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Research Article

On Prime-Gamma-Near-Rings with Generalized Derivations

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Let N be a 2-torsion free prime Γ-near-ring with center $Z(N)$. Let (f,d) and (g,h) be two generalized derivations on N. We prove the following results: (i) if $f([x,y]_\alpha)=0$ or $f([x,y]_\alpha)=\pm[x,y]_\alpha$ or $f^2(x)\in Z(N)$ for all $x,y\in N$, $\alpha\in\Gamma$, then N is a commutative Γ-ring. (ii) If $a\in N$ and $[f(x),a]_\alpha=0$ for all $x\in N$, $\alpha\in\Gamma$, then $d(a)\in Z(N)$. (iii) If (fg,dh) acts as a generalized derivation on N, then $f=0$ or $g=0$.

1. Introduction

The derivations in Γ-near-rings have been introduced by Bell and Mason [1]. They studied basic properties of derivations in Γ-near-rings. Then Aşci [2] obtained commutativity conditions for a Γ-near-ring with derivations. Some characterizations of Γ-near-rings and regularity conditions were obtained by Cho [3]. Kazaz and Alkan [4] introduced the notion of two-sided Γ-α-derivation of a Γ-near-ring and investigated the commutativity of a prime and semiprime Γ-near-rings. Uçkun et al. [5] worked on prime Γ-near-rings with derivations and they found conditions for a Γ-near-ring to be commutative. In [6] Dey et al. studied commutativity of prime Γ-near-ring with generalized derivations.

In this paper, we obtain the conditions of a prime Γ-near-ring to be a commutative Γ-ring. If $a\in N$, and $[f(x),a]_\alpha=0$ for all $x\in N,\alpha\in\Gamma$, then d is central. Also we prove that if (fg,dh) is the generalized derivation on N, then f and g are trivial.

2. Preliminaries

A Γ-near-ring is a triple $(N, +, \Gamma)$, where

(i) $(N, +)$ is a group (not necessarily abelian);

(ii) Γ is a nonempty set of binary operations on N such that for each $\alpha \in \Gamma, (N, +, \alpha)$ is a left near-ring;

(iii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in N$ and $\alpha, \beta \in \Gamma$.

We will use the word Γ-near-ring to mean left Γ-near-ring. For a near-ring N, the set $N_0 = \{x \in N : 0\alpha x = 0, \alpha \in \Gamma\}$ is called the zero-symmetric part of N. A Γ-near-ring N is said to be zero-symmetric if $N = N_0$. Throughout this paper, N will denote a zero symmetric left Γ-near-ring with multiplicative centre $Z(N)$. Recall that a Γ-near-ring N is prime if $x\Gamma N\Gamma y = 0$ implies $x = 0$ or $y = 0$. An additive mapping $d : N \to N$ is said to be a derivation on N if $d(x\alpha y) = x\alpha d(y) + d(x)\alpha y$ for all $x, y \in N, \alpha \in \Gamma$, or equivalently, as noted in [1], that $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for all $x, y \in N, \alpha \in \Gamma$. Further, an element $x \in N$ for which $d(x) = 0$ is called a constant. For $x, y \in N, \alpha \in \Gamma$, the symbol $[x, y]_\alpha$ will denote the commutator $x\alpha y - y\alpha x$, while the symbol (x, y) will denote the additive-group commutator $x+y-x-y$. An additive mapping $f : N \to N$ is called a generalized derivation if there exits a derivation d of N such that $f(x\alpha y) = f(x)\alpha y + x\alpha d(y)$ for all $x, y \in N, \alpha \in \Gamma$. The concept of generalized derivation covers also the concept of a derivation.

3. Derivations on Γ-Near-Rings

In this section we prove that a few subsidiary results (Lemmas 3.1, 3.2, 3.4, 3.8, 3.9, 3.10 and 3.11) to use them for proving of our main results (Theorems 3.3, 3.5, 3.6, 3.12 and 3.13).

Lemma 3.1. *Let d be an arbitrary derivation on a Γ-near-ring N. Then N satisfies the following partial distributive law: $(a\alpha d(b) + d(a)\alpha b)\beta c = a\alpha d(b)\beta c + d(a)\alpha b\beta c$ and $(d(a)\alpha b + a\alpha d(b))\beta c = d(a)\alpha b\beta c + a\alpha d(b)\beta c$ for all $a, b, c \in N, \alpha, \beta \in \Gamma$.*

Proof. For all $a, b, c \in N, \alpha, \beta \in \Gamma$, we get $d((a\alpha b)\beta c) = a\alpha b\beta d(c) + (a\alpha d(b) + d(a)\alpha b)\beta c$ and $d(a\alpha(b\beta c)) = a\alpha d(b\beta c) + d(a)\alpha b\beta c = a\alpha(b\beta d(c) + d(b)\beta c) + d(a)\alpha b\beta c = a\alpha b\beta d(c) + a\alpha d(b)\beta c + d(a)\alpha b\beta c$. Equating these two relations for $d(a\alpha b\beta c)$ now yields the required partial distributive law. □

Lemma 3.2. *Let d be a derivation on a Γ-near-ring N and suppose $u \in N$ is not a left zero divisor. If $[u, d(u)]_\alpha = 0, \alpha \in \Gamma$, then (x, u) is a constant for every $x \in N$.*

Proof. From $u\alpha(u + x) = u\alpha u + u\alpha x$, for all $x \in N$, $\alpha \in \Gamma$, we obtain $u\alpha d(u + x) + d(u)\alpha(u + x) = u\alpha d(u) + d(u)\alpha u + u\alpha d(x) + d(u)\alpha x$, which reduces $u\alpha d(x) + d(u)\alpha u = d(u)\alpha u + u\alpha d(x)$, for all $\alpha \in \Gamma$.

Since $d(u)\alpha u = u\alpha d(u)$, $\alpha \in \Gamma$, this equation is expressible as $u\alpha(d(x) + d(u) - d(x) - d(u)) = 0 = u\alpha d((x,\ u))$. Thus $d((x, u)) = 0$. □

Theorem 3.3. *Let N be a Γ-near-ring having no nonzero divisors of zero. If N admits a nontrivial commuting derivation d, then $(N, +)$ is abelian.*

Proof. Let c be any additive commutator. Then c is a constant by Lemma 3.2. Moreover, for any $w \in N$, $\alpha \in \Gamma$, $w\alpha c$ is an additive commutator, hence also a constant. Thus, $0 = d(w\alpha c) = w\alpha d(c) + d(w)\alpha c$ and $d(w)\alpha c = 0$, for all $\alpha \in \Gamma$. Since $d(w) \neq 0$ for all $w \in N$, we conclude that $c = 0$. □

Lemma 3.4. *Let N be a prime Γ-near-ring.*

(i) *If $z \in Z(N) - \{0\}$, then z is not a zero divisor in N.*

(ii) *If $Z(N) - \{0\}$ contains an element z for which $z + z \in Z(N)$, then $(N, +)$ is abelian.*

(iii) *Let d be a nonzero derivation on N. Then $x\Gamma d(N) = \{0\}$ implies $x = 0$, and $d(N)\Gamma x = \{0\}$ implies $x = 0$.*

(iv) *If N is 2-torsion free and d is a derivation on N such that $d^2 = 0$, then $d = 0$.*

Proof. (i) If $z \in Z(N) - \{0\}$ and $z\alpha x = 0$, $x \in N$, $\alpha \in \Gamma$, then $z\alpha r\beta x = 0$, $x, r \in N$, $\alpha \in \Gamma$. Thus we get $z\Gamma N\Gamma x = 0$, by primeness of N, $x = 0$.

(ii) Let $z \in Z(N) - \{0\}$ be an element such that $z + z \in Z(N)$, and let $x, y \in N, \alpha \in \Gamma$. Since $z + z$ is distributive, we get $(x + y)\alpha(z + z) = x\alpha(z + z) + y\alpha(z + z) = x\alpha z + x\alpha z + y\alpha z + y\alpha z = z\alpha(x + x + y + y)$.

On the other hand, $(x + y)\alpha(z + z) = (x + y)\alpha z + (x + y)\alpha z = z\alpha(x + y + x + y)$. Thus, $x + x + y + y = x + y + x + y$ and therefore $x + y = y + x$. Hence $(N, +)$ is abelian.

(iii) Let $x\Gamma d(N) = 0$, and let r, s be arbitrary elements of N and $\alpha, \beta \in \Gamma$. Then $0 = x\alpha d(r\beta s) = x\alpha r\beta d(s) + x\alpha d(r)\beta s = x\alpha r\beta d(s)$. Thus $x\Gamma N\Gamma d(N) = \{0\}$, and since $d(N) \neq \{0\}, x = 0$.

A similar argument works if $d(N)\Gamma x = \{0\}$, since Lemma 3.1 provides enough distributivity to carry it through.

(iv) For arbitrary $x, y \in N, \alpha \in \Gamma$, we have $0 = d^2(x\alpha y) = d(x\alpha d(y) + d(x)\alpha y) = x\alpha d^2(y) + d(x)\alpha d(y) + d(x)\alpha d(y) + d^2(x)\alpha y = 2d(x)\alpha d(y)$. Since N is 2-torsion free, $d(x)\alpha d(y) = 0, x,\ y \in N, \alpha \in \Gamma$. Thus $d(x)\Gamma d(N) = \{0\}$ for each $x \in N$, and (iii) yields $d(a) = 0$. Thus $d = 0$. □

Theorem 3.5. *If a prime Γ-near-ring N admits a nontrivial derivation d for which $d(N) \in Z(N)$, then $(N, +)$ is abelian. Moreover, if N is 2-torsion free, then N is a commutative Γ-ring.*

Proof. Let c be an arbitrary constant, and let x be anon-constant. Then $d(x\alpha c) = x\alpha d(c) + d(x)\alpha c = d(x)\alpha c \in Z(N)$, $\alpha \in \Gamma$. Since $d(x) \in Z(N) - \{0\}$, it follows easily that $c \in Z(N)$. Since $c+c$ is a constant for all constants c, it follows from Lemma 3.4(ii) that $(N, +)$ is abelian, provided that there exists a nonzero constant.

Assume, then, that 0 is the only constant. Since d is obviously commuting, it follows from Lemma 3.2 that all u which are not zero divisors belong to the center of $(N, +)$, denoted by $Z(N)$. In particular, if $x \neq 0$, $d(x) \in Z(N)$. But then for all $y \in N$, $d(y)+d(x)-d(y)-d(x) = d((y, x)) = 0$, hence $(y, x) = 0$.

Now we assume that N is 2-torsion free. By Lemma 3.1, $(a\alpha d(b) + d(a)\alpha b)\beta c = a\alpha d(b)\beta c + d(a)\alpha b\beta c$ for all $a, b, c \in N$, $\alpha, \beta \in \Gamma$, and using the fact that $d(a\alpha b) \in Z(N)$, $\alpha \in \Gamma$, we get $c\alpha a\beta d(b) + c\alpha d(a)\beta b = a\alpha d(b)\beta c + a\alpha d(b)\beta c, \alpha, \beta \in \Gamma$. Since $(N, +)$ is abelian and $d(N) \subseteq Z(N)$, this equation can be rearranged to yield $d(b)\alpha[c, a]_\beta = d(a)\alpha[b, c]_\beta$ for all $a, b, c \in N, \alpha, \beta \in \Gamma$.

Suppose now that N is not commutative. Choosing $b, c \in N$, with $[b, c]_\beta \neq 0$, $\beta \in \Gamma$, and letting $a = d(x)$, we get $d^2(x)\alpha[b, c]_\beta = 0$, for all $x \in N, \alpha, \beta \in \Gamma$, and since the central elements $d^2(x)$ cannot be a nonzero divisor of zero, we conclude that $d^2(x) = 0$ for all $x \in N$. But by Lemma 3.4(iv), this cannot happen for nontrivial d. □

Theorem 3.6. *Let N be a prime Γ-near-ring admitting a nontrivial derivation d such that $[d(x), d(y)]_\alpha = 0$ for all $x, y \in N, \alpha \in \Gamma$. Then $(N, +)$ is abelian. Moreover, if N is 2-torsion free, then N is a commutative Γ-ring.*

Proof. By Lemma 3.4(ii), if both z and $z+z$ commute element-wise with $d(N)$, then $z\alpha d(c) = 0, \alpha \in \Gamma$, for all additive commutators c. Thus, taking $z = d(x)$, we get $d(x)\alpha d(c) = 0$ for all $x \in N, \alpha \in \Gamma$, so $d(c) = 0$ by Lemma 3.4(iii). Since $w\alpha c$ is also an additive commutator for any $w \in N$, $\alpha \in \Gamma$, we have $d(w\alpha c) = 0\ d(w)\alpha c$, and another application of Lemma 3.4(iii) gives $c = 0$.

Now we assume that N is 2-torsion free. By the partial distributive law, $d(d(x)\alpha y)\beta d(z) = d(x)\alpha d(y)\beta d(z) + d^2(x)\alpha y\beta d(z)$ for all $x, y, z \in N, \alpha, \beta \in \Gamma$, hence, $d^2(x)\alpha y\beta d(z) = d(d(x)\alpha y)\beta d(z) - d(x)\alpha d(y)\beta d(z) = d(x)\alpha d(y)\beta d(z) = d(z)\alpha(d(d(x)\beta y) - d(x)\beta d(y)) = d(z)\alpha d^2(x)\beta y = d^2(x)\alpha d(z)\beta y, \alpha, \beta \in \Gamma$. Thus $d^2(x)\alpha(y\beta d(z) - d(z)\beta y) = 0$ for all $x, y\ z \in N$, $\alpha,\ \beta \in \Gamma$.

Replacing $y\delta t, \delta \in \Gamma$, we obtain $d^2(x)\alpha y\delta t\beta d(z) = d^2(x)\alpha d(z)\beta y\delta t = d^2(x)\alpha y\beta d(z)\delta t$, for all $x, y, z, t \in N$, $\alpha, \beta, \delta \in \Gamma$, so that $d^2(x)\alpha y\beta[t, d(z)]_\delta = 0$ for all $x, y, z, t \in N$, $\alpha, \beta, \delta \in \Gamma$. The primeness of N shows that either $d^2 = 0$ or $d(N) \subseteq Z(N)$, and since the first of these conditions is impossible by Lemma 3.4(iv), the second must hold and N is a commutative Γ-ring by Theorem 3.5. □

Definition 3.7. Let N be a Γ-near-ring and d a derivation of N. An additive mapping $f : N \to N$ is said to be a right generalized derivation of N associated with d if

$$f(x\alpha y) = f(x)\alpha y + x\alpha d(y) \quad \forall x, y \in R,\ \alpha \in \Gamma, \tag{3.1}$$

and f is said to be a left generalized derivation of N associated with d if

$$f(x\alpha y) = d(x)\alpha y + x\alpha f(y) \quad \forall x, y \in R,\ \alpha \in \Gamma. \tag{3.2}$$

Finally, f is said to be a generalized derivation of N associated with d if it is both a left and right generalized derivation of N associated with d.

Lemma 3.8. *Let f be a right generalized derivation of a Γ-near ring N associated with d. Then*

(i) $f(x\alpha y) = x\alpha d(y) + f(x)\alpha y$ *for all* $x, y \in N, \alpha \in \Gamma$;

(ii) $f(x\alpha y) = x\alpha f(y) + d(x)\alpha y$ *for all* $x, y \in N, \alpha \in \Gamma$.

Proof. (i) For any $x, y \in N, \alpha \in \Gamma$, we get

$$\begin{gathered} f(x\alpha(y+y)) = f(x)\alpha(y+y) + x\alpha d(y+y) = f(x)\alpha y + f(x)\alpha y + x\alpha d(y) + x\alpha d(y), \\ f(x\alpha y + x\alpha y) = f(x)\alpha y + x\alpha d(y) + f(x)\alpha y + x\alpha d(y). \end{gathered} \tag{3.3}$$

Comparing these two expressions, we obtain

$$f(x)\alpha y + x\alpha d(y) = x\alpha d(y) + f(x)\alpha y \quad \forall x, y \in N,\ \alpha \in \Gamma, \tag{3.4}$$

and so,

$$f(x\alpha y) = x\alpha d(y) + f(x)\alpha y \quad \forall x,\ y \in N,\ \alpha \in \Gamma. \tag{3.5}$$

(ii) In a similar way. □

Lemma 3.9. *Let f be a right generalized derivation of a Γ-near ring N associated with d. Then*

(i) $(f(x)\alpha y + x\alpha d(y))\beta z = f(x)\alpha y\beta z + x\alpha d(y)\beta z$, *for all $x, y \in N$, $\alpha, \beta \in \Gamma$.*

(ii) $(d(x)\alpha y + x\alpha f(y))\beta z = d(x)\alpha y\beta z + x\alpha f(y)\beta z$, *for all $x, y \in N$, $\alpha, \beta \in \Gamma$.*

Proof. (i) For any $x, y, z \in N$, $\alpha, \beta \in \Gamma$, we get $f((x\alpha y)\beta z) = f(x\alpha y)\beta z + x\alpha y\beta d(z)$.
On the other hand,

$$f(x\alpha(y\beta z)) = f(x)\alpha y\beta z + x\alpha d(y\beta z) = f(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta d(z). \tag{3.6}$$

From these two expressions of $f(x\alpha y\beta z)$, we obtain that, for all $x, y, z \in N$, $\alpha, \beta \in \Gamma$,

$$(f(x)\alpha y + x\alpha d(y))\beta z = f(x)\alpha y\beta z + x\alpha d(y)\beta z. \tag{3.7}$$

(ii) The proof is similar. □

Lemma 3.10. *Let N be a prime Γ-near-ring, f a nonzero generalized derivation of N associated with the nonzero derivation d and $a \in N$.* (i) *If $a\Gamma f(N) = 0$, then $a = 0$.* (ii) *If $f(N)\Gamma a = 0$, then $a = 0$.*

Proof. (i) For any $x, y \in N, \alpha, \beta \in \Gamma$, we get $0 = a\beta f(x\alpha y) = a\beta f(x)\alpha y + a\beta x\alpha d(y) = a\beta x\alpha d(y)$. Hence $a\Gamma N\Gamma d(N) = 0$. Since N is a prime Γ-near-ring and $d \neq 0$, we obtain $a = 0$.

(ii) A similar argument works if $f(N)\Gamma a = 0$. □

Lemma 3.11. *Let N be a prime Γ-near-ring. Let f be a generalized derivation of N associated with the nonzero derivation d. If N is a 2-torsion free Γ-near-ring and $f^2 = 0$, then $f = 0$.*

Proof. (i) For any $x, y \in N, \alpha \in \Gamma$, we get

$$0 = f^2(x\alpha y) = f(f(x\alpha y)) = f(f(x)\alpha y + x\alpha d(y)) = f^2(x)\alpha y + 2f(x)\alpha d(y) + x\alpha d^2(y). \tag{3.8}$$

By the hypothesis,

$$2f(x)\alpha d(y) + x\alpha d^2(y) = 0 \quad \forall x, y \in N,\ \alpha \in \Gamma. \tag{3.9}$$

Writing $f(x)$ by x in (3.9), we get $f(x)\alpha d^2(y) = 0$, for all $x, y \in N, \alpha \in \Gamma$.

By Lemma 3.9(ii), we obtain that $d^2(N) = 0$ or $f = 0$. If $d^2(N) = 0$ then $d = 0$ from Lemma 3.4(iv), a contradiction. So we find $f = 0$. □

Theorem 3.12. *Let N be a prime Γ-near-ring with a nonzero generalized derivation f associated with d. If $f(N) \subseteq Z(N)$, then $(N,+)$ is abelian. Moreover, if N is 2-torsion free, then N is commutative Γ-ring.*

Proof. Suppose that $a \in N$, such that $f(a) \neq 0$. So, $f(a) \in Z(N) - \{0\}$ and $f(a) + f(a) \in Z(N) - \{0\}$. For all $x, y \in N$, $\alpha \in \Gamma$, we have $(x+y)\alpha(f(a+f(a)) = (f(a+f(a))\alpha(x+y)$.

That is, $x\alpha f(a) + x\alpha f(a) + y\alpha f(a) + y\alpha f(a) = f(a)\alpha x + f(a)\alpha x + f(a)\alpha y + f(a)\alpha y$, for all $x, y \in N$, $\alpha \in \Gamma$.

Since $f(a) \in Z(N)$, we get $f(a)\alpha x + f(a)\alpha y = f(a)\alpha y + f(a)\alpha x$, and so, $f(a)\alpha(x,y) = 0$ for all $x, y \in N, \alpha \in \Gamma$.

Since $f(a) \in Z(N) - \{0\}$ and N is a prime Γ-near-ring, it follows that $(x,y) = 0$, for all $x, y \in N$. Thus $(N,+)$ is abelian.

Using the hypothesis, for any $x, y, z \in N, \alpha, \beta \in \Gamma$, $z\alpha f(x\beta y) = f(x\beta y)\alpha z$. By Lemma 3.4(ii), we have $z\alpha d(x)\beta y + z\alpha x f(y) = d(x)\alpha y\beta z + x\alpha f(y)\beta z$. Using $f(N) \subset Z(N)$ and $(N,+)$ being abelian, we obtain that

$$z\alpha d(x)\beta y - d(x)\alpha y\beta z = [x,z]_\alpha \beta f(y), \quad \forall x, y \in N,\ \alpha, \beta \in \Gamma. \tag{3.10}$$

Substituting $f(z)$ for z in (3.10), we get $f(z)\beta[d(x),y]_\alpha = 0$ for all $x, y \in N, \alpha, \beta \in \Gamma$.

Since $f(z) \in Z(N)$ and f a nonzero generalized derivation with associated with d, we get $d(N) \subset Z(N)$. So, N is a commutative Γ-ring by Theorem 3.3. □

Theorem 3.13. *Let N be a prime Γ-near-ring with a nonzero generalized derivation f associated with d. If $[f(N), f(N)]_\alpha = 0$, $\alpha \in \Gamma$, then $(N,+)$ is abelian. Moreover, if N is 2-torsion free, then N is commutative Γ-ring.*

Proof. By the same argument as in Theorem 3.12, it is shown that if both z and $z+z$ commute elementwise with $f(N)$, then we have

$$z\alpha f(x,y) = 0 \quad \forall x, y \in N,\ \alpha \in \Gamma. \tag{3.11}$$

Substituting $f(t), t \in N$ for z in (3.11), we get $f(t)\alpha f(x,y) = 0$, $\alpha \in \Gamma$. By Lemma 3.9(i), we obtain that $f(x,y) = 0$ for all $x, y \in N$, $\alpha \in \Gamma$. For any $w \in N$, $\beta \in \Gamma$, we have $0 = f(w\beta x, w\beta y) = f(w\beta(x,y)) = d(w)\beta(x,y) + w\beta f(x,y)$ and so, we obtain $d(w)\beta(x,y) = 0$, for any $w \in N, \beta \in \Gamma$. From Lemma 3.4(iii), we get $(x,y) = 0$ for any $x, y \in N$.

Now we assume that N is 2-torsion free. By the assumption $[f(N), f(N)]_\alpha = 0$, $\alpha \in \Gamma$, we have

$$f(z)\alpha f(f(x)\beta y) = f(f(x)\beta y)\alpha f(z) \quad \forall x, y, z \in N,\ \alpha, \beta \in \Gamma. \tag{3.12}$$

Hence we get

$$\begin{aligned} &f(z)\alpha d(f(x))\beta y + f(z)\alpha f(x)\beta f(y) = d(f(x))\alpha y\beta f(z) + f(x)\alpha f(y)\beta f(z),\\ &f(z)\alpha d(f(x))\beta y + f(x)\alpha f(z)\beta f(y) = d(f(x))\alpha y\beta f(z) + f(x)\alpha f(z)\beta f(y), \end{aligned} \tag{3.13}$$

and so,

$$f(z)\alpha d(f(x)\beta y) = d(f(x))\alpha y\beta f(z) \quad \forall x,y,z \in N,\ \alpha,\beta \in \Gamma. \tag{3.14}$$

If we take $y\delta w$ instead of y in (3.14), then

$$\begin{aligned} d(f(x))\alpha y\delta w\beta f(z) = f(z)\alpha d(f(x)\beta y\delta w) = d(f(x))\alpha y\delta f(z)\beta w \\ \forall x,y,z \in N,\ \alpha,\beta,\delta \in \Gamma, \end{aligned} \tag{3.15}$$

and so,

$$\begin{aligned} d(f(x))\alpha y\delta w\beta f(z) - d(f(x))\alpha y\delta f(z)\beta w = d(f(x))\alpha y\delta [f(z),w]_{\delta} = 0 \\ \forall x,y,z \in N, \alpha,\beta,\delta \in \Gamma. \end{aligned} \tag{3.16}$$

Thus we get $d(f(x))\Gamma N\Gamma[f(z),w]_{\delta} = 0$, for all $x,y,z \in N$, $\alpha,\beta,\delta \in \Gamma$. Since N is a prime Γ-near-ring, we have $d(f(N)) = 0$ or $f(N) \subset Z(N)$. Let us assume that $d(f(N)) = 0$. Then

$$0 = d(f(x\alpha y)) = d(d(x)\alpha y + x\alpha f(y)) \tag{3.17}$$

and so,

$$d^2(x)\alpha y + d(x)\alpha d(y) + d(x)\alpha f(y) = 0, \quad \forall x,y \in N,\ \alpha \in \Gamma. \tag{3.18}$$

Replacing y by $y\beta z$, $\beta \in \Gamma$, in (3.18), we get

$$\begin{aligned} 0 &= d^2(x)\alpha y\beta z + d(x)\alpha d(y\beta z) + d(x)\alpha f(y\beta z) \\ &= d^2(x)\alpha y\beta z + d(x)\alpha d(y)\beta z + d(x)\alpha y\beta d(z) + d(x)\alpha f(y)\beta z + d(x)\alpha y\beta d(z) \\ &= \left\{d^2(x)\alpha y + d(x)\alpha d(y) + d(x)\alpha f(y)\right\}\beta z + 2d(x)\alpha y\beta d(z) \quad \forall x,y,z \in N,\ \alpha,\beta \in \Gamma. \end{aligned} \tag{3.19}$$

Using (3.18) and N being 2-torsion free Γ-near-ring, we get $d(N)\Gamma N\Gamma d(N) = 0$.

Thus we obtain that $d = 0$. It contradicts by $d \neq 0$. The theorem is proved. □

4. Generalized Derivations of Γ-Near-Rings

We denote a generalized derivation $f : N \to N$ determined by a derivation d of N by (f,d). We assume that d is a nonzero derivation of N unless stated otherwise.

Theorem 4.1. *Let (f,d) be a generalized derivation of N. If $f([x,y]_{\alpha}) = 0$ for all $x,y \in N, \alpha \in \Gamma$, then N is a commutative Γ-ring.*

Proof. Assume that $f([x,y]_\alpha)=0$ for all $x,y\in N$, $\alpha\in\Gamma$. Substitute $x\beta y$ instead of y, obtaining

$$f([x,x\beta y]_\alpha)=f(x\beta[x,y]_\alpha)=d(x)\beta[x,y]_\alpha+x\beta f([x,y]_\alpha)=0. \tag{4.1}$$

Since the second term is zero, it is clear that

$$d(x)\alpha x\beta y=d(x)\alpha y\beta x \quad \forall x,y\in N,\ \alpha,\beta\in\Gamma. \tag{4.2}$$

Replacing y by $y\delta z$ in (4.2) and using this equation, we get

$$d(x)\alpha y\beta[x,z]_\delta=0 \quad \forall x,y,z\in N,\ \alpha,\beta,\delta\in\Gamma. \tag{4.3}$$

Hence either $x\in Z(N)$ or $d(x)=0$. Let $L=\{x\in N\mid d(x)=0\}$. Then $Z(N)$ and L are two additive subgroups of $(N,+)=Z(N)\cup L$. However, a group cannot be the union of proper subgroups, hence either $N=Z(N)$ or $N=L$. Since $d\neq 0$, we are forced to conclude that N is a commutative Γ-ring. □

Theorem 4.2. *Let (f,d) be a generalized derivation of N. If $f([x,y]_\alpha)=\pm[x,y]_\alpha$ for all $x,y\in N$, $\alpha\in\Gamma$, then N is a commutative Γ-ring.*

Proof. Assume that $f([x,y]_\alpha)=\pm[x,y]_\alpha$ for all $x,y\in N$, $\alpha\in\Gamma$. Replacing y by $x\beta y$, $\beta\in\Gamma$, in the hypothesis, we have

$$f([x,x\beta y]_\alpha)=\pm(x\alpha x\beta y-x\alpha y\beta x)=\pm x\beta[x,y]_\alpha. \tag{4.4}$$

On the other hand,

$$f([x,x\beta y]_\alpha)=f(x\beta[x,y]_\alpha)=d(x)\beta[x,y]_\alpha+x\beta f([x,y]_\alpha)=d(x)\beta[x,y]_\alpha+x\beta(\pm[x,y]_\alpha). \tag{4.5}$$

It follows from the two expressions for $f([x,x\beta y]_\alpha)$ that

$$d(x)\alpha x\beta y=d(x)\alpha y\beta x \quad \forall x,y\in N,\ \alpha,\beta\in\Gamma. \tag{4.6}$$

Using the same argument as in the proof of Theorem 4.1, we get that N is a commutative Γ-ring. □

Theorem 4.3. *Let (f,d) be a nonzero generalized derivation of N. If f acts as a homomorphism on N, then f is the identity map.*

Proof. Assume that f acts as a homomorphism on N. Then one obtains

$$f(x\alpha y)=f(x)\alpha f(y)=d(x)\alpha y+x\alpha f(y) \quad \forall x,y\in N,\ \alpha\in\Gamma. \tag{4.7}$$

Replacing y by $y\beta z$ in (4.7), we arrive at

$$f(x)\alpha f(y\beta z) = d(x)\alpha y\beta z + x\alpha f(y\beta z). \tag{4.8}$$

Since (f,d) is a generalized derivation and f acts as a homomorphism on N, we deduce that

$$f(x\alpha y)\beta f(z) = d(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta f(z). \tag{4.9}$$

By Lemma 3.9(ii), we get

$$d(x)\alpha y\beta f(z) + x\alpha f(y)\beta f(z) = d(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta f(z), \tag{4.10}$$

and so

$$d(x)\alpha y\beta f(z) + x\alpha f(y\beta z) = d(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta f(z). \tag{4.11}$$

That is,

$$d(x)\alpha y\beta f(z) + x\alpha d(y)\beta z + x\alpha y\beta f(z) = d(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta f(z). \tag{4.12}$$

Hence, we deduce that

$$d(x)\alpha y\beta(f(z) - z) = 0 \quad \forall x,y,z \in N,\ \alpha,\beta \in \Gamma. \tag{4.13}$$

Because N is prime and $d \neq 0$, we have $f(z) = z$ for all $z \in N$. Thus, f is the identity map. □

Theorem 4.4. *Let (f,d) be a nonzero generalized derivation of N. If f acts as an antihomomorphism on N, then f is the identity map.*

Proof. By the hypothesis, we have

$$f(x\alpha y) = f(y)\alpha f(x) = d(x)\alpha y + x\alpha f(y) \quad \forall x,y \in N,\ \alpha \in \Gamma. \tag{4.14}$$

Replacing y by $x\beta y$ in the last equation, we obtain

$$f(x\beta y)\alpha f(x) = d(x)\beta x\alpha y + x\beta f(x\alpha y). \tag{4.15}$$

Since (f,d) is a generalized derivation and f acts as an antihomomorphism on N, we get

$$(d(x)\beta y + x\beta f(y))\alpha f(x) = d(x)\alpha x\beta y + x\alpha f(y)\beta f(x). \tag{4.16}$$

By Lemma 3.9(ii), we conclude that

$$d(x)\alpha y\beta f(x) + x\alpha f(y)\beta f(x) = d(x)\alpha x\beta y + x\alpha f(y)\beta f(x), \tag{4.17}$$

and so

$$d(x)\alpha y\beta f(x) = d(x)\alpha x\beta y \quad \forall x, y \in N,\ \alpha, \beta \in \Gamma. \tag{4.18}$$

Replacing y by $y\delta z$ and using this equation, we have

$$d(x)\alpha y\beta [f(x), z]_\alpha = 0 \quad \forall x, z \in N,\ \alpha, \beta \in \Gamma. \tag{4.19}$$

Hence we obtain the following alternatives: $d(x) = 0$ or $f(x) \in Z(N)$, for all $x \in N$. By a standard argument, one of these must hold for all $x \in N$. Since $d \neq 0$, the second possibility gives that N is commutative Γ-ring by Theorem 3.12, and so we deduce that f is the identity map by Theorem 4.3. □

Theorem 4.5. *Let (f, d) be a generalized derivation of N such that $d(Z(N)) \neq 0$, and $a \in N$. If $[f(x), a]_\alpha = 0$ for all $x \in N$, $\alpha, \beta \in \Gamma$, then $a \in Z(N)$.*

Proof. Since $d(Z(N)) \neq 0$, there exists $c \in Z(N)$ such that $d(c) \neq 0$. Furthermore, as d is a derivation, it is clear that $d(c) \in Z(N)$. Replacing x by $c\beta x$, $\beta \in \Gamma$, in the hypothesis and using Lemma 3.9(ii), we have

$$\begin{gathered} f(c\beta x)\alpha a = a\alpha f(c\beta x), \\ d(c)\alpha x\beta a + c\alpha f(x)\beta a = a\alpha d(c)\beta x + a\alpha c\beta f(x). \end{gathered} \tag{4.20}$$

Since $c \in Z(N)$ and $d(c) \in Z(N)$, we get

$$d(c)\alpha x\beta [y, a]_\alpha = 0 \quad \forall y \in N,\ \alpha, \beta, \delta \in \Gamma. \tag{4.21}$$

By the primeness of N and $0 \neq d(c) \in Z(N)$, we obtain that $a \in Z(N)$. □

Theorem 4.6. *Let (f, d) be a generalized derivation of N, and $a \in N$. If $[f(x), a]_\alpha = 0$ for all $x \in N$, then $d(a) \in Z(N)$.*

Proof. If $a = 0$, then there is nothing to prove. Hence, we assume that $a \neq 0$.

Replacing x by $a\beta x$ in the hypothesis, we have

$$\begin{gathered} f(a\beta x)\alpha a = a\alpha f(a\beta x), \\ d(a)\alpha x\beta a + a\alpha f(x)\beta a = a\alpha d(a)\beta x + a\alpha a\beta f(x). \end{gathered} \tag{4.22}$$

Using $f(x)\alpha a = a\alpha f(x)$, we have

$$d(a)\alpha x\beta a = a\alpha d(a)\beta x \quad \forall x \in N,\ \alpha, \beta \in \Gamma. \tag{4.23}$$

Taking $x\delta y$ instead of x in the last equation and using this, we conclude that

$$d(a)\alpha N\beta [a, y]_\alpha = 0 \quad \forall y \in N,\ \alpha, \beta \in \Gamma. \tag{4.24}$$

Since N is a prime Γ-near-ring, we have either $d(a) = 0$ or $a \in Z(N)$. If $0 \neq a \in Z(N)$, then $(N, +)$ is abelian by Lemma 3.2(ii). Thus

$$\begin{gathered} f(x\alpha a) = f(a\alpha x) \\ f(x)\alpha a + x\alpha d(a) = d(a)\alpha x + a\alpha f(x) \end{gathered} \tag{4.25}$$

and so

$$[d(a), x]_\alpha = 0 \ \forall x \in N,\ \alpha \in \Gamma. \tag{4.26}$$

That is, $d(a) \in Z(N)$. Hence in either case we have $d(a) \in Z(N)$. This completes the proof. □

Theorem 4.7. *Let (f, d) be a generalized derivation of N. If N is a 2-torsion free Γ-near-ring and $f^2(N) \subset Z(N)$, then N is a commutative Γ-ring.*

Proof. Suppose that $f^2(N) \subset Z(N)$. Then we get

$$f^2(x\alpha y) = f^2(x)\alpha y + 2f(x)\alpha d(y) + x\alpha d^2(y) \in Z(N) \quad \forall x, y \in N,\ \alpha \in \Gamma. \tag{4.27}$$

In particular, $f^2(x)\alpha c + 2f(x)\alpha d(c) + x\alpha d^2(c) \in Z(N)$ for all $x \in N$, $c \in Z(N)$, $\alpha \in \Gamma$. Since the first summand is an element of $Z(N)$, we have

$$2f(x)\alpha d(c) + x\alpha d^2(c) \in Z(N) \quad \forall x \in N,\ c \in Z(N),\ \alpha \in \Gamma. \tag{4.28}$$

Taking $f(x)$ instead of x in (4.28), we obtain that

$$2f^2(x)\alpha d(c) + f(x)\alpha d^2(c) \in Z(N) \quad \forall x \in N,\ c \in Z(N),\ \alpha \in \Gamma. \tag{4.29}$$

Since $d(c) \in Z(N)$, $f^2(x) \in Z(N)$, and so $f^2(x)\alpha d(c) \in Z(N)$ for all $x \in N$, $c \in Z(N)$, $\alpha \in \Gamma$, we conclude $f(x)\alpha d^2(c) \in Z(N)$ for all $x \in N$, $c \in Z(N)$, $\alpha \in \Gamma$.

Since N is prime, we get $d^2(Z(N)) = 0$ or $f(N) \subseteq Z(N)$. If $f(N) \subseteq Z(N)$, then N is a commutative Γ-ring by Lemma 3.8. Hence, we assume $d^2(Z) = 0$. By (4.28), we get $2f(x)\alpha d(c) \in Z(N)$ for all $x \in N$, $c \in Z(N)$, $\alpha \in \Gamma$.

Since N is a 2-torsion free near-ring and $d(c) \in Z(N)$, we obtain that either $f(N) \subset Z(N)$ or $d(Z(N)) = 0$. If $f(N) \subset Z(N)$, then we are already done. So, we may assume that $d(Z(N)) = 0$. Then

$$\begin{gathered} f(c\alpha x) = f(x\alpha c), \\ f(c)\alpha x + c\alpha d(x) = f(x)\alpha c + x\alpha d(c), \end{gathered} \tag{4.30}$$

and so

$$f(c)\alpha x + c\alpha d(x) = f(x)\alpha c \quad \forall x \in N,\ c \in Z(N). \tag{4.31}$$

Now replacing x by $f(x)$ in (4.31), and using the fact that $f^2(N) \subset Z(N)$, we get

$$f(c)\alpha f(x) + c\alpha d(f(x)) = f^2(x)\alpha c \quad \forall x \in N,\ c \in Z(N). \tag{4.32}$$

That is,

$$f(c)\alpha f(x) + c\alpha d(f(x)) \in Z \quad \forall x \in N,\ c \in Z(N),\ \alpha \in \Gamma. \tag{4.33}$$

Again taking $f(x)$ instead of x in this equation, one can obtain

$$f(c)\alpha f^2(x) + c\alpha d\left(f^2(x)\right) \in Z \quad \forall x \in N,\ c \in Z(N),\ \alpha \in \Gamma. \tag{4.34}$$

The second term is equal to zero because of $d(Z) = 0$. Hence we have $f(c)\alpha f^2(x) \in Z(N)$ for all $x \in N$, $c \in Z(N)$, $\alpha \in \Gamma$.

Since $f^2(N) \subset Z(N)$ by the hypothesis, we get either $f^2(N) = 0$ or $f(Z(N)) \subset Z(N)$. If $f^2(N) = 0$, then the theorem holds by Definition 3.7. If $f(Z) \subset Z(N)$, then $f(x\alpha f(c)) = f(f(c)\alpha x)$ for all $x \in N$, $c \in Z(N)$, and so

$$d(x)\alpha f(c) = f(c)\alpha f(x) \quad \forall x \in N,\ c \in Z(N). \tag{4.35}$$

Using $f(c) \in Z(N)$, we now have $f(c)\alpha(d(x) - f(x)) = 0$ for all $x \in N$, $c \in Z(N)$, $\alpha \in \Gamma$. Since $f(Z(N)) \subset Z(N)$, we have either $f(Z(N)) = 0$ or $d = f$. If $d = f$, then f is a derivation of N and so N is commutative Γ-ring by Lemma 3.11.

Now assume that $f(Z(N)) = 0$. Returning to the equation (4.31), we have

$$c\alpha(d(x) - f(x)) = 0 \quad \forall x \in N,\ c \in Z(N),\ \alpha \in \Gamma. \tag{4.36}$$

Since $c \in Z(N)$, we have either $d = f$ or $Z(N) = 0$. Clearly, $d = f$ implies the theorem holds. If $Z(N) = 0$, then $f^2(N) = 0$ by the hypothesis, and so N is a commutative Γ-ring by Lemma 3.4(iv). Hence, the proof is completed. □

Corollary 4.8. *Let N be a 2-torsion free near-ring, and (f, d) a nonzero generalized derivation of N. If $[f(N), f(N)]_\alpha = 0$, $\alpha \in \Gamma$, then N is a commutative Γ-ring.*

Lemma 4.9. *Let (f, d) and (g, h) be two generalized derivations of N. If h is a nonzero derivation on N and $f(x)\alpha h(y) = -g(x)\alpha d(y)$ for all $x, y \in N$, then $(N, +)$ is abelian.*

Proof. Suppose that $f(x)\alpha h(y) + g(x)\alpha d(y) = 0$ for all $x, y \in N$, $\alpha \in \Gamma$.

We substitute $y + z$ for y, thereby obtaining

$$f(x)\alpha h(y) + f(x)\alpha h(z) + g(x)\alpha d(y) + g(x)\alpha d(z) = 0. \tag{4.37}$$

Using the hypothesis, we get

$$f(x)\alpha h(y, z) = 0 \quad \forall x, y, z \in N,\ \alpha \in \Gamma. \tag{4.38}$$

It follows by Lemma 3.10(ii) that $h(y,z) = 0$ for all $y,z \in N$. For any $w \in N$, we have $h(w\alpha y, w\alpha z) = h(w\alpha(y,z)) = h(w)\alpha(y,z) + w\alpha h(y,z) = 0$ and so $h(w)\alpha(y,z) = 0$ for all $w,y,z \in N$, $\alpha \in \Gamma$.

An appeal to Lemma 3.4(iii) yields that $(N,+)$ is abelian. □

Theorem 4.10. *Let (f,d) and (g,h) be two generalized derivations of N. If N is 2-torsion free and $f(x)\alpha h(y) = -g(x)\alpha d(y)$ for all $x,y \in N, \alpha \in \Gamma$, then $f = 0$ or $g = 0$.*

Proof. If $h = 0$ or $d = 0$, then the proof of the theorem is obvious. So, we may assume that $h \neq 0$ and $d \neq 0$. Therefore, we know that $(N,+)$ is abelian by Lemma 4.9.

Now suppose that

$$f(x)\alpha h(y) + g(x)\alpha d(y) = 0 \quad \forall x,y \in N,\ \alpha \in \Gamma. \tag{4.39}$$

Replacing x by $u\beta v$ in this equation and using the hypothesis, we get

$$\begin{aligned} & f(u\beta v)\alpha h(y) + g(u\beta v)\alpha d(y) \\ & \quad = u\alpha f(v)\beta h(y) + d(u)\alpha v\beta h(y) + u\alpha g(v)\beta d(y) + h(u)\alpha v\beta d(y) \\ & \quad = 0, \end{aligned} \tag{4.40}$$

and so

$$d(u)\alpha v\beta h(y) = -h(u)\alpha v\beta d(y) \quad \forall u,v,y \in N,\ \alpha \in \Gamma. \tag{4.41}$$

Taking $y\delta t$ instead of y in the above relation, we obtain

$$d(u)\alpha v\beta h(y)\delta t + d(u)\alpha v y\beta\delta h(t) = -h(u)\alpha v\beta d(y)\delta t - h(u)\alpha v\beta y\delta d(t). \tag{4.42}$$

That is,

$$d(u)\alpha v\beta y\delta h(t) = -h(u)\alpha v\beta y\delta d(t) \quad \forall u,v,y,t \in N,\ \alpha,\beta,\delta \in \Gamma. \tag{4.43}$$

Replacing y by $h(y)$ in (4.43) and using this relation, we have

$$h(u)\alpha N\beta\big(d(y)\delta h(t) - h(y)\alpha d(t)\big) = 0 \quad \forall u,y,t \in N. \tag{4.44}$$

Since N is a prime Γ-near-ring and $h \neq 0$, we obtain that

$$d(y)\alpha h(t) = h(y)\alpha d(t), \quad \forall y,t \in N. \tag{4.45}$$

Now again taking $u\lambda v$ instead of x in the initial hypothesis, we get

$$f(u)\alpha v\beta h(y) + u\alpha d(v)\beta h(y) + g(u)\alpha v\beta d(y) + u\alpha h(v)\beta d(y) = 0. \tag{4.46}$$

Using (4.45) yields that

$$f(u)\alpha v\beta h(y) + 2u\alpha h(v)\beta d(y) + g(u)\alpha v\beta d(y) = 0 \quad \forall u, v, y \in N, \tag{4.47}$$

Taking $h(v)$ instead of v in this equation, we arrive at

$$f(u)\alpha h(v)\beta h(y) + 2u\alpha h^2(v)\beta d(y) + g(u)\alpha h(v)\beta d(y) = 0. \tag{4.48}$$

By the hypothesis and (4.45), we have

$$\begin{aligned} 0 &= -g(u)\alpha d(v)\beta h(y) + 2u\alpha h^2(v)\beta d(y) + g(u)\alpha h(v)\beta d(y) \\ &= -g(u)\alpha h(v)\beta d(y) + 2u\alpha h^2(v)\beta d(y) + g(u)\alpha h(v)\beta d(y), \end{aligned} \tag{4.49}$$

and so

$$2u\alpha h^2(v)\beta d(y) = 0 \quad \forall u, v, y \in N, \ \alpha, \beta \in \Gamma. \tag{4.50}$$

Since N is a 2-torsion free prime Γ-near-ring, we obtain that $h^2(N)\Gamma d(N) = 0$. An appeal to Lemma 3.4(iii) and (iv) gives that $h = 0$. This contradicts by our assumption. Thus the proof is completed. □

Theorem 4.11. *Let (f, d) and (g, h) be two generalized derivations of N. If (fg, dh) acts as a generalized derivation on N, then $f = 0$ or $g = 0$.*

Proof. By calculating $fg(x\alpha y)$ in two different ways, we see that $g(x)\alpha d(y) + f(x)\alpha h(y) = 0$ for all $x, y \in N$, $\alpha \in \Gamma$. The proof is completed by using Theorem 4.10. □

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