_1}; \tag{2.8}$$

(4) for every $j, k = 1, \ldots, n$, $W_j \cap W_k = \emptyset$, the set $\Omega \setminus (\bigcup_{j=1}^{n} W_j)$ is not empty and the function

$$\begin{gathered} p_0(\zeta) = 1 \quad \forall \zeta \in \left(\bigcup_{j=1}^{n} W_j \right)^c \cap \Omega, \\ p_0(\zeta) = 0 \quad \forall \zeta \in U_{\epsilon_k}, \quad k = 1, \ldots, n, \end{gathered} \tag{2.9}$$

(5) for any $\zeta \in \Omega$ the following equation

$$\sum_{k=0}^{n} p_k(\zeta) = 1 \tag{2.10}$$

holds.

We also need the following.

Definition 2.2. A function $\varphi : \Omega = \bigcap_{i=1}^{n} \Omega_i \to \mathbb{C}$ is said to be essentially radial if there exists a conformally equivalent canonical bounded domain $D = \bigcap_{i=1}^{n} D_i$ such that, if the map $\Theta : \Omega \to D$ is the conformal mapping from Ω onto D, then

(1) for every $k = 2, \ldots, n$ and for some $\epsilon_k > 0$, one has

$$\varphi \circ \Theta^{-1}(z) = \varphi \circ \Theta^{-1}(|z - a_k|) \tag{2.11}$$

when $z \in U_{\epsilon_k} = \{\zeta \in \Omega : r_k < |\zeta - a_k| < r_k + \epsilon_k\}$,

(2) for $k = 1$ and for some $\epsilon_1 > 0$, one has

$$\varphi \circ \Theta^{-1}(z) = \varphi \circ \Theta^{-1}(|z|) \tag{2.12}$$

when $z \in U_{\epsilon_1} = \{\zeta \in \Omega : 1 - \epsilon_1 < |\zeta| < 1\}$.

The reader should note that, in the case where it is necessary to stress the use of a specific conformal equivalence, we will say that the map φ is essentially radial *via* $\Theta : \bigcap_{\ell=1}^{n} \Omega_\ell \to \bigcap_{\ell=1}^{n} D_\ell$. Moreover, we stress that in what follows, when we are working with a general multiply-connected domain and we have a conformal equivalence $\Theta : \bigcap_{\ell=1}^{n} \Omega_\ell \to \bigcap_{\ell=1}^{n} D_\ell$, we always assume that the ∂-partition is given on $\bigcap_{\ell=1}^{n} D_\ell$ and transferred to $\bigcap_{\ell=1}^{n} \Omega_\ell$ through Θ in the natural way.

Definition 2.3. If $\varphi \in L^2(\Omega)$ is an essentially radial function via $\Theta : \bigcap_{i=1}^{n} \Omega_i \to \bigcap_{i=1}^{n} D_i$, $\varphi_j = \varphi \cdot p_j$ for any $j = 1, \ldots, n$ where $\mathfrak{P} = \{p_0, p_1, \ldots, p_n\}$ is a ∂-partition for Ω then one defines the n sequences

$$a_{\varphi_1} = \left\{a_{\varphi_1}(k)\right\}_{k\in\mathbb{Z}_+}, a_{\varphi_2} = \left\{a_{\varphi_2}(k)\right\}_{k\in\mathbb{Z}_+}, \ldots, a_{\varphi_n} = \left\{a_{\varphi_n}(k)\right\}_{k\in\mathbb{Z}_+} \tag{2.13}$$

as follows: if $j = 2, \ldots, n$,

$$a_{\varphi_j}(k) = r_j \int_{r_j}^{\infty} \varphi_j \circ \Theta^{-1}\left(r_j s + a_j\right)(k+1)\frac{r_j^{2k+1}}{s^{2k+1}}\frac{1}{s^2}ds \quad \forall k \in \mathbb{Z}_+, \tag{2.14}$$

and if $j = 1$,

$$a_{\varphi_1}(k) = \int_0^1 \varphi_1 \circ \Theta^{-1}(s)(k+1)s^{2k+1}ds \quad \forall k \in \mathbb{Z}_+. \tag{2.15}$$

At this point we can state the main result.

Theorem 2.4. *Let $\varphi \in L^2(\Omega)$ be an essentially radial function via $\Theta : \bigcap_{\ell=1}^{n} \Omega_\ell \to \bigcap_{\ell=1}^{n} D_\ell$ and $\varphi_j = \varphi \cdot p_j$ for any $j = 1, \ldots, n$ where $\mathfrak{P} = \{p_0, p_1, \ldots, p_n\}$ is a ∂-partition for Ω. If the operator $T_\varphi : L_a^2(\Omega, dv) \to L_a^2(\Omega, dv)$ is bounded and if for any $j = 1, \ldots, n$ the sequence $a_{\varphi_j} = \{a_{\varphi_j}(k)\}_{k\in\mathbb{Z}_+}$ satisfies the following*

$$\sup_{k\in\mathbb{Z}_+}\left\{\left|(k+1)a_{\varphi_j}(k) - k a_{\varphi_j}(k-1)\right|\right\} < \infty, \tag{2.16}$$

then the operator $T_\varphi : L_a^2(\Omega, dv) \to L_a^2(\Omega, dv)$ is compact if and only if

$$\lim_{w\to\partial\Omega} \widetilde{T_\varphi}(w) = 0. \tag{2.17}$$

3. Canonical Multiply-Connected Domains and Essentially Radial Symbols

We concentrate on the relationship between compact Toeplitz operators and the Berezin transform. As we said in the introduction, Axler and Zheng have proved (see [5]) that if $\mathbf{D}$ is the disk, $S = \sum_i^m \prod_k^{m_j} T_{\varphi_{i,k}}$, where $\varphi_{i,k} \in L^\infty(\mathbf{D})$, then S is compact if and only if its Berezin transform vanishes at the boundary of the disk. Their fundamental result has been extended in several directions, in particular when Ω is a general smoothly bounded multiply-connected planar domain [6]. In this section we try to characterize the compactness in terms of the Berezin transform. In the next theorem, under a certain condition, we will show that the Berezin transform characterization of compactness still holds in this context.

In the case of the disk, it is possible to show that when the operator is radial then its Berezin transform has a very special form. In fact, if $\varphi : \mathbb{D} \to \mathbb{C}$ is radial, then

$$\widetilde{T_\varphi}(z) = \left(1 - |z|^2\right)^2 \sum (n+1)\langle T_\varphi e_n, e_n\rangle |z|^{2n}, \tag{3.1}$$

where, by definition,

$$e_n(z) = \sqrt{n+1}z^n \quad \forall n \in \mathbb{Z}_+. \tag{3.2}$$

Therefore to show that the vanishing of the Berezin transform implies compactness is equivalent, given that T_φ is diagonal and to show that $\lim_{|z|\to 1}(1-|z|^2)^2\sum(n+1)\langle T_\varphi e_n, e_n\rangle |z|^{2n} = 0$ implies $\lim_{n\to\infty}\langle T_\varphi e_n, e_n\rangle = 0$, Korenblum and Zhu realized this fact in their seminal paper [3], and, along this line, more was discovered by Zorboska (see [2]) and Grudsky and Vasilevski (see [1]).

In the case of a multiply-connected domain, it is not possible to write things so neatly; however, we can exploit our estimates near the boundary to use similar arguments. In fact, for an essentially radial function, the values depend essentially on the distance from the boundary. Moreover, we can simplify our analysis if we use the fact that every multiply-connected domain is conformally equivalent to a canonical bounded multiply-connected domain whose boundary consists of finitely many circles. It is important to stress that in the case of essentially radial symbol it is possible to exploit what has been done in the case of the disk, but the operator is not a diagonal operator, and the Berezin transform is not particularly simple to write in an explicit way.

In what follows the punctured disk $\Delta = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ plays a very important role; for this reason we need the following.

Theorem 3.1. *There exists an isomorphism $I : L^2(\Delta) \to L^2(\Omega_1)$ such that*

$$I\left(L_a^2(\Delta)\right) = L_a^2(\Omega_1). \tag{3.3}$$

Moreover, for any $p \geq 2$ one has that $L_a^p(\Delta) = L_a^p(\Omega_1)$, and, for any $(z, w) \in \Delta^2$, the Bergman kernels K^Δ and K^{Ω_1} satisfy the following equation:

$$K^\Delta(z, w) = K^{\Omega_1}(z, w). \tag{3.4}$$

Proof. Suppose that $f \in L_a^2(\Delta)$; this means that f is holomorphic on Δ, then we can write down the Laurent expansion of f about 0, and we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n. \tag{3.5}$$

This implies that $|f(z)|^2 = \sum_{n,m=-\infty}^{\infty} a_n \overline{a_m} z^n \overline{z^m}$; therefore we have

$$\begin{aligned}
\int_\Delta \left|f(z)\right|^2 d\nu(z) &= \int_\Delta \sum_{n,m=-\infty}^{\infty} a_n \overline{a_m} z^n \overline{z^m} d\nu(z) \\
&= \int_0^{2\pi} \int_0^1 \sum_{n,m=-\infty}^{\infty} a_n \overline{a_m} r^{n+m+1} e^{i(n-m)\theta} dr\, d\theta \\
&= \sum_{n,m=-\infty}^{\infty} a_n \overline{a_m} \int_0^{2\pi} e^{i(n-m)\theta} d\theta \int_0^1 r^{n+m+1} dr \\
&= 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 \int_0^1 r^{2n+1} dr \\
&= 2\pi \left(\sum_{n\neq -1}^{\infty} |a_n|^2 \left[\frac{r^{2n+2}}{2n+2} \right]_0^1 + |a_{-1}|^2 \int_0^1 \frac{1}{r} dr \right).
\end{aligned} \tag{3.6}$$

The last equation, together with the fact that f is square-integrable, implies that $a_n = 0$ if $n \leq -1$. Then we can conclude that f has an holomorphic extension on Ω_1. We define

$$\mathcal{I} : L^2(\Delta) \longrightarrow L^2(\Omega_1) \tag{3.7}$$

in this way: if $g \in L^2(\Delta)$, then $\mathcal{I}g(z) = g(z)$ if $z \neq 0$ and

$$\mathcal{I}g(0) = \int_\Delta g(z) d\nu(z). \tag{3.8}$$

Then $\mathcal{I}g \in L^2(\Omega_1)$ and $\|\mathcal{I}g\|_{\Omega_1} = \|g\|_\Delta$. If $f \in L_a^2(\Delta)$, we have just shown that $\mathcal{I}f \in L_a^2(\Omega_1)$. Clearly $\mathcal{I}$ is injective and surjective, in fact if $G \in L^2(\Omega_1)$, then $g = G_{|\Delta}$ is an element of $L^2(\Delta)$ and $\mathcal{I}(g) = G$. Then $\mathcal{I}$ is an isomorphism of $L^2(\Delta)$ onto $L^2(\Omega_1)$ and $\mathcal{I}(L_a^2(\Delta)) = L_a^2(\Omega_1)$. Moreover, observing that $p > 2$ implies $\|f\|_{\Delta,2} \leq \|f\|_{\Delta,p}$ for any $f \in L^p(\Delta)$, we conclude that $L_a^p(\Delta) = L_a^p(\Omega_1)$.

Finally, it is easy to verify that for any $f, g \in L_a^2(\Delta)$ we have

$$\langle f, g \rangle_\Delta = \langle \mathcal{I}f, \mathcal{I}g \rangle_{\Omega_1}, \tag{3.9}$$

and this fact implies, by the definition of the Bergman reproducing kernel, that

$$K^\Delta(z, \mathrm{w}) = K^{\Omega_1}(z, w), \tag{3.10}$$

for any $(z, w) \in \Delta^2$. $\square$

In order to better explain our intuition, we remind the reader that we proved that, if $\varphi \in L^2(D)$ is an essentially radial function where Ω is a bounded multiply-connected domain and if we define $\varphi_j = \varphi \cdot p_j$ where $j = 1, \ldots, n$ where $\mathfrak{P} = \{p_0, p_1, \ldots, p_n\}$ is a ∂-partition for Ω, then the fact that the operator $T_\varphi : L_a^2(\Omega, d\nu) \rightarrow L_a^2(\Omega, d\nu)$ is bounded (compact) is equivalent to fact, that for any $j = 1, \dots, n$, the operators $T_{\varphi_j} : L^2_a(\Omega_j, d\nu) \to L^2_a(\Omega_j, d\nu)$ are bounded (compact) (see [4]).

We start our investigation by focusing our attention on the case of bounded symbols. In fact, we prove the following.

Theorem 3.2. *Let $\varphi \in L^\infty(D)$ be an essentially radial function, if one defines $\varphi_j = \varphi \cdot p_j$ where $j = 1, \dots, n$ and $\mathfrak{P}$ is a ∂-partition for D. Then for the bounded operator T_φ the following are equivalent:*

(1) *the operator $T_\varphi : L^2_a(D, d\nu) \to L^2_a(D, d\nu)$ is compact;*

(2) *for any $j = 1, \dots, n$ one has*

$$\lim_{k \to \infty} a_{\varphi_j}(k) = 0. \tag{3.11}$$

Proof. Since $\varphi \in L^\infty(D)$, we know that the operator $T_\varphi : L^2_a(\Omega, d\nu) \to L^2_a(\Omega, d\nu)$ is bounded, and we know that the boundedness (compactness) is equivalent to the fact that for any $j = 1, \dots, n$ the operators $T_{\varphi_j} : L^2(D_j, d\nu) \to L^2_a(D_j, d\nu)$ are bounded (compact). If $j = 2, \dots, n$, we observe that if we consider the following sets $\Delta_{0,1} = \{z \in \mathbb{C} : 0 < |z - a| < 1\}$ and $\Delta_{a_j, r_j} = \{z \in \mathbb{C} : 0 < |z - a_j| < r_j\}$ and the maps $\Delta_{0,1} \xrightarrow{\alpha} \Delta_{a_j,r_j} \xrightarrow{\beta} D_j$ where $\alpha(z) = a_j + r_j z$ and $\beta(w) = (w - a_j)^{-1} r_j^2 + a_j$ and we use Proposition 1.1 in [8], we can claim that $T_{\varphi_j} = V^{-1}_{\beta\circ\alpha} T_{\varphi_j \circ \beta \circ \alpha} V_{\beta\circ\alpha}$ where $V_{\beta\circ\alpha} : L^2(\Delta_{0,1}) \to L^2(D_j)$ is an isomorphism of the Hilbert spaces. Therefore T_{φ_j} is compact if and only if $T_{\varphi_j \circ\beta\circ\alpha}$ is compact. We also notice that the previous theorem implies that function $\{\sqrt{k+1} z^k\}$ is an orthonormal basis for $L^2(\Delta_{0,1})$, and this, in turn, implies that the compactness of the operator $T_{\varphi_j\circ\beta\circ\alpha}$ is equivalent to the fact that for the sequence $a_{\varphi_j} = \{a_{\varphi_j}(k)\}_{k\in\mathbb{N}}$ we have $\lim_{k\to\infty} a_{\varphi_j}(k) = 0$ where, by definition,

$$a_{\varphi_j}(k) = \int_{\Delta_{0,1}} \varphi_j \circ \beta \circ \alpha(z)(k+1) z^k \overline{z}^k dz \quad \forall m \in \mathbb{Z}_+. \tag{3.12}$$

To complete the proof we observe that, since φ_j is radial and $\beta \circ \alpha(r) = r^{-1} r_j + a_j$, then, after a change of variable, we can rewrite the last integral, and hence the formula

$$a_{\varphi_j}(k) = r_j \int_{r_j}^{\infty} \varphi_j(r_j s + a_j)(k+1) \frac{r_j^{2k+1}}{s^{2k+1}} \frac{1}{s^2} ds \quad \forall m \in \mathbb{Z}_+ \tag{3.13}$$

must hold for any $j = 2, \dots, n$. For the case $j = 1$ the proof is similar. □

Now we can prove the following.

Theorem 3.3. *Let $\varphi \in L^\infty(\Omega)$ be an essentially radial function via $\Theta : \bigcap_{\ell=1}^n \Omega_\ell \to \bigcap_{\ell=1}^n D_\ell$, if one defines $\varphi_j = \varphi \cdot p_j$ where $j = 1, \dots, n$ and $\mathfrak{P}$ is a ∂-partition for Ω. Then for the bounded operator T_φ the following are equivalent:*

(1) *the operator $T_\varphi : L^2_a(\Omega, d\nu) \to L^2_a(\Omega, d\nu)$ is compact;*

(2) *for any $j = 1, \dots, n$ one has*

$$\lim_{k \to \infty} a_{\varphi_j}(k) = 0. \tag{3.14}$$

Proof. We know that Ω is a regular domain, and therefore if Θ is a conformal mapping from Ω onto D then the Bergman kernels of Ω and $\Theta(\Omega) = D$ are related via $K^D(\Theta(z),\Theta(w))\Theta'(z)\overline{\Theta'(w)} = K^\Omega(z,w)$ and the operator $V_\Theta f = \Theta' \cdot f \circ \Theta$ is an isometry from $L^2(D)$ onto $L^2(\Omega)$ (see [8, Proposition 1.1]). In particular we have $V_\Theta P^D = P^\Omega V_\Theta$ and this implies that $V_\Theta T_\varphi = T_{\varphi\circ\Theta^{-1}} V_\Theta$. Therefore the operator T_φ is bounded if and only if for any $j = 1,\dots,n$ the operators $T_{\varphi_j\circ\Theta^{-1}} : L^2_a(D_j, d\nu) \to L^2_a(D_j, d\nu)$ are bounded (compact). Hence we can conclude that the operator is bounded (compact) if for any $j = 1,\dots,n$ we have

$$\lim_{k\to\infty} a_{\varphi_j}(k) = 0, \tag{3.15}$$

where, by definition, if $j = 2,\dots,n$,

$$a_{\varphi_j}(k) = r_j \int_{r_j}^{\infty} \varphi_j \circ \Theta^{-1}(r_j s + a_j)(k+1)\frac{r_j^{2k+1}}{s^{2k+1}}\frac{1}{s^2}ds \quad \forall k \in \mathbb{Z}_+, \tag{3.16}$$

and if $j = 1$,

$$a_{\varphi_1}(k) = \int_0^1 \varphi_1 \circ \Theta^{-1}(s)(k+1)s^{2k+1}ds \quad \forall k \in \mathbb{Z}_+. \tag{3.17}$$

□

Theorem 3.4. *Let $\varphi \in L^2(D)$ be an essentially radial function, if one defines $\varphi_j = \varphi \cdot p_j$ where $j = 1,\dots,n$ and $\mathfrak{P}$ is a ∂-partition for Ω and the operator $T_\varphi : L^2_a(D, d\nu) \to L^2_a(D, d\nu)$ is bounded (compact) and if for any $j = 1,\dots,n$ the sequences $a_{\varphi_j} = \{a_{\varphi_j}(k)\}_{k\in\mathbb{N}}$ are such that*

$$\sup_{k\in\mathbb{N}}\left\{\left|(k+1)a_{\varphi_j}(k) - k a_{\varphi_j}(k-1)\right|\right\} \tag{3.18}$$

is finite, then the operator $T_\varphi : L^2_a(D, d\nu) \to L^2_a(D, d\nu)$ is compact if and only if

$$\lim_{w\to\partial D} \widetilde{T_\varphi}(w) = 0. \tag{3.19}$$

Proof. We know that the operator under examination is bounded (compact) if and only if for any $j = 1,\dots,n$ the operators

$$T_{\varphi_j} : L^2(D_j, d\nu) \longrightarrow L^2_a(D_j, d\nu) \tag{3.20}$$

are bounded (compact). If $j = 2,\dots,n$, we observe that if we consider the following sets $\Delta_{0,1} = \{z \in \mathbb{C} : 0 < |z-a| < 1\}$ and $\Delta_{a_j,r_j} = \{z \in \mathbb{C} : 0 < |z-a_j| < r_j\}$ and the following maps

$$\Delta_{0,1} \xrightarrow{\alpha} \Delta_{a_j,r_j} \xrightarrow{\beta} D_j, \tag{3.21}$$

where $\alpha(z) = a_j + r_j z$ and $\beta(w) = (w - a_j)^{-1} r_j^2 + a_j$ and we use Proposition 1.1 in [8], we can claim that

$$T_{\varphi_j} = V_{\beta\circ\alpha}^{-1} T_{\varphi_j\circ\beta\circ\alpha} V_{\beta\circ\alpha}, \tag{3.22}$$

where $V_{\beta\circ\alpha} : L^2(\Delta_{0,1}) \to L^2(D_j)$ is an isomorphism of Hilbert's spaces. Therefore T_{φ_j} is compact if and only if $T_{\varphi_j\circ\beta\circ\alpha}$ is compact. Since $T_{\varphi_j\circ\beta\circ\alpha} : L_a^2(\Delta_{0,1}) \to L_a^2(\Delta_{0,1})$ and $\{\sqrt{k+1}z^k\}_{k=0}^{\infty}$ is an orthonormal basis, a simple calculation shows that

$$a_{\varphi_j}(k) = \left\langle T_{\varphi_j\circ\beta\circ\alpha}\sqrt{k+1}z^k, \sqrt{k+1}z^k \right\rangle; \tag{3.23}$$

therefore our assumption on $a_{\varphi_j} = \{a_{\varphi_j}(k)\}_{k\in\mathbb{N}}$ that

$$\sup_{k\in\mathbb{Z}_+}\left\{\left|(k+1)a_{\varphi_j}(k) - k a_{\varphi_j}(k-1)\right|\right\} < \infty \tag{3.24}$$

implies (see [9, Theorem 6]) that the compactness of $T_{\varphi_j\circ\beta\circ\alpha}$ is equivalent to the fact that the Berezin transform vanishes at the boundary. Since the case $j = 1$ is immediate, we can conclude, from what we proved so far, that for any $j = 1, 2, \ldots, n$ we have that compactness is equivalent to the fact that $\lim_{z\to\partial D_\ell}\widetilde{\varphi_\ell}^{D_\ell}(z) = 0$. To complete the proof we set, for any $j = 1, \ldots, n$, $\mathcal{S}_j = \{w \in \Omega \mid p_j(w) = 1\}$ where $\{p_1, \ldots, p_n\}$ is in the ∂-partition for Ω. By definition of ∂-partition, it follows that $\mathcal{S}_j \cap \mathcal{S}_i = \emptyset$ is $j \neq i$, and we can write

$$\begin{aligned}
\widetilde{\varphi}(z) &= \left\langle T_\varphi k_z^D, k_z^D \right\rangle \\
&= \int_D \varphi(w)\left|k_z^D(w)\right|^2 dw \\
&= \int_{\mathcal{S}_j} \varphi_\ell(w)\left|k_z^D(w)\right|^2 dw + \int_{\Omega\cap\mathcal{S}_j^c} \varphi_\ell(w)\left|k_z^D(w)\right|^2 dw.
\end{aligned} \tag{3.25}$$

Since for any $\ell = 1, \ldots, n$ the quantity $\int_{\mathcal{S}_\ell} \varphi_\ell(w)|k_z^D(w)|^2 dw$ can be written as

$$\int_{\mathcal{S}_\ell}\left(\varphi_\ell(w)\frac{\left(K_z^{D_\ell}\right)^2}{\left\|K_z^{D_\ell}\right\|_2^2}\right)\frac{\left\|K_z^{D_\ell}\right\|_2^2}{\|K_z^D\|_2^2}dw + \int_{\mathcal{S}_\ell}\varphi_\ell(w)\left(\left|k_z^D(w)\right|^2 - \frac{\left(K_z^{D_\ell}\right)^2}{\|K_z^D\|_2^2}\right)dw, \tag{3.26}$$

we observe that the function $\theta_\ell(z,w) = (|k_z^D(w)|^2 - (K_z^{D_\ell})^2/\|K_z^D\|_2^2)$ is bounded on the set $\mathcal{S}_\ell$ and vanishes at the boundary.

In fact, to prove this we remind the reader that there exists an isomorphism $I : L^2(\Delta_{0,1}) \to L^2(\Omega_1)$ such that $I(L_a^2(\Delta_{0,1})) = L_a^2(\Omega_1)$ and the Bergman kernels K^Δ and K^{Ω_1} satisfy the following equation $K^\Delta(z,w) = K^{\Omega_1}(z,w)$. If we define $\Delta_{a,r} = \{z \in \mathbb{C} : 0 < |z - \mathrm{a}| < r\}$ and $O_{a,r} = \{z \in \mathbb{C} : |z - a| > r\}$, then $K^{O_{a,r}}(z,w) = r^2/(r^2 - (z-a)\cdot\overline{(w-a)})^2$

for all $(z,w) \in O_{a,r} \times O_{a,r}$. The well-known fact that the reproducing kernel of the unit disk is given by $(1-z\overline{w})^{-2}$ implies that $K^{\Delta_{0,1}}(z,w) = 1/(1-z\cdot\overline{w})^2$ for all $(z,w) \in \Delta_{0,1}\times\Delta_{0,1}$ therefore, by conformal mapping, that the reproducing kernel of $\Delta_{a,r}$ is $K^{\Delta_{a,r}}(z,w) = r^2/(r^2-(z-a)\cdot\overline{(w-a)})^2$ for all $(z,w) \in \Delta_{a,r}\times\Delta_{a,r}$. If we define $\phi : \Delta_{a,r} \to O_{a,r}$ by $\phi(z) = (z-a)^{-1}r^2 + a$ and using the fact that $K^{O_{a,r}}(\phi(z),\phi(w))\phi'(z)\overline{\phi'(w)} = K^{\Delta_{a,r}}(z,w)$, we obtain that $K^{O_{a,r}}(z,w) = r^2/(r^2-(z-a)\cdot\overline{(w-a)})^2$ for all $(z,w)\in O_{a,r}\times O_{a,r}$. Since $\Omega_1 = O_{0,1}$ and, for $j=2,\ldots,n$, $O_{a_j,r_j} = \Omega_j$, then we prove that $K^{D_j}(z,w) = r_j^2/(r_j^2-(z-a_j)\cdot\overline{(w-a_j)})^2$ if $j=2,\ldots,n$.

Hence, for the function $\theta_\ell(z,w)$, we have

$$\begin{aligned}
|\theta_\ell(\zeta,z)| &= \left|\left(\frac{\sum_{m=0}^{n} K_m^D(\zeta,z)}{\|K_z^D\|_2^2}\right)^2 - \frac{\left(K_z^{D_\ell}\right)^2}{\|K_z^D\|_2^2}\right| \\
&= \left|\frac{\left(K_z^{D_\ell}\right)^2}{\|K_z^D\|_2^2}\left(1+\sum_{m\neq\ell}^{n}\frac{K_m^D(\zeta,z)}{K_z^{D_\ell}(\zeta,z)}\right)^2 - \frac{\left(K_z^{D_\ell}\right)^2}{\|K_z^D\|_2^2}\right| \\
&= \left|\frac{\left(K_z^{D_\ell}\right)^2}{\|K_z^D\|_2^2}\left(1+\sum_{\substack{m\neq\ell\\ m>0}}^{n}\frac{r_m^2/\left(r_m^2-(z-a_m)\cdot\overline{(\zeta-a_m)}\right)^2}{r_\ell^2/\left(r_\ell^2-(z-a_\ell)\cdot\overline{(\zeta-a_\ell)}\right)^2}+\frac{K_0^D(\zeta,z)}{K_z^{D_\ell}(\zeta,z)}\right)^2 - \frac{\left(K_z^{D_\ell}\right)^2}{\|K_z^D\|_2^2}\right|.
\end{aligned} \tag{3.27}$$

Hence it follows that $\theta_\ell(z,w)$ is bounded on the set $\mathcal{S}_\ell$ and goes to zero as it approaches the boundary. Since $\varphi \in L^2(D)$ and D has a finite measure, we can conclude that, by the dominated convergence theorem, we have

$$\lim_{z\to\partial D_\ell}\int_{\mathcal{S}_\ell}\varphi_\ell(w)\left(\left|k_z^D(w)\right|^2 - \frac{\left(K_z^{D_\ell}\right)^2}{\|K_z^D\|_2^2}\right)dw = 0. \tag{3.28}$$

Now we observe that Lemma 2 in [4] implies that

$$\lim_{z\to\partial D_\ell}\int_{D\cap\mathcal{S}_\ell}\left(\varphi_\ell(w)\left(\frac{K_z^{D_\ell}}{\left\|K_z^{D_\ell}\right\|_2}\right)^2\right)\left(\frac{\left\|K_z^{D_\ell}\right\|_2}{\left\|K_z^D\right\|_2}\right)^2 dw \tag{3.29}$$

goes to zero if and only $\lim_{z\to\partial D_\ell}\widetilde{\varphi_\ell}^{D_\ell}(z) = 0$, and a simple calculation shows that

$$\int_{\Omega\cap\mathcal{S}_j^c}\varphi_\ell(w)\left|k_z^D(w)\right|^2 dw = 0. \tag{3.30}$$

Hence, as a consequence, we have shown that, if the conditions in the hypothesis hold, then

$$\lim_{z \to \partial D_\ell} \widetilde{\varphi}^D(z) = 0, \tag{3.31}$$

and this completes the proof since $\partial D = \bigcup_1^n \partial D_\ell$. □

Now we can prove the following.

Theorem 3.5. *Let $\varphi \in L^2(\Omega)$ be an essentially radial function via $\Theta : \bigcap_{\ell=1}^n \Omega_\ell \to \bigcap_{\ell=1}^n D_\ell$ and $\varphi_j = \varphi \cdot p_j$ for any $j = 1, \dots, n$ where $\mathfrak{P} = \{p_0, p_1, \dots, p_n\}$ is a ∂-partition for Ω. If the operator $T_\varphi : L_a^2(\Omega, d\nu) \to L_a^2(\Omega, d\nu)$ is bounded and if for any $j = 1, \dots, n$ the sequence $a_{\varphi_j} = \{a_{\varphi_j}(k)\}_{k \in \mathbb{Z}_+}$ satisfies the following*

$$\sup_{k \in \mathbb{Z}_+} \left\{ \left| (k+1) a_{\varphi_j}(k) - k a_{\varphi_j}(k-1) \right| \right\} < \infty, \tag{3.32}$$

then the operator $T_\varphi : L_a^2(\Omega, d\nu) \to L_a^2(\Omega, d\nu)$ is compact if and only if

$$\lim_{w \to \partial\Omega} \widetilde{T_\varphi}(w) = 0. \tag{3.33}$$

Proof. We know that Ω is a regular domain, and therefore, if Θ is a conformal mapping from Ω onto D then the Bergman kernels of Ω and $\Theta(\Omega) = D$ are related via $K^D(\Theta(z), \Theta(w))\Theta'(z)\overline{\Theta'(w)} = K^\Omega(z, w)$ and the operator $V_\Theta f = \Theta' \cdot f \circ \Theta$ is an isometry from $L^2(D)$ onto $L^2(\Omega)$ (see [8, Proposition 1.1]). In particular we have $V_\Theta P^D = P^\Omega V_\Theta$ and this implies that $V_\Theta T_\varphi = T_{\varphi \circ \Theta^{-1}} V_\Theta$. Therefore the operator T_φ is bounded (compact) if and only if the operator $T_{\varphi \circ \Theta^{-1}} : L^2(D, d\nu) \to L_a^2(D, d\nu)$ is bounded (compact). In the previous theorem we proved that the operator in exam is bounded (compact) if and only if for any $j = 1, \dots, n$ the operators $T_{\varphi_j \circ \Theta^{-1}} : L_a^2(D_j, d\nu) \to L_a^2(D_j, d\nu)$ are bounded (compact). Hence, since the sequences $a_{\varphi_j} = \{a_{\varphi_j}(m)\}_{m \in \mathbb{N}}$ satisfy the stated properties, we can conclude that the operators $T_{\varphi_j \circ \Theta^{-1}} : L_a^2(D_j, d\nu) \to L_a^2(D_j, d\nu)$ are compact if and only if for any $j = 1, \dots, n$ we have

$$\lim_{z \to \partial D_j} \widetilde{\varphi_j \circ \Theta^{-1}}^{D_j}(z) = 0. \tag{3.34}$$

Therefore it follows that

$$\lim_{z \to \partial D} \widetilde{\varphi \circ \Theta^{-1}}^{D}(z) = 0, \tag{3.35}$$

and, since Θ is a conformal mapping, this implies that

$$\lim_{z \to \partial\Omega} \widetilde{\varphi}^\Omega(z) = 0. \tag{3.36}$$

□

Finally, we also observe that as a simple consequence we obtain the following.

Theorem 3.6. *Let* $\varphi \in L^2(\Omega)$ *be an essentially radial function via* $\Theta : \bigcap_{\ell=1}^{n} \Omega_\ell \rightarrow \bigcap_{\ell=1}^{n} D_\ell$ *and* $\varphi_j = \varphi \cdot p_j$ *for any* $j = 1, \ldots, n$ *where* $\mathfrak{P} = \{p_0, p_1, \ldots, p_n\}$ *is a* ∂*-partition for* Ω*. If the operator* $T_\varphi : L_a^2(\Omega, d\nu) \rightarrow L_a^2(\Omega, d\nu)$ *is bounded and if for any* $j = 1, \ldots, n$ *the sequence* $a_{\varphi_j} = \{a_{\varphi_j}(k)\}_{k\in\mathbb{Z}_+}$ *satisfies the following*

$$\sup_{k\in\mathbb{Z}_+}\left\{\left|k\left(a_{\varphi_j}(k) - a_{\varphi_j}(k-1)\right)\right|\right\} < \infty, \tag{3.37}$$

then the operator $T_\varphi : L_a^2(\Omega, d\nu) \rightarrow L_a^2(\Omega, d\nu)$ *is compact if and only if*

$$\lim_{w\to\partial\Omega} \widetilde{T_\varphi}(w) = 0. \tag{3.38}$$

Finally, we observe that it is also to recover as corollary the following.

Corollary 3.7. *Let* $\varphi \in L^2(\Omega)$ *be an essentially radial symbol via the conformal equivalence* $\Theta : \Omega \rightarrow D$*. If one defines* $\varphi_j = \varphi \cdot p_j$ *where* $j = 1, \ldots, n$ *and* $\mathfrak{P}$ *is a* ∂*-partition for* Ω*. Let us assume that* $\gamma_{\phi_j} = \{\gamma_{\phi_j}(m)\}_{m\in\mathbb{N}}$ *is in* $\ell_\infty(\mathbb{Z}_+)$ *and that there is a constant* C_3 *such that for* $j = 2, \ldots, n$

$$\sup_{\tau\in[a_j+r_j,\infty)}\left|\varphi_j \circ \Theta(\tau) - \frac{\tau - a_j}{\tau - r_j - a_j}\int_{a_j+r_j}^{\tau} \varphi_j \circ \Theta(y)\left(\frac{r_j}{(y-a_j)^2}\right)dy\right| < C_3 \tag{3.39}$$

and for $j = 1$

$$\sup_{\tau\in[0,1]}\left|\varphi_1 \circ \Theta(\tau) - \frac{1}{1-\tau}\int_{\tau}^{1} \varphi_1 \circ \Theta(s)ds\right| < C_3. \tag{3.40}$$

Then the operator $T_\varphi : L_a^2(\Omega, d\nu) \rightarrow L_a^2(\Omega, d\nu)$ *is compact if and only if*

$$\lim_{w\to\partial\Omega} \widetilde{T_\varphi}(w) = 0. \tag{3.41}$$

The last corollary was also proved, in different way, in [4].

References

[1] S. Grudsky and N. Vasilevski, "Bergman-Toeplitz operators: radial component influence," *Integral Equations and Operator Theory*, vol. 40, no. 1, pp. 16–33, 2001.

[2] N. Zorboska, "The Berezin transform and radial operators," *Proceedings of the American Mathematical Society*, vol. 131, no. 3, pp. 793–800, 2003.

[3] B. Korenblum and K. H. Zhu, "An application of Tauberian theorems to Toeplitz operators," *Journal of Operator Theory*, vol. 33, no. 2, pp. 353–361, 1995.

[4] R. C. Raimondo, "Toeplitz operators on the bergman space of planar domains with essentially radial symbols," *International Journal of Mathematics and Mathematical Sciences*, vol. 2011, Article ID 164843, 26 pages, 2011.

[5] S. Axler and D. Zheng, "Compact operators via the Berezin transform," *Indiana University Mathematics Journal*, vol. 47, no. 2, pp. 387–400, 1998.
[6] R. Raimondo, "Compact operators on the Bergman space of multiply-connected domains," *Proceedings of the American Mathematical Society*, vol. 129, no. 3, pp. 739–747, 2001.
[7] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Translations of Mathematical Monographs, Vol. 26, American Mathematical Society, Providence, RI, USA, 1969.
[8] J. Arazy, S. D. Fisher, and J. Peetre, "Hankel operators on planar domains," *Constructive Approximation*, vol. 6, no. 2, pp. 113–138, 1990.
[9] Z. -H. Zhou, W. -L. Chen, and X. -T. Dong, "The Berezin transform and radial operators on the Bergman space of the unit ball," *Complex Analysis and Operator Theory*. In press.