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Research Article

Norm for Sums of Two Basic Elementary Operators

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We give necessary and sufficient conditions under which the norm of basic elementary operators attains its optimal value in terms of the numerical range.

1. Introduction

Let E be a normed space over $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$, S_E its unit sphere, and E^* its dual topological space. Let D be the normalized duality mapping form E to E^* given by

$$D(x) = \left\{\varphi \in E^* : \varphi(x) = \|x\|^2, \|\varphi\| = \|x\|\right\}, \quad \forall x \in E. \tag{1.1}$$

Let $B(E)$ be the normed space of all bounded linear operators acting on E. For any operator $A \in B(E)$ and $x \in E$,

$$\begin{gathered} W_x(A) = \{\varphi(Ax) : \varphi \in D(x)\}, \\ W(A) = \cup\{W_x(A) : x \in S_E\} \end{gathered} \tag{1.2}$$

is called the spatial numerical range of A, which may be defined as

$$W(A) = \{\varphi(Ax) : x \in S_E;\ \varphi \in D(x)\}. \tag{1.3}$$

This definition was extended to arbitrary elements of a normed algebra $\mathcal{A}$ by Bonsall [1–3] who defined the numerical range of $a \in \mathcal{A}$ as

$$V(a) = W(A_a), \tag{1.4}$$

where A_a is the left regular representation of A in $B(\mathcal{A})$, that is, $A_a = ab$ for all $b \in \mathcal{A}$. $V(a)$ is known as the algebra numerical range of $a \in \mathcal{A}$, and, according to the above definitions, $V(a)$ is defined by

$$V(a) = \{\varphi(ab) : b \in S_{\mathcal{A}};\ \varphi \in D(b)\}. \tag{1.5}$$

For an operator $A \in B(E)$, Bachir and Segres [4] have extended the usual definitions of numerical range from one operator to two operators in different ways as follows.

The spatial numerical range $W(A)_B$ of $A \in B(E)$ relative to B is

$$W(A)_B = \{\varphi(Ax) : x \in S_E;\ \varphi \in D(Bx)\}. \tag{1.6}$$

The spatial numerical range $G(A)_B$ of $A \in B(E)$ relative to B is

$$G(A)_B = \{\varphi(Ax) : x \in E;\ \|Bx\| = 1,\ \varphi \in D(Bx)\}. \tag{1.7}$$

The maximal spatial numerical range of $A \in B(E)$ relative to B is

$$M(A)_B = \{\varphi(Ax) : x \in S_E;\ \|Bx\| = \|B\|,\ \varphi \in D(Bx)\}. \tag{1.8}$$

For $A, B \in B(E)$, let $S_E(B) = \{(x_n)_n : x_n \in S_E, \|Bx_n\| \to \|B\|\}$, then the set

$$\mathcal{M}(A)_B = \{\lim \varphi_n(Ax_n) : (x_n)_n \in S_E(B),\ \varphi_n \in D(Bx_n)\} \tag{1.9}$$

is called the generalized maximal numerical range of A relative to B. It is known that $\mathcal{M}(A)_B$ is a nonempty closed subset of $\mathbb{K}$ and $M(A)_B \subseteq \mathcal{M}(A)_B \subseteq \overline{W(A)_B}$. The definition of $\mathcal{M}(A)_B$ can be rewritten, with respect to the semi-inner product $[\cdot,\cdot]$ as

$$\mathcal{M}(A)_B = \{\lim[Ax_n, Bx_n] : (x_n)_n \in S_E(B)\}, \tag{1.10}$$

with respect to an inner product $(\cdot,\cdot)$ as

$$\mathcal{M}(A)_B = \{\lim(Ax_n, Bx_n) : (x_n)_n \in S_E(B)\}. \tag{1.11}$$

We shall be concerned to estimate the norm of the elementary operator $M_{A_1,B_1} + M_{A_2,B_2}$, where A_1, A_2, B_1, B_2 are bounded linear operators on a normed space E and M_{A_1,B_1} is the basic elementary operator defined on $B(E)$ by

$$M_{A_1,B_1}(X) = A_1XB_1. \tag{1.12}$$

We also give necessary and sufficient conditions on the operators A_1, A_2, B_1, B_2 under which $M_{A_1,B_1} + M_{A_2,B_2}$ attaints its optimal value $\|A_1\|\|B_1\| + \|A_2\|\|B_2\|$.

2. Equality of Norms

Our next aim is to give necessary and sufficient conditions on the set $\{A_1, A_2, B_1, B_2\}$ of operators for which the norm of $M_{A_1,B_1} + M_{A_2,B_2}$ equals $\|A_1\|\|B_1\| + \|A_2\|\|B_2\|$.

Lemma 2.1. *For any of the operators $A, B, C \in B(E)$ and all $\alpha, \beta \in \mathbb{K}$, one has*

$$\begin{aligned} \mathcal{M}(\alpha A + \beta B)_B &= \alpha \mathcal{M}(A)_B + \beta \|B\|^2; \\ \mathcal{M}(\alpha A + \beta C)_B &\subseteq \alpha \mathcal{M}(A)_B + \beta \mathcal{M}(C)_B. \end{aligned} \tag{2.1}$$

Proof. The proof is elementary. □

Theorem 2.2. *Let A_1, A_2, B_1, B_2 be operators in $B(E)$.*
If $\|A_1\|\|A_2\| \in \mathcal{M}(A_1)_{A_2} \cup \mathcal{M}(A_2)_{A_1}$ and $\|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2} \cup \mathcal{M}(B_2)_{B_1}$, then

$$\|M_{A_1,B_1} + M_{A_2,B_2}\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|. \tag{2.2}$$

Proof. The proof will be done in four steps; we choose one and the others will be proved similarly. Suppose that $\|A_1\|\|A_2\| \in \mathcal{M}(A_1)_{A_2}$ and $\|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2}$, then there exist $(x_n)_n \in S_E(A_2)$, $\varphi_n \in D(A_2 x_n)$ such that $\|A_1\|\|A_2\| = \lim_n \varphi_n(A_1 x_n)$ and there exist $(y_n)_n \in S_E(B_2)$, $\varphi_n \in D(B_2 y_n)$ such that $\|B_1\|\|B_2\| = \lim_n \varphi_n(B_1 y_n)$. Define the operators $X_n \in B(E)$ as follows:

$$X_n(y_n) = (\varphi_n \otimes x_n)(y_n) = \varphi_n(y_n) x_n, \quad \forall n. \tag{2.3}$$

Then $\|X_n\| \leq \|B_2\|$, for all $n \geq 1$, and

$$\begin{aligned} \left\|(M_{A_1+B_1} + M_{A_2+B_2}) X_n(y_n)\right\| &= \left\|(A_1 X_n B_1 + A_2 X_n B_2) y_n\right\| \\ &= \left\|A_1 X_n(B_1 y_n) + A_2 X_n(B_2 y_n)\right\| \\ &= \frac{\|\varphi_n\|}{\|\varphi_n\|} \left\|A_1 \varphi_n(B_1 y_n) x_n + A_2 \varphi_n(B_2 y_n) x_n\right\| \\ &\geq \frac{1}{\|\varphi_n\|} \left\|\varphi_n(\varphi_n(B_1 y_n) A_1 x_n + \varphi_n(B_2 y_n) A_2 x_n)\right\| \\ &= \frac{1}{\|\varphi_n\|} \left\|\varphi_n(B_1 y_n)\varphi_n(A_1 x_n) + \left\|B_2 y_n\right\|^2 \|A_2 x_n\|^2\right\|. \end{aligned} \tag{2.4}$$

$$\|M_{A_1,B_1} + M_{A_2,B_2}\| \geq \frac{\left\|(M_{A_1,B_1} + M_{A_2,B_2}) X_n(y_n)\right\|}{\|X_n\|}, \quad \forall n \geq 1. \tag{2.5}$$

Hence

$$\|M_{A_1,B_1}+M_{A_2,B_2}\| \geq \frac{\left\|\varphi_n(B_1y_n)\varphi_n(A_1x_n)+\|B_2y_n\|^2\|A_2x_n\|^2\right\|}{\|A_2\|\|B_2\|}, \quad \forall n \geq 1. \tag{2.6}$$

Letting $n \longrightarrow \infty$,

$$\|M_{A_1,B_1}+M_{A_2,B_2}\| \geq \|A_1\|\|B_1\|+\|A_2\|\|B_2\|. \tag{2.7}$$

Since

$$\|M_{A_1,B_1}+M_{A_2,B_2}\| \leq \|A_1\|\|B_1\|+\|A_2\|\|B_2\|, \tag{2.8}$$

therefore

$$\|M_{A_1,B_1}+M_{A_2,B_2}\| = \|A_1\|\|B_1\|+\|A_2\|\|B_2\|. \tag{2.9}$$

□

Corollary 2.3. *Let E be a normed space and $A, B \in B(E)$. Then, the following assertions hold:*

(1) *if $\|A\|\|B\| \in \mathcal{M}(A)_B$, then $\|A+B\| = \|A\|+\|B\|$;*

(2) *if $\|A\| \in \mathcal{M}(I)_A$ and $\|B\| \in \mathcal{M}(I)_B$, then $\|M_{A,B}+I\| = 1+\|A\|\|B\|$.*

Remark 2.4. In the previous corollary, if we set $B = I$, then we obtain an important equation called the Daugavet equation:

$$\|A+I\| = 1+\|A\|. \tag{2.10}$$

It is well known that every compact operator on $C[0,1]$ [5] or on $L_1[0,1]$ [6] satisfies (2.10).

A Banach space E is said to have the Daugavet property if every rank-one operator on E satisfies (2.10). So that from our Corollary 2.3 if $1 \in \mathcal{M}(I)_A$ or $1 \in \mathcal{M}(A)_I$ for every rank-one operator A, then E has the Daugavet property.

The reverse implication in the previous theorem is not true, in general, as shown in the following example which is a modification of that given by the authors Bachir and Segres [4, Example 3.17].

Example 2.5. Let c_0 be the classical space of sequences $(x_n)_n \subset \mathbb{C} : x_n \longrightarrow 0$, equipped with the norm $\|(x_n)_n\| = \max_n |x_n|$ and let L be an infinite-dimensional Banach space. Taking the Banach space $E = L \oplus c_0$ equipped with the norm, for $x = (x_1 + x_2) \in E$, $\|x\| = \|x_1 + x_2\| = \max\{\|x_1\|, \|Tx_1\| + \|x_2\|\}$, where T is any norm-one operator from L to c_0 which does not attain its norm (by Josefson-Nissenzweig's theorem [7]), we can find a sequence $(\varphi_n)_n \subset S_{E^*}$ such that φ_n converges weakly to 0. Therefore we get the desired operator $T : L \longrightarrow c_0$ defined by

$$(Tx)_n = \frac{n}{n+1}\varphi_n(x). \tag{2.11}$$

Let A_1, A_2, B_1, B_2 be operators defined on E as follows:

$$\begin{aligned} A_1(x_1 + x_2) &= 0 + Tx_1; \\ A_2x = A_2(x_1 + x_2) &= x_1 + 0; \\ B_1(x_1 + x_2) &= x_1 - x_2; \\ B_2 = I, \quad \forall x &= (x_1 + x_2) \in L \times c_0, \end{aligned} \tag{2.12}$$

where I is the identity operator on E. It easy to check that A_1, A_2, B_1 are linear bounded operators and $\|A_1\| = \|A_2\| = \|B_1\| = \|B_2\| = 1$. If we choose $X_0 = I$ and $x_0 = x_1 + 0$ such that $1 = \|Tx_1\| \geq \|x_1\|$, then $\|X_0\| = \|x_0\| = 1$ and

$$\begin{aligned} \|M_{A_1,B_1} + M_{A_2,B_2}\| &\geq \|(M_{A_1,B_1} + M_{A_2,B_2})X_0(x_0)\| \\ &= \|(A_1X_0B_1 + A_2X_0B_2)(x_0)\| \\ &= \|0 + Tx_1 + x_1 + 0\| \\ &= \max\{\|x_1\|, 2\|Tx_1\|\} \\ &= 2, \end{aligned} \tag{2.13}$$

and from

$$\|M_{A_1,B_1} + M_{A_2,B_2}\| \leq \|A_1\|\|A_2\| + \|B_1\|\|B_2\| = 2 \tag{2.14}$$

we get

$$\|M_{A_1,B_1} + M_{A_2,B_2}\| = 2 = \|A_1\|\|A_2\| + \|B_1\|\|B_2\|. \tag{2.15}$$

It is clear from the definitions of $\mathcal{M}(A_1)_{A_2}$ and $W(A_1)_{A_2}$ that

$$\mathcal{M}(A_1)_{A_2} \subseteq \overline{W(A_1)_{A_2}} \tag{2.16}$$

(for details, see [4]).

The next result shows that the reverse is true under certain conditions, before that we recall the definition of Birkhoff-James orthogonality in normed spaces.

Definition 2.6. Let E be a normed space and $x, y \in E$. We say that x is orthogonal to y in the sense of Birkhoff-James ([8, 9]), in short $x \perp_{B-J} y$, iff

$$\forall \lambda \in \mathbb{K} : \left\|x + \lambda y\right\| \geq \|x\|. \tag{2.17}$$

If F, G are linear subspaces of E, we say that F is orthogonal to G in the sense of $\perp_{B-J}$, written as $F \perp_{B-J} G$ iff $x \perp_{B-J} y$ for all $x \in F$ and all $y \in G$.

If $T \in B(E)$, we will denote by $\mathrm{Ran}(T)$ and $T^\dagger$ the range and the dual adjoint, respectively, of the operator T.

Theorem 2.7. *Let A_1, A_2, B_1, B_2 be operators in $B(E)$.*
If $\|M_{A_1,B_1} + M_{A_2,B_2}\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|$,

$$\mathrm{Ran}\left(A_2^\dagger\right)\perp_{B-J} \mathrm{Ran}\left(A_1^\dagger - \frac{\|A_1\|}{\|A_2\|}A_2^\dagger\right), \qquad \mathrm{Ran}\,(B_2)\perp_{B-J} \mathrm{Ran}\left(B_1 - \frac{\|B_1\|}{\|B_2\|}B_2\right), \tag{2.18}$$

then

$$\|A_1\|\|A_2\| \in \mathcal{M}\left(A_1^\dagger\right)_{A_2^\dagger}, \quad \|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2}. \tag{2.19}$$

Moreover, if

$$\mathrm{Ran}\left(A_1^\dagger\right)\perp_{B-J} \mathrm{Ran}\left(A_2^\dagger - \frac{\|A_2\|}{\|A_1\|}A_1^\dagger\right), \qquad \mathrm{Ran}\,(B_1)\perp_{B-J} \mathrm{Ran}\left(B_2 - \frac{\|B_2\|}{\|B_1\|}B_1\right), \tag{2.20}$$

then

$$\|A_1\|\|A_2\| \in \mathcal{M}\left(A_1^\dagger\right)_{A_2^\dagger} \cap \mathcal{M}\left(A_2^\dagger\right)_{A_1^\dagger}, \qquad \|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2} \cap \mathcal{M}(B_2)_{B_1}. \tag{2.21}$$

Proof. If $\|M_{A_1,B_1}+M_{A_2,B_2}\| = \|A_1\|\|B_1\|+\|A_2\|\|B_2\|$, then we can find two normalized sequences $(X_n)_n \subseteq B(E)$ and $(x_n)_n \subseteq E$ such that

$$\lim_n \|A_1X_nB_1x_n + A_2X_nB_2x_n\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|. \tag{2.22}$$

We have for all $n \geq 1$

$$\begin{aligned} \|A_1X_nB_1x_n\| &\leq \|A_1\|\|B_1x_n\| \leq \|A_1\|\|B_1\| \\ \|A_2X_nB_2x_n\| &\leq \|A_2\|\|B_2x_n\| \leq \|A_2\|\|B_2\|, \end{aligned} \tag{2.23}$$

so we can deduce from the above inequalities and (2.10) that $\lim_n\|B_1x_n\| = \|B_1\|$ and $\lim_n\|B_2x_n\| = \|B_2\|$. From the assumptions $\mathrm{Ran}(B_2)\perp_{B-J} \mathrm{Ran}(B_1 - (\|B_1\|/\|B_2\|)B_2)$ we get

$$\overline{\mathrm{Ran}\left(B_1 - \frac{\|B_1\|}{\|B_2\|}B_2\right)} \cap \overline{\mathrm{Ran}(B_2)} = \{0\}. \tag{2.24}$$

Set $\chi_n = (B_1 - (\|B_1\|/\|B_2\|)B_2)x_n$ and $y_n = B_2x_n$ for all n and define the function ϕ_n on the closed subspace F spanned by $\{x_n, y_n\}$ for all n as

$$\phi_n(a\chi_n + by_n) = b\left\|y_n\right\|^2 = b\|B_2x_n\|, \quad \forall a, b \in \mathbb{K}. \tag{2.25}$$

It is clear that ϕ_n is linear for all n and

$$\left|\phi_n(a\chi_n + by_n)\right| = |b|\|B_2x_n\|^2 = \left\|a\chi_n + by_n\right\|\|B_2x_n\|\frac{\|by_n\|}{\left\|a\chi_n + by_n\right\|}. \tag{2.26}$$

From the assumptions $\mathrm{Ran}(B_2)\perp_{B-J}\ \mathrm{Ran}(B_1 - (\|B_1\|/\|B_2\|)B_2)$ it follows that

$$\left|\phi(a\chi_n + by_n)\right| \leq \|B_2x_n\|\left\|a\chi_n + by_n\right\|, \quad \forall a,b \in \mathbb{K}, \forall n. \tag{2.27}$$

This means that ϕ_n is continuous for each n on the subspace F with $\|\phi_n\| = \|B_2x_n\|$ (by (2.27) and $\phi_n(y_n) = \|y_n\|\|B_2x_n\|$). Then by Hahn-Banach theorem there is $\widetilde{\phi_n} \in E^*$ with $\widetilde{\phi_n}|_F = \phi_n$ and $\|\phi_n\| = \|\widetilde{\phi_n}\|$, for each n. So

$$\widetilde{\phi_n}(\chi_n) = \widetilde{\phi_n}\left(\left(B_1 - \frac{\|B_1\|}{\|B_2\|}B_2\right)x_n\right) = 0, \tag{2.28}$$

hence

$$\begin{gathered}\lim_n \widetilde{\phi_n}(\chi_n) = \widetilde{\phi_n}\left(\left(B_1 - \frac{\|B_1\|}{\|B_2\|}B_2\right)x_n\right) = 0,\\ \widetilde{\phi_n}(B_2x_n) = \|B_2x_n\|^2, \qquad \left\|\widetilde{\phi_n}\right\| = \|B_2x_n\|.\end{gathered} \tag{2.29}$$

Thus, $0 \in \mathcal{M}(B_1 - (\|B_1\|/\|B_2\|)B_2)_{B_2}$ and by Lemma 2.1

$$0 \in \left(\mathcal{M}(B_1)_{B_2} - \frac{\|B_1\|}{\|B_2\|}\|B_2\|^2\right) = \mathcal{M}(B_1)_{B_2} - \|B_1\|\|B_2\|. \tag{2.30}$$

Therefore,

$$\|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2}. \tag{2.31}$$

□

From $\|M_{A_1,B_1} + M_{A_2,B_2}\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|$ we can find a normalized sequences $(X_n)_n \subseteq B(E)$ such that

$$\lim_n \|A_1X_nB_1 + A_2X_nB_2\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|. \tag{2.32}$$

Since $\|A_1X_nB_1 + A_2X_nB_2\| = \|B_1^\dagger X_n^\dagger A_1^\dagger + B_2^\dagger X_n^\dagger A_2^\dagger\|$, for each n, then we can find a normalized $\phi_{n_k} \in E^\dagger$ such that

$$\lim_{k,n}\left\|B_1^\dagger X_n^\dagger A_1^\dagger \phi_{n_k} + B_2^\dagger X_n^\dagger A_2^\dagger \phi_{n_k}\right\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|. \tag{2.33}$$

We argue similarly and get

$$\lim_{k,n}\left\|A_1^{\dagger}\phi_{n_k}\right\| = \left\|A_1^{\dagger}\right\|, \qquad \lim_{k,n}\left\|A_2^{\dagger}\phi_{n_k}\right\| = \left\|A_2^{\dagger}\right\|. \tag{2.34}$$

Following the same steps as in the previous case we obtain $\|A_1\|\|A_2\| \in \mathcal{M}(A_1^{\dagger})_{A_2^{\dagger}}$.

Moreover, if we have $\operatorname{Ran}(A_1^{\dagger})\perp_{B-J}\operatorname{Ran}(A_2^{\dagger} - (\|A_2\|/\|A_1\|)A_1^{\dagger})$ and $\operatorname{Ran}(B_1)\perp_{B-J}\operatorname{Ran}(B_2 - (\|B_2\|/\|B_1\|)B_1)$, it suffices to reverse, in the proof of the previous case, the role of $A_1^{\dagger}$ into $A_2^{\dagger}$ and B_1 into B_2.

For the completeness of the previous theorem we need to prove the following result which is very interesting.

We recall that Phelps [10] has proved that, for a Banach space E, $\cup\{D(x) : x \in E\}$ is dense in E^*; this property is called subreflexivity of the space E. Using this fact, Bonsall and Duncan [2] has proved that for any operator $T \in B(E)$ we have $\overline{W(T)} = \overline{W(T^{\dagger})}$. The following result generalizes the Bollobas result in the case $\mathcal{M}(A)_B$, where $A, B \in B(E)$.

Proposition 2.8. *Let E be a Banach space with smooth dual and let $A, B \in B(E)$ such that B is a surjective operator. Then $\mathcal{M}(A^{\dagger})_{B^{\dagger}} \subseteq \mathcal{M}(A)_B$.*

Proof. Let $a \in \mathcal{M}(A^{\dagger})_{B^{\dagger}}$, then there are $\varphi_n \in D(B^{\dagger}\varphi_n)$, $(\varphi_n)_n \in S_{E^*}(B^{\dagger})$ such that $a = \lim_n \varphi_n(A^{\dagger}\varphi_n)$.

By the subreflexivity of E there exist sequences $(\varphi_{n_k})_{n_k} \subseteq E^*$ and $(x_{n_k}) \subseteq E$ such that $\varphi_{n_k} \in D(x_{n_k})$ and $\|\varphi_{n_k} - \|Bx_{n_k}\|\varphi_n\|$ to 0. It follows that the sequence $(\widehat{x}_{n_k}) \subseteq E^{**}$ has an E^{**}-weak convergent subsequence $(\widehat{x}_{n_m})_{n_m}$, that is,

$$\widehat{x}_{n_m}(f) \longrightarrow \Psi(f), \quad \forall f \in E^*,\ \Psi \in E^{**}. \tag{2.35}$$

On the one hand, we have

$$\|Bx_{n_m}\|^2 = \left[B^{\dagger}(\varphi_{n_m} - \|Bx_{n_m}\|\varphi_n)\right](x_{n_m}) + \|Bx_{n_m}\|\left(B^{\dagger}\varphi_n\right)(x_{n_m}). \tag{2.36}$$

Then

$$\|Bx_{n_m}\|^2 \le \left\|B^{\dagger}(\varphi_{n_m} - \|Bx_{n_m}\|\varphi_n)\right\| + \|Bx_{n_m}\|\left\|B^{\dagger}\varphi_n\right\|. \tag{2.37}$$

Thus

$$\|Bx_{n_m}\|\left|\|Bx_{n_m}\| - \left\|B^{\dagger}\varphi_n\right\|\right| \le \left\|B^{\dagger}\right\|\|\varphi_{n_m} - \|Bx_{n_m}\|\varphi_n\|. \tag{2.38}$$

On the other hand,

$$\begin{aligned}\left|\widehat{x}_{n_m}\left(B^{\dagger}\varphi_n\right)-\left\|B^{\dagger}\varphi_n\right\|\right| &\le \left|\widehat{x}_{n_m}\left(B^{\dagger}\varphi_n\right)-\widehat{x}_{n_m}\left(\frac{B^{\dagger}\varphi_{n_m}}{\|Bx_{n_m}\|}\right)\right| \\ &\quad+\left|\frac{1}{\|Bx_{n_m}\|}\widehat{x}_{n_m}\left(B^{\dagger}\varphi_{n_m}\right)-\left\|B^{\dagger}\varphi_n\right\|\right| \\ &=\left|\widehat{x}\left(B^{\dagger}\varphi_n-\frac{1}{\|Bx_{n_m}\|}B^{\dagger}\varphi_{n_m}\right)\right|+\left|\left\|Bx_{n_m}-\left\|B^{\dagger}\right\|\varphi_n\right\|\right| \\ &\longrightarrow 0 \quad \text{as } m\longrightarrow\infty.\end{aligned} \tag{2.39}$$

So $\lim_m \widehat{x}_{n_m}(B^{\dagger}\varphi_n)=\|B^{\dagger}\varphi_n\|$ and $\|B^{\dagger}\varphi_n\|\Psi_n\in D(B^{\dagger}\varphi_n)$. Then by smoothness of the space E^* we get $\|B^{\dagger}\varphi_n\|\Psi_n=\Psi_n$, for all n. Next,

$$\begin{aligned}\left|\widehat{x}_{n_m}\left(A^{\dagger}\varphi_{n_m}\right)-\frac{\|Bx_{n_m}\|}{\|B^{\dagger}\varphi_n\|}\varphi_n\left(A^{\dagger}\varphi_n\right)\right| &\le \left|\widehat{x}_{n_m}\left(A^{\dagger}\varphi_{n_m}\right)-\widehat{x}_{n_m}\left(\|Bx_{n_m}\|A^{\dagger}\varphi_n\right)\right| \\ &\quad+\|Bx_{n_m}\|\left|\widehat{x}_{n_m}\left(A^{\dagger}\varphi_n\right)-\frac{1}{\|B^{\dagger}\varphi_n\|}\varphi_n\left(A^{\dagger}\varphi_n\right)\right| \\ &=\left|\widehat{x}_{n_m}\left(A^{\dagger}\varphi_{n_m}-A^{\dagger}\varphi_n\right)\right| \\ &\quad+\|Bx_{n_m}\|\left|\widehat{x}_{n_m}\left(A^{\dagger}\varphi_n\right)-\varphi_n\left(A^{\dagger}\varphi_n\right)\right| \\ &\longrightarrow 0 \quad \text{as } m\longrightarrow\infty.\end{aligned} \tag{2.40}$$

Then $\lim_m \widehat{x}_{n_m}(A^{\dagger}\varphi_{n_m})=\varphi_n(A^{\dagger}\varphi_n)$ or $\lim_m \varphi_{n_m}(Ax_{n_m}=\varphi(A^{\dagger}\varphi_n)$ and therefore

$$\lim_n\left[\lim_m \varphi_{n_m}(Ax_{n_m})\right]=\lim_n \varphi_n\left(A^{\dagger}\varphi_n\right)=a \tag{2.41}$$

which means that $a\in\mathcal{M}(A)_B$. □

Corollary 2.9. *Let E be a Banach space with smooth dual and $A_1, A_2, B_1, B_2 \in B(E)$.*

If $\|M_{A_1,B_1}+M_{A_2,B_2}\| = \|A_1\|\|B_1\|+\|A_2\|\|B_2\|$ and $\operatorname{Ran}(A_2^{\dagger})\perp_{B-J}\operatorname{Ran}(A_1^{\dagger}-(\|A_1\|/\|A_2\|)A_2^{\dagger})$ with A_2 being surjective, and $\operatorname{Ran}(B_2)\perp_{B-J}\operatorname{Ran}(B_1-(\|B_1\|/\|B_2\|)B_2)$, then

$$\|A_1\|\|A_2\|\in\mathcal{M}(A_1)_{A_2}, \qquad \|B_1\|\|B_2\|\in\mathcal{M}(B_1)_{B_2}. \tag{2.42}$$

Moreover, if $\operatorname{Ran}(A_1^{\dagger})\perp_{B-J}\operatorname{Ran}(A_2^{\dagger}-(\|A_2\|/\|A_1\|)A_1^{\dagger})$, A_2 is surjective, and $\operatorname{Ran}(B_1)\perp_{B-J}\operatorname{Ran}(B_2-(\|B_2\|/\|B_1\|)B_1)$, then

$$\|A_1\|\|A_2\|\in\mathcal{M}(A_1)_{A_2}\cap\mathcal{M}(A_2)_{A_1}, \qquad \|B_1\|\|B_2\|\in\mathcal{M}(B_1)_{B_2}\cap\mathcal{M}(B_2)_{B_1}. \tag{2.43}$$

Corollary 2.10. *Let E be a Banach space with smooth dual and $A_1, A_2, B_1, B_2 \in B(E)$ such that A_1, A_2 are surjective operators. If* $\mathrm{Ran}(A_i^\dagger)\perp_{B-J} \mathrm{Ran}(A_j^\dagger - (\|A_j\|/\|A_i\|)A_i^\dagger)$ *and* $\mathrm{Ran}(B_i)\perp_{B-J} \mathrm{Ran}\,(B_j - (\|B_j\|/\|B_i\|)B_i)$, $(i, j = 1, 2$ *such that* $i \neq j)$ *then the following assertions are equivalent:*

(1) $\|M_{A_1,B_1} + M_{A_2,B_2}\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|$;

(2) $\|A_1\|\|A_2\| \in \mathcal{M}(A_1)_{A_2} \cap \mathcal{M}(A_2)_{A_1}$ *and* $\|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2} \cap \mathcal{M}(B_2)_{B_1}$.

As a particular case, we obtain the following.

Corollary 2.11. *Let E be a Banach space with smooth dual and A, B are surjective operators in $B(H)$. If*

$$\mathrm{Ran}\left(B^\dagger\right)\perp_{B-J} \mathrm{Ran}\left(A^\dagger - \frac{\|A\|}{\|B\|}B^\dagger\right), \qquad \mathrm{Ran}(A)\perp_{B-J} \mathrm{Ran}\left(B - \frac{\|B\|}{\|A\|}A\right), \tag{2.44}$$

then the following assertions are equivalent:

(1) $\|A\|\|B\| \in \mathcal{M}(A)_B \cap \mathcal{M}(B)_A$;

(2) $\|M_{A,B} + M_{B,A}\| = 2\|A\|\|B\|$.

3. Hilbert Space Case

Let $E = \mathcal{H}$ be a complex Hilbert space and $A \in B(\mathcal{H})$. The maximal numerical range of A [11] denoted by $W_0(A)$ is defined by

$$\{\lambda \in \mathbb{C} : \exists (x_n), \|x_n\| = 1, \text{ such that } \lim\langle Ax_n, x_n\rangle = \lambda \text{ and } \lim\|Ax_n\| = \|A\|\}, \tag{3.1}$$

and its normalized maximal range, denoted by $W_N(A)$, is given by

$$W_N(A) = \begin{cases} W_0\left(\dfrac{A}{\|A\|}\right) & \text{if } A \neq 0 \\ 0 & \text{if } A = 0. \end{cases} \tag{3.2}$$

The set $W_0(A)$ is nonempty, closed, convex, and contained in the closure of the numerical range of A.

In this section we prove that if $E = \mathcal{H}$, the conditions

$$\|A_1\|\|A_2\| \in \mathcal{M}(A_1)_{A_2} \cap \mathcal{M}(A_2)_{A_1}, \qquad \|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2} \cap \mathcal{M}(B_2)_{B_1} \tag{3.3}$$

would imply that

$$\begin{gathered} \left\|A_2^* A_1\right\| = \|A_1\|\|A_2\|, \qquad \left\|B_2 B_1^*\right\| = \|B_1\|\|B_2\|, \\ W_N\left(A_2^* A_1\right) \cap W_N\left(B_2 B_1^*\right) \neq \emptyset. \end{gathered} \tag{3.4}$$

Proposition 3.1. *Let $\mathscr{H}$ be a complex Hilbert space, $A_1, A_2, B_1, B_2 \in B(\mathscr{H})$.*

If $\|A_1\|\|A_2\| \in \mathcal{M}(A_1)_{A_2} \cap \mathcal{M}(A_2)_{A_1}$ and $\|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2} \cap \mathcal{M}(B_2)_{B_1}$, then $\|A_2^* A_1\| = \|A_1\|\|A_2\|$ and $\|B_2 B_1^*\| = \|B_1\|\|B_2\|$ and $W_N(A_2^* A_1) \cap W_N(B_2 B_1^*) \neq \emptyset$.

Proof. If $A_1 = 0$ or $A_2 = 0$ and $B_1 = 0$ or $B_2 = 0$, the result is obvious.

The proof will be done in four steps, we choose one and the others will be proved similarly. Suppose that $A_1 \neq 0$ and $A_2 \neq 0$, if $\|A_1\|\|A_2\| \in \mathcal{M}(A_1)_{A_2}$, then there exists a sequence $(x_n)_n \in S_{\mathscr{H}}(A_2)$ such that

$$\|A_1\|\|A_2\| = \lim \langle A_1 x_n, A_2 x_n \rangle. \tag{3.5}$$

We have $|\langle A_2^* A_1 x_n, x_n \rangle| \leq \|A_2^* A_1\| \leq \|A_1\|\|A_2\|$; this yields

$$\lim \left\| A_2^* A_1 x_n \right\| = \left\| A_2^* A_1 \right\| = \|A_1\|\|A_2\|. \tag{3.6}$$

From (3.5) and (3.6) we get

$$\left\| A_2^* A_1 \right\| = \|A_1\|\|A_2\|, \quad 1 \in W_0\left(\frac{A_2^* A_1}{\left\| A_2^* A_1 \right\|} \right). \tag{3.7}$$

Suppose now that $B_1 \neq 0$ and $B_2 \neq 0$, if $\|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2}$, then there exists a sequence $(y_n)_n \in S_{\mathscr{H}}(B_2)$ such that

$$\|B_1\|\|B_2\| = \lim \langle B_1 y_n, B_2 y_n \rangle. \tag{3.8}$$

Since $\lim_n \|B_1 y_n\| = \|B_1\|$, then $\lim_n (B_1^* B_1 y_n - \|B_1\|^2 y_n) = 0$.

Suppose that $w_n = B_1 y_n / \|B_1\|$, then $y_n = B_1^* w_n / \|B_1\| + z_n$ such that $\lim_n z_n = 0$.

Hence

$$\begin{aligned} \langle B_2 y_n, B_1 y_n \rangle &= \left\langle B_2 \left(\frac{B_1^* w_n}{\|B_1\|} \right), \|B_1\| w_n \right\rangle \\ &= \langle B_2 B_1^* w_n, w_n \rangle + \langle B_2 z_n, \|B_1\| w_n \rangle. \end{aligned} \tag{3.9}$$

From this, we derive that

$$\lim \left\| B_2 B_1^* w_n \right\| = \left\| B_2 B_1^* \right\| = \|B_1\|\|B_2\|. \tag{3.10}$$

From (3.8) and (3.10) we have

$$\left\| B_2 B_1^* \right\| = \|B_1\|\|B_2\|, \quad 1 \in W_0\left(\frac{B_2 B_1^*}{\left\| B_2 B_1^* \right\|} \right). \tag{3.11}$$

From (3.7) and (3.11) we get $\|A_2^* A_1\| = \|A_1\|\|A_2\|$ and $\|B_2 B_1^*\| = \|B_1\|\|B_2\|$ and $W_N(A_2^* A_1) \cap W_N(B_2 B_1^*) \neq \emptyset$. □

Remark 3.2. We remark that in the case $E = \mathcal{H}$ we obtain an implication given by Boumazgour [12].

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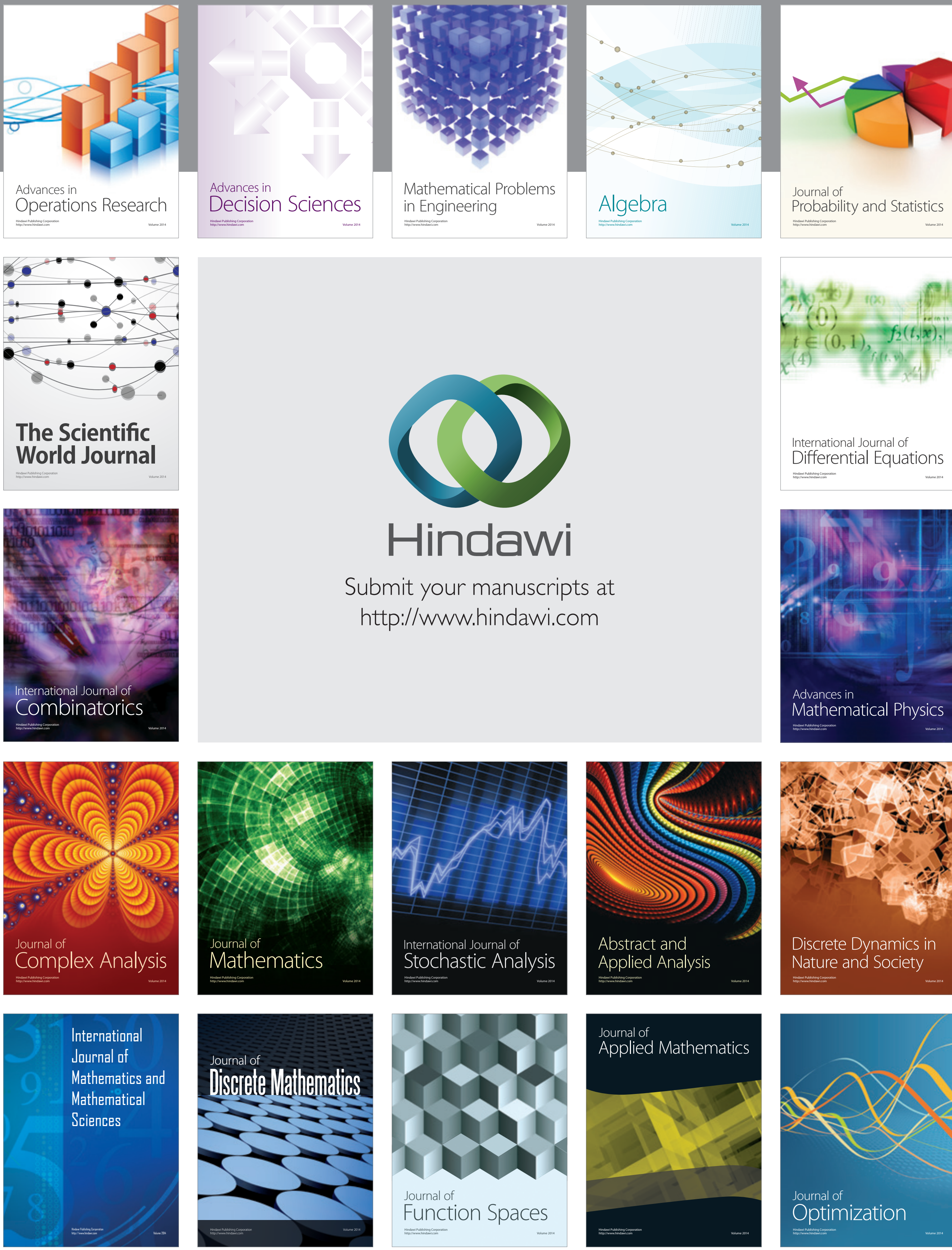
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