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Research Article

Positive Solutions for $(k, n-k)$ Conjugate Multipoint Boundary Value Problems in Banach Spaces

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By means of the fixed point index theory of strict-set contraction operator, we study the existence of positive solutions for the multipoint singular boundary value problem $(-1)^{n-k}u^{(n)}(t) = f(t, u(t))$, $0 < t < 1$, $n \geq 2$, $1 \leq k \leq n-1$, $u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, $u^{(i)}(0) = u^{(j)}(1) = \theta$, $1 \leq i \leq k-1$, $0 \leq j \leq n-k-1$ in a real Banach space E, where θ is the zero element of E, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in [0, +\infty)$, $i = 1, 2, \ldots, m-2$. As an application, we give two examples to demonstrate our results.

1. Introduction

The theory of ordinary differential equations in Banach spaces has become a new important branch (see, e.g., [1–13] and the references cited therein). In 1988, Guo and Lakshmikantham [4] discussed multiple solutions for two-point boundary value problems of second-order ordinary differential equations in Banach spaces. In [7], Guo obtained the existence of positive solutions for a boundary value problem of nth-order nonlinear impulsive integrodifferential equations in a Banach space by means of fixed point index theory and fixed point theory of completely continuous operators, respectively. Liu et al. in [6] obtained the existence of unbounded nonnegative solutions of a boundary value problem for nth-order impulsive integrodifferential equations on an infinite interval in Banach spaces by means of the M*ddoto*ch fixed point theory in a Banach space. Zhang et al. in [9] dealt with the existence, nonexistence, and multiplicity of positive solutions for a class of nonlinear three-point boundary value problems of nth-order differential equations in Banach spaces. Zhao and Chen in [8, 12] investigated the existence of at least triple positive solutions for nonlinear boundary value problem by upper and low solution methods.

In this paper, the author considers the existence of positive solutions of the following higher-order $(k, n-k)$ conjugate multipoint boundary value problems (BVPs):

$$
\begin{gathered}
(-1)^{n-k}u^{(n)}(t) = f(t,u(t)), \quad t \in (0,1), n \geq 2, 1 \leq k \leq n-1, \\
u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\
u^{(i)}(0) = u^{(j)}(1) = \theta, \quad 1 \leq i \leq k-1, 0 \leq j \leq n-k-1
\end{gathered}
\tag{1.1}
$$

in a real Banach space E, where θ is the zero element of $E, 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, a_i \in [0,+\infty), i = 1,2,\ldots,m-2$. $f : (0,1) \times E \to E$ is continuous and allowed to be singular at $t = 0$ and $t = 1$.

In scalar space, because of the widely applied background in mechanics and engineering, the nonlinear higher-order boundary value problems have received much attention (see Chyan and Henderson [Appl. Math. Letters 15 (2002) 767–774]). In [14], Eloe and Ahmad had solved successfully the existence of positive solution to the following nth-order boundary value problems:

$$
\begin{gathered}
u^{(n)}(t) + a(t)f(u) = 0, \quad t \in (0,1), \\
u^{(i)}(0) = 0, \quad i = 0,1,\ldots,n-2, \quad u(1) = au(\eta).
\end{gathered}
\tag{1.2}
$$

Recently, the existence of solutions and positive solutions of nonlinear $(k, n-k)$ focal boundary value problem

$$
\begin{gathered}
(-1)^{n-k}u^{(n)}\ (t) = f\left(t, u(t), u'(t), \ldots, u^{(n-1)}(t)\right), \quad t \in (0,1), \\
u^{(i)}(0) = 0, \quad i = 0,1,\ldots,k; \qquad u^{(j)}(1) = 0, \quad j = k+1, k+2, \ldots, n-1,
\end{gathered}
\tag{1.3}
$$

and its special cases has been studied by many authors (see, e.g., [15–23]).

By using the Krasnoselskii fixed point theorem, Eloe and Henderson in [15], Agarwal and O'Regan in [16], and Kong and Wang in [20] have established the existence of solutions for following the $(k, n-k)$ conjugate boundary value problem:

$$
\begin{gathered}
(-1)^{n-k}u^{(n)}(t) = f(t,u(t)), \quad t \in (0,1), \\
u^{(i)}(0) = u^{(i)}(1) = 0, \quad 0 \leq i \leq k-1, 0 \leq j \leq n-k-1.
\end{gathered}
\tag{1.4}
$$

Very recently, by using the fixed point theory in a cone for strict-set contraction operators, Jiang and Zhang in [11] have discussed the existence of positive solutions for the above boundary value problem (1.4) in a Banach space, where the nonlinear term $f(t,u(t)) : I \times E \to E$ is continuous but not allowed to have singularity at $t = 0,1$, where $I = [0,1]$.

The organization of this paper is as follows. We shall introduce some lemmas and notations in the rest of this section. The preliminary lemmas are in Section 2. The main results are given in Section 3. Finally, two examples are presented to demonstrate our main results in Section 4.

Let the real Banach space E with norm $\|\cdot\|$ be partially ordered by a cone P of E, that is, $u \leq v$ if and only if $v - u \in P$, and P^* denotes the dual cone of P, that is, $P^* = \{\varphi \mid \varphi \in E^*, \varphi(u) \geq 0, u \in P\}$. A cone is called a solid cone if the set of interior points is not empty.

The closed balls in spaces E and $C[I, E]$ are denoted by $P_r = \{u \in E : \|u\| \leq r\}$ $(r > 0)$ and $B_r = \{u \in C[I, E] : \|u\|_C \leq r\}$, respectively.

The basic space used in this paper is $C[I, E]$. For any $u \in C[I, E]$, evidently, $(C[I, E], \|\cdot\|_c)$ is a Banach space with norm $\|u\|_C = \max_{t \in I}\|u(t)\|$, and $Q = \{u \in C[I, E] : u(t) \geq \theta$ for $t \in I\}$ is a cone of the Banach space $C[I, E]$. A function $u \in C^n[I, E]$ is called a positive solution of the boundary value problem (1.1) if it satisfies (1.1) and $u \in Q, u(t) \not\equiv \theta$.

Let $u(t) : I \to E$ be continuous, and the abstract generalized integral $\int_0^a u(t)dt (a \in I)$ can be similarly defined as in the scalar spaces and $\int_0^a u(t)dt \in E$. If $\lim_{\varepsilon \to 0^+} \int_0^\varepsilon u(t)dt$ exist, then we say that the abstract integral is convergent, otherwise the abstract integral is divergent.

At the end of this section, we state some definitions and lemmas which will be used in Sections 2 and 3 (for details, see [1–3]).

Definition 1.1 (Kuratovski noncompactness measure). Let E be a real Banach space, and S is a bounded set in E. We denote $\alpha(S) = \inf\{\delta > 0 : S = \bigcup_{i=1}^m S_i$, all the diameters of $S_i \leq \delta\}$.

In the following, $\alpha(\cdot)$ denotes the Kuratowski measure of noncompactness in E and $C[I, E]$.

Definition 1.2 (strict-set contraction operator). Let E_1, E_2 be real Banach spaces, and $S \subset E_1$. $T : S \to E_2$ is a continuous and bounded operator. If there exists a constant k, such that $\alpha(T(S)) \leq k\alpha(S)$, then T is called a k-set contraction operator. When $k < 1, T$ is called a strict-set contraction operator.

Lemma 1.3. *If $D \subset C[I, E]$ is bounded and equicontinuous, then $\alpha(D(t))$ is continuous on I and $\alpha(D) = \alpha(D(I)) = \sup_{t \in I}\alpha(D(t))$, where $D(I) = \{u(t) : u \in D, t \in I\}, D(t) = \{u(t) : u \in D\}$.*

Lemma 1.4. *Let P be a cone in a real Banach space E and let Ω be a nonempty bounded open convex subset of P. Suppose that $T : \overline{\Omega} \to P$ is a strict-set contraction operator and $T(\overline{\Omega}) \subset \Omega$, where $\overline{\Omega}$ denotes the closure of in P. Then the fixed-point index $i(T, \Omega, P) = 1$.*

Lemma 1.5. *Let P be a cone in a real Banach space E and let Ω be a bounded open subset of P, and suppose that $T : P \cap \overline{\Omega} \to P$ is a strict-set contraction operator.*

(i) *If $\theta \in \Omega$, and $Tu \neq \lambda u$, for all $u \in \partial\Omega \cap P, \lambda \geq 1$. Then $i(T, \Omega \cap P, P) = 1$.*

(ii) *If there exists $u_0 \in P \setminus \{\theta\}$, such that $u - Tu \neq \lambda u$, for all $u \in \partial\Omega \cap P, \lambda \geq 0$. Then $i(T, \Omega \cap P, P) = 0$.*

2. The Preliminary Lemmas

To prove the main results, we need the following lemmas.

Lemma 2.1 (see [20]). *Let $k(t,s)$ be the Green function for the $(k,n-k)$ conjugate BVP* (1.4). *Then*

$$k(t,s) = \frac{1}{(k-1)!(n-k-1)!}\begin{cases}\int_0^{t(1-s)} v^{k-1}(v+s-t)^{n-k-1}dv, & 0 \le t \le s \le 1,\\ \int_0^{s(1-t)} v^{n-k-1}(v+t-s)^{k-1}dv, & 0 \le s \le t \le 1.\end{cases} \tag{2.1}$$

Obviously, $k(t,s)$ is continuous on $I \times I$ and has the following properties.

(G_1) There exist nonnegative functions $p(t), m(t), q(t) \in C[0,1]$ such that

$$p(t)q(s) \le k(t,s) \le m(t)q(s) \le q(s), \quad \forall t,s \in I, \tag{2.2}$$

where

$$p(t) = \frac{t^k(1-t)^{n-k}}{n-1}, \qquad m(t) = \frac{t^{k-1}(1-t)^{n-k-1}}{\min\{k,n-k\}}, \qquad q(s) = \frac{s^{n-k}(1-s)^K}{(k-1)!(n-k-1)!}. \tag{2.3}$$

(G_2) For any $\tau \in (0,1/2)$, $k(t,s)$ satisfies

$$k(t,s) \ge \rho(\tau)k(v,s), \quad t \in I_\tau = [\tau, 1-\tau], s, v \in I, \tag{2.4}$$

where

$$\rho(\tau) := \frac{\min_{t\in I_\tau} p(t)}{\max_{s\in[0,1]} m(s)}. \tag{2.5}$$

Setting

$$\Phi(t) = \frac{(n-1)!}{(k-1)!(n-k-1)!}\int_t^1 s^{k-1}(1-s)^{n-k-1}ds. \tag{2.6}$$

It is obvious that $0 \le \Phi(t) \le 1, t \in [0,1]$, and by the properties of the Euler integral, we have

$$\Phi(0) = 1, \quad \Phi(1) = 0, \quad \|\Phi\| = 1. \tag{2.7}$$

In order to abbreviate our discussion, we give the following assumptions.

(C_0) $\sum_{i=1}^{m-2} a_i\Phi(\xi_i) < 1$.

(C_1) $f \in C[(0,1) \times P, P]$ and $\|f(t,u)\| \le g(t)\|h(u)\|, t \in (0,1), u \in P$, where $h : P \to P$ is continuous and bounded and $g : (0,1) \to (0,+\infty)$ is continuous and satisfies $\int_0^1 g(s)ds < +\infty$.

(C_2) For any $r > 0$ and $[a,b] \subset (0,1)$, $f(t,u)$ is uniformly continuous on $[a,b] \times P_r$.

(C_3) There exists constant $L \geq 0$ such that for any $t \in (0,1)$ and the bounded set $D \subset P$

$$\alpha(f(t,D)) \leq L\alpha(D), \tag{2.8}$$

where

$$2L\left\{1 + \left(1 - \sum_{i=1}^{m-2} a_i \Phi(\xi_i)\right)^{-1} \sum_{i=1}^{m-2} a_i\right\} \cdot \max_{s\in I} q(s) < 1. \tag{2.9}$$

Lemma 2.2. *Let* $\sum_{i=1}^{m-2} a_i \Phi(\xi_i) \neq 1$ *and* (C_1) *be satisfied, then the problem*

$$(-1)^{n-k} u^{(n)}(t) = f(t,u(t)), \quad t \in I = [0,1], n \geq 2, 1 \leq k \leq n-1, \tag{2.10}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{2.11}$$

$$u^{(i)}(0) = u^{(j)}(1) = \theta, \quad 1 \leq i \leq k-1, 0 \leq j \leq n-k-1 \tag{2.12}$$

has a unique solution

$$u(t) = \int_0^1 K(t,s) f(s,u(s))ds, \quad t \in I, \tag{2.13}$$

where

$$K(t,s) = k(t,s) + \left(1 - \sum_{i=1}^{m-2} a_i \Phi(\xi_i)\right)^{-1} \Phi(t) \sum_{i=1}^{m-2} a_i k(\xi_i, s), \quad 0 \leq t, s \leq 1. \tag{2.14}$$

Proof. According to the definitions of generalized integral in abstract space, the proof of this lemma is similar to the proof in scalar spaces, so we omit it. □

For $u \in C[I,E]$, we define an operator T by

$$(Tu)(t) = \int_0^1 K(t,s) f(s,u(s))ds, \quad t \in I. \tag{2.15}$$

Lemma 2.3. *Suppose that* (C_0)–(C_3) *hold. Then* $T : C[I,E] \cap B_r \rightarrow C[I,E]$ *is a strict-set contraction operator.*

Proof. For $u \in u \in C[I,P]$, it follows from (C_1) that T is well defined and bounded operator. If $B \in C[I,P]$ is a bounded subset of $C[I,P]$, then TB is bounded.

Next we prove that T is continuous on $C[I,E]\cap B_r$. Let $\{u_j\},\{u\}\subset C[I,E]\cap B_r$, and $\|u_j-u\|_C\to 0$ $(j\to\infty)$. Hence $\{u_j\}$ is a bounded subset of $C[I,E]\cap B_r$. Thus, there exists $r>0$ such that $r=\sup_j\|u_j\|_C<\infty$ and $\|u\|_C\le r$.

According to continuity of f, for all $\varepsilon>0$, there exists $J>0$ such that

$$\|f(t,x_j(t))-f(t,x(t))\|<\frac{\varepsilon}{\left[1+\left(1-\sum_{i=1}^{m-2}a_i\Phi(\xi_i)\right)^{-1}\sum_{i=1}^{m-2}a_i\right]\int_0^1 q(s)ds}, \tag{2.16}$$

for $j\ge J$, for all $t\in I$.

Then,

$$\begin{aligned}\|(Tu_j)(t)-(Tu)(t)\| &\le \int_0^1 K(t,s)\|f(s,x_j(s))-f(s,(s))\|ds\\ &\le \left[1+\left(1-\sum_{i=1}^{m-2}a_i\Phi(\xi_i)\right)^{-1}\sum_{i=1}^{m-2}a_i\right]\int_0^1 q(s)\|f(s,x_j(s))\\ &\qquad -f(s,x(s))\|ds<\varepsilon.\end{aligned} \tag{2.17}$$

Therefore, for all $\varepsilon>0$, for any $t\in I$ and $j\ge J$, we get

$$\|(Tu_j)(t)-(Tu)(t)\|\longrightarrow 0. \tag{2.18}$$

This implies T is continuous on $C[I,E]\cap B_r$. By the properties of continuity of $G(t,s)$, it is easy to see that T is equicontinuous on I.

For any $S\subset C[I,E]\cap B_r$, it is easy to get that functions $T(S)=\{Tu\mid u\in S\}$ are uniformly bounded. By Lemma 1.3, we get

$$\alpha(TS)=\sup_{t\in I}\alpha((TS)(t)), \tag{2.19}$$

where $(TS)(t)=\{(Tu)(t):u\in S, t\in I \text{ is fixed}\}$.

Write

$$D=\left\{\int_0^1 K(t,s)f(s,u(s))ds : u\in S\right\}, \tag{2.20}$$

$$D_\varepsilon=\left\{\int_\varepsilon^{1-\varepsilon} K(t,s)f(s,u(s))ds : u\in S\right\},\quad \left(0<\varepsilon<\frac{1}{2}\right). \tag{2.21}$$

By (C_1), for any $u \in S$, we have

$$H_\varepsilon = \left\| \int_\varepsilon^{1-\varepsilon} K(t,s) f(s,u(s)) ds - \int_0^1 K(t,s) f(s,u(s)) ds \right\| \\ \leq c_0 \int_0^\varepsilon K(t,s) g(s) ds + c_0 \int_{1-\varepsilon}^1 K(t,s) g(s) ds, \tag{2.22}$$

where $c_0 := \max_{u\in B_r} \|h(u)\|$.

It follows from H_ε and (C_1) that the Hausdorff metric $d_H(D_\varepsilon, D) \to 0$, $(\varepsilon \to 0^+)$. Thus

$$\lim_{\varepsilon\to 0^+} \alpha(D_\varepsilon) = \alpha(D). \tag{2.23}$$

We next shall estimate $\alpha(D_\varepsilon)$. For any $u \in C[I,E]$, by $\int_\varepsilon^{1-\varepsilon} u(t)dt \in (1-2\varepsilon)\overline{co}\{u(t) : t \in I\}$, then

$$\begin{aligned} \alpha(D_\varepsilon) &= \alpha\left(\left\{\int_\varepsilon^{1-\varepsilon} K(t,s) f(s,u(s)) ds : u \in S\right\}\right) \\ &\leq (1-2\varepsilon)\alpha\left(\overline{co}\left(\left\{K(t,s) f(s,u(s)) : s \in [\varepsilon, 1-\varepsilon], u \in S\right\}\right)\right) \\ &\leq \alpha\left(\left\{k(t,s) f(s,u(s)) : s \in [\varepsilon, 1-\varepsilon], u \in S\right\}\right) \\ &\quad + \left(1 - \sum_{i=1}^{m-2} a_i \Phi(\xi_i)\right)^{-1} \alpha\left(\left\{\sum_{i=1}^{m-2} a_i k(\xi_i, s) f(s,u(s)) : s \in [\varepsilon, 1-\varepsilon], u \in S\right\}\right) \\ &\leq \left(1 + \left(1 - \sum_{i=1}^{m-2} a_i \Phi(\xi_i)\right)^{-1} \sum_{i=1}^{m-2}\right) \cdot \max_{s\in I} q(s) \cdot \alpha\left(f(I_\varepsilon \times S(I_\varepsilon))\right), \end{aligned} \tag{2.24}$$

where $I_\varepsilon = [\varepsilon, 1-\varepsilon]$, $S(I_\varepsilon) = \{u(t) : t \in I_\varepsilon\}$.

On the other hand, using a similar method as in the proof of Lemma 2 in [11] we can get that

$$\alpha(S(I)) \leq 2\alpha(S). \tag{2.25}$$

Therefore, it follows from (2.19), (2.25) that

$$\alpha\left(f(I_\varepsilon \times S(I_\varepsilon))\right) = \sup_{t\in I_\varepsilon} \alpha\left(f(t, S(I_\varepsilon))\right) \leq L \cdot \alpha(S(I_\varepsilon)) \leq L \cdot \alpha(S(I)) \leq 2L \cdot \alpha(S). \tag{2.26}$$

Thus, we have

$$\alpha(D_\varepsilon) \leq 2L\left(1 + \left(1 - \sum_{i=1}^{m-2} a_i \Phi(\xi_i)\right)^{-1} \sum_{i=1}^{m-2} a_i\right) \cdot \max_{s\in I} q(s) \cdot \alpha(S). \tag{2.27}$$

Combining with (2.24), we get

$$\alpha(D) \le 2L\left(1+\left(1-\sum_{i=1}^{m-2} a_i\Phi(\xi_i)\right)^{-1}\sum_{i=1}^{m-2} a_i\right)\cdot \max_{s\in I} q(s)\cdot\alpha(S). \tag{2.28}$$

Therefore, we have

$$\alpha(TS) \le 2L\left(1+\left(1-\sum_{i=1}^{m-2} a_i\Phi(\xi_i)\right)^{-1}\sum_{i=1}^{m-2} a_i\right)\cdot \max_{s\in I} q(s)\cdot\alpha(S), \quad \forall S\in P\cap B_r. \tag{2.29}$$

Notice that by (2.9) we claim that $T : C[I,E]\cap B_r \to C[I,E]$ is a strict-set contraction. The proof is complete. □

Further, we construct a cone Ω by

$$\Omega = \left\{u\in Q : u(t)\ge \rho^*(\tau)u(s), \forall t\in I_\tau, \forall s\in I\right\}, \tag{2.30}$$

where $I_\tau = [\tau, 1-\tau]$, and

$$\rho^*(\tau) = \min\left\{\rho(\tau), \min_{t\in I_\tau}\Phi(t)\right\}, \tag{2.31}$$

where $\rho(\tau)$ is defined in (G_2). It is easy to see that Ω is a closed convex cone of $C[I,E]$ and $\Omega\subset Q$.

Lemma 2.4. *Suppose that* (C_0)–(C_3) *hold. Then,* $T(\Omega)\subset\Omega$.

Proof. From (G_2), (2.6), (2.15), and (2.30), for any $u\in\Omega, t\in T_\tau, t'\in I$, we obtain

$$\begin{aligned}(Tu)(t) &= \int_0^1 K(t,s)f(s,u(s))ds \\ &\ge \rho(\tau)\int_0^1 k(t',s)f(s,u(s))ds + \left(1-\sum_{i=1}^{m-2} a_i\Phi(\xi_i)\right)^{-1}\Phi(t)\int_0^1\sum_{i=1}^{m-2} a_i k(\xi_i,s)f(s,u(s))ds \\ &\ge \rho^*(\tau)\left(\int_0^1 k(t',s)f(s,u(s))\right)ds + \left(1-\sum_{i=1}^{m-2} a_i\Phi(\xi_i)\right)^{-1}\int_0^1\sum_{i=1}^{m-2} a_i k(\xi_i,s)f(s,u(s))ds \\ &\ge \rho^*(\tau)\left[\int_0^1 K(t',s)f(s,u(s))ds\right] = \rho^*(\tau)(Tu)(t'), \qquad t'\in I.\end{aligned} \tag{2.32}$$

Therefore,$T(u)\in\Omega$, that is, $T(\Omega)\subset\Omega$. □

3. The Main Results

Let

$$M^* := \max_{s\in I} q(s) \cdot \left(1 + \left(1 - \sum_{i=1}^{m-2} a_i \Phi(\xi_i)\right)^{-1} \sum_{i=1}^{m-2} a_i\right) \int_0^1 g(s)ds. \tag{3.1}$$

Theorem 3.1. *Let* (C_0)–(C_3) *hold. In addition, assume that the following conditions are satisfied:*

(C_4) $d_i := M^* c_i < 1, (i = 1, 2)$, where

$$c_1 := \overline{\lim_{\|u\|\to\infty}} \frac{\|h(u)\|}{\|u\|}, \qquad c_2 := \overline{\lim_{\|u\|\to\infty}} \frac{\|h(u)\|}{\|u\|}, \tag{3.2}$$

(C_5) P is a solid cone, and there exist $u_0 \in Int(P)$ and $\tau \in (0, 1/2)$ such that $f(t, u) \geq l(t)u_0, l(t) \in [I_\tau, R^+]$, and

$$l_0 = \min_{t\in I_\tau} p(t) \int_\tau^{1-\tau} q(s)l(s)ds \geq 1, \tag{3.3}$$

for all $t \in I_\tau = [\tau, 1-\tau], u \geq u_0$.

Then problem (1.1) *has at least two positive solutions u_1, u_2, and $u_1(t) \geq l_0 u_0$, for $t \in I_\tau, u \geq u_0$.*

Proof. We first show that there exists $\beta > 0$, such that

$$\|v\| \geq \beta, \quad \text{for any } v \geq u_0. \tag{3.4}$$

If (3.4) is not true, then there exist the sequences $\{v_n\}_{n=1}^{\infty}$ satisfying $v_n \geq u_0$, and $\|v_n\| < 1/n, (n = 1, 2, \ldots)$. Thus $u_0 \leq \theta$, which is contradiction with $u_0 \in \text{int}(P)$.

Let

$$c_i' = \frac{1 + c_i M^*}{2M^*}, \quad (i = 1, 2). \tag{3.5}$$

Then

$$c_i' > c_i, \qquad d_i' = c_i' M^* < 1. \tag{3.6}$$

From (C_4) and (3.6), there exist two positive constants r_1, r_2 with

$$0 < r_1 < \beta, \qquad r_2 > \max\{\beta, 2\|u_0\|\} \tag{3.7}$$

such that

$$\|h(u)\| \leq c_i'\|u\|, \quad u \in B_{r_1} \cap P, \qquad \|h(u)\| \leq c_i'\|u\|, \quad u \in P \setminus B_{r_2}. \tag{3.8}$$

Therefore, for any $u \in P$, we have

$$\|h(u)\| \leq c_i' \|u\| + M, \tag{3.9}$$

where $M = \sup\{\|h(u)\| : u \in B_{r_2}\}$.

Choose

$$r_3 = r_2 + \left(1 - d_2'\right)^{-1} M^* M. \tag{3.10}$$

In the following, let

$$\Omega_1 = \{u \in Q :| \ \|u\| < r_1\}, \tag{3.11}$$

$$\Omega_2 = \{\in Q :| \ \|u\| < r_3\}, \tag{3.12}$$

$$\Omega_3 = \{u \in Q :| \ \|u\| < r_3, \text{if } t \in I_\tau, u(t) \geq u_0\}. \tag{3.13}$$

It is to see from (3.7) that Ω_3 is nonempty for $2u_0 \in \Omega_3$, which implies $\Omega_3 \neq \emptyset$. Obviously, $\Omega_i \subset Q$ $(i = 1, 2, 3)$ are nonempty, convex, open sets and $\overline{\Omega}_1 = Q \cap B_{r_1}, \overline{\Omega}_2 = Q \cap B_{r_2}, \overline{\Omega}_3 = \{u \in \overline{\Omega}_2 : u(t) \geq u_0, t \in I_\tau\}$. So

$$\Omega_1 \subset \Omega_2, \qquad \Omega_3 \subset \Omega_2, \qquad \Omega_1 \cap \Omega_3 = \emptyset. \tag{3.14}$$

For any $u \in \overline{\Omega}_2$, form (3.8) and (3.9), we have

$$\begin{aligned} \|(Tu)(t)\| &\leq \left(\int_0^1 K(t,s) g(s) ds\right) \left(c_2' \|u\| + M\right) \\ &\leq \int_0^1 g(s) ds \cdot \left(c_2' \|u\| + M\right) \cdot \left(1 + \left(1 - \sum_{i=1}^{m-2} a_i \Phi(\xi_i)\right)^{-1} \sum_{i=1}^{m-2} a_i\right) \max_{s \in I} q(s) \\ &= d_2' \|u\| + M^* M < d_2' r_3 + M^* M < r_3, \end{aligned} \tag{3.15}$$

which implies

$$T\left(\overline{\Omega}_2\right) \subset \Omega_2. \tag{3.16}$$

From (2.15) and (3.7), we get

$$T\left(\overline{\Omega}_1\right) \subset \Omega_1. \tag{3.17}$$

For any $u \in \overline{\Omega}_3$, it follows from (3.16) that $\|Tu\| < r_3$. According to (C_5) and (G_1), we can obtain

$$\begin{aligned}(Tu)(t) &\geq \int_{\tau}^{1-\tau} K(t,s)f(s,u(s))ds \\ &\geq \int_{\tau}^{1-\tau} K(t,s)l(s)u_0 ds \\ &\geq \min_{t\in I_\tau} p(t)\int_{\tau}^{1-\tau} q(s)l(s)ds \cdot u_0 = l_0 u_0 \geq u_0, \quad \forall t \in I_\tau,\end{aligned} \tag{3.18}$$

which implies

$$T\left(\overline{\Omega}_3\right) \subset \Omega_3. \tag{3.19}$$

Combining (3.16)–(3.19) with Lemma 1.4, we have $i(T,\Omega_i,P) = 1, i = 1,2,3$. Furthermore, using the fixed point index theory, we obtain successively

$$i\left(T,\Omega_2 \setminus \left(\overline{\Omega}_1 \cup \overline{\Omega}_3\right),P\right) = i(\mathrm{T},\Omega_2,P) - i(T,\Omega_1,P) - i(T,\Omega_3,P) = -1. \tag{3.20}$$

Then T has at least two fixed points u_1 and u_2 which satisfy $u_1 \in \Omega_3$ and $u_2 \in \Omega_2 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_3)$. Then Theorem 3.1 is proved. □

Theorem 3.2. *Assume that (C_0)–(C_3), $d_2 < 1$ hold. Suppose further that*

(C_5') P is a cone of the real Banach space E, and there exist $u_0 \in P \setminus \{\theta\}$ and $\tau \in (0,1/2)$ such that $f(t,u) \geq l(t)u_0, l(t) \in [I_\tau, R^+]$, for for all $t \in I_\tau, u \geq u_0$.

Then problem (1.1) *has at least one positive solutions u with $u(t) \geq l_0 u_0, t \in I_\tau, u \geq u_0$, where l_0 is defined in* (3.3).

Proof. As in the proof of Theorem 3.1, we need only to show that T has one positive fixed point u with $u(t) \geq l_0 u_0, t \in I_\tau, l_0 \geq 1$.

Choose r_0 satisfying $r_0 > \max\{\beta, 2\|u_0\|\}$ and let

$$\Omega_0 = \{u \in C[I,E] : \|u\| < r_0, u(t) \geq u_0, \forall t \in I_\tau\}. \tag{3.21}$$

Obviously, Ω_0 is a bounded closed convex set in $C[I,E]$. $\Omega_0 \neq \emptyset$, for $u^*(t) \equiv u_0 \in \Omega_0$. Let $u \in \Omega_0$. As the proof of (3.19), we have $T(\Omega_0) \subset \Omega_0$, where T is given by (2.15). Thus, it follows from Lemmas 2.3 and 1.4 that T has a fixed point $u \in \Omega_0$. The proof is complete. □

Theorem 3.3. *Assume that (C_0)–(C_3) and the following conditions hold.*

(C_6) There exists $R_0 > 0$ such that $\sup_{P_{R_0}} \|h(u)\| < R_0(M^*)^{-1}$, where $P_{R_0} = \{u \in P : \|u\| < R_0\}$.

(C_7) *There exists* $\varphi \in P^*$ *such that* $\varphi(u) > 0$ *for any* $u > \theta$, *and*

$$\lim_{\|u\| \to +\infty} \min_{t \in I_\tau} \frac{\varphi(f(t,u))}{\varphi(u)} > M_0, \tag{3.22}$$

where $\tau \in (0, 1/2)$, *and*

$$M_0 := \left(\min_{t \in I_\tau} p(t) \cdot \rho^*(\tau) \int_\tau^{1-\tau} q(s) ds \right)^{-1}. \tag{3.23}$$

(C_8) *There exists* $\varphi \in P^*$ *such that* $\varphi(u) > 0$ *for any* $u > \theta$, *and*

$$\lim_{\|u\| \to 0} \min_{t \in I_\tau} \frac{\varphi(f(t,u))}{\varphi(u)} > M_0, \tag{3.24}$$

where $\tau \in (0, 1/2)$, *and* M_0 *is defined in* (3.23). *Then problem* (1.1) *has at least two positive solutions.*

Proof. If (C_6) holds, then there exists ε_1 satisfying $0 < \varepsilon_1 < R_0 (M^*)^{-1}$ such that $\|h(u)\| \leq (R_0 (M^*)^{-1} - \varepsilon_1)$. For any $u \in \Omega \cap \partial P_{R_0}, t \in I_\tau$, it follows from (C_1) and (2.15) that

$$\begin{aligned} \|Tu\| &\leq \int_0^1 q(s) g(s) \|h(u)\| ds + \left(1 - \sum_{i=1}^{m-2} a_i \Phi(\xi_i)\right)^{-1} \sum_{i=1}^{m-2} a_i \int_0^1 q(s) g(s) \|h(u)\| ds \\ &\leq \max_{s \in I_\tau} q(s) \left(1 + \left(1 - \sum_{i=1}^{m-2} a_i \Phi(\xi_i)\right)^{-1} \sum_{i=1}^{m-2} a_i \right) \int_0^1 g(s) ds \cdot \left(R_0 (M^*)^{-1} - \varepsilon_1\right) \\ &= \left(R_0 (M^*)^{-1} - \varepsilon_1\right) M^* < R_0, \end{aligned} \tag{3.25}$$

which implies $Tu \neq \lambda u$, for all $u \in \Omega \cap \partial P_{R_0}$, and $\lambda \geq 1$. From Lemma 1.5 (i), we have

$$i(T, \Omega \cap \partial P_{R_0}, \Omega) = 1. \tag{3.26}$$

According to (C_7), there exist $\varepsilon_2 > 0$ and $R_2 > R_0 > 0$, for any $t \in I_\tau$, such that

$$\varphi(f(t,u)) \geq (M_0 + \varepsilon_2) \varphi(u), \quad \|u\| \geq R_2. \tag{3.27}$$

Let $U_2 = \{u \in C[I, E] : \|u\| < R_2\}$. We need only to show that $u - Tu \neq \lambda e$, for any $u \in \Omega \cap \partial U_2$, and $\lambda \geq 0, e \in P$ with $\|e\| = 1$. If it is false, then there exists $u^* \in \Omega \cap \partial U_2$ and $\lambda_0 \geq 0$, such that

$$u^* - Tu^* = \lambda_0 e, \tag{3.28}$$

which implies

$$\begin{aligned} u^*(t) \geq Tu^*(t) &= \int_0^1 K(t,s) f(s,u^*(s)) ds \\ &\geq p(t) \int_\tau^{1-\tau} q(s) f(s,u^*(s)) ds. \end{aligned} \tag{3.29}$$

Hence, for any $\varphi \in P^*$, we have

$$\begin{aligned} \varphi(u^*(t)) \geq \varphi(Tu^*(t)) &= \int_0^1 K(t,s) \varphi\big(f(s,u^*(s))\big) ds \\ &\geq \int_\tau^{1-\tau} p(t) q(s)(M_0+\varepsilon_2)\varphi(u^*(s)) ds \\ &\geq \min_{t\in I_\tau} p(t) \int_\tau^{1-\tau} q(s)(M_0+\varepsilon_2)\varphi(u^*(s)) ds \\ &\geq (M_0+\varepsilon_2)\min_{t\in I_\tau} p(t)\cdot \rho^*(\tau) \int_\tau^{1-\tau} q(s) ds \cdot \varphi(u^*(t)). \end{aligned} \tag{3.30}$$

It is easy to see that $\varphi(u^*(t)) > 0$. In fact, if $\varphi(u^*(t)) = 0$, since $u^* \in \Omega \cap \partial U_2$, then we have $\varphi(u^*(t)) \geq \rho^*(\tau)\varphi(u^*(s)) \geq 0$ and consequently $\|u^*\| = 0$, which contradicts with $\|u^*\| = R_2$. By (3.29) and (3.30), we get $1 < (M_0+\varepsilon_2) \min_{t\in I_\tau} p(t)\rho^*(\tau)\int_\tau^{1-\tau} q(s)ds \leq 1$, which is a contradiction. It follows from Lemma 1.5 (ii) that

$$i(T, \Omega \cap \partial U_2, \Omega) = 0. \tag{3.31}$$

If (C_8) holds, then there exist $\varepsilon_3 > 0$ and $R_1 > 0$ with $R_1 < R_0$, such that

$$\varphi\big(f(t,u)\big) \geq (M_0+\varepsilon_3)\varphi(u), \quad \|u\| \geq R_1. \tag{3.32}$$

Let $U_1 = \{u \in C[I,E] : \|u\| < R_1\}$. As in the proof of (3.31), for any $u \in \Omega \cap \partial U_1, t \in I_\tau$, we get

$$i(T, \Omega \cap \partial U_1, \Omega) = 0. \tag{3.33}$$

Notice that $\overline{U}_1 \subset P_{R_0}, \overline{P}_{R_0} \subset U_2$. Thus, it follows from (3.26), (3.31), and (3.33) that

$$\begin{aligned} \Big(i\Big(T, \Omega \cap \Big(P_{R_0} \setminus \overline{U}_1\Big)\Big), \Omega\Big) &= i(T, \Omega \cap P_{R_0}, \Omega) - i(T, \Omega \cap U_1, \Omega) = 1, \\ \Big(i\Big(T, \Omega \cap \Big(U_2 \setminus \overline{P}_{R_0}\Big)\Big), \Omega\Big) &= i(T, \Omega \cap U_2, \Omega) - i(T, \Omega \cap P_{R_0}, \Omega) = -1. \end{aligned} \tag{3.34}$$

Then T has at least two fixed points u_1 and u_2 which satisfy $u_1 \in \Omega \cap (P_{R_0} \setminus \overline{U}_1)$ and $u_2 \in \Omega \cap (U_2 \setminus \overline{P}_{R_0})$. Then Theorem 3.3 is proved.

Similarly to the proofs of Theorem 3.3, we can easily get the following corollaries. □

Corollary 3.4. *Assume that* (C_0)*–*(C_3)*,* $d_1 < 1$ *and* (C_7) *hold. Then problem* (1.1) *has at least one positive solution.*

Corollary 3.5. *Assume that* (C_0)*–*(C_3)*,* $d_2 < 1$ *and* (C_8) *hold. Then problem* (1.1) *has at least one positive solution.*

4. Examples

Now we present two examples to illustrate our main results.

Example 4.1. Consider the boundary value problems in $E = R^N$ (N-dimensional Euclidean space and $\|u\| = \max_{1\le p\le N}|u_p| < +\infty$)

$$\begin{gathered} (-1)^6 u_p^{(10)}(t) = \frac{1+t}{\sqrt{t(1-t)}}\left(\sin^2 u_{p+1} + \ln\left(1+u_p^2\right) + \sqrt{t(1-t)}H(u_p)\right), \quad 0<t<1, \\ u_p(0) = \frac{1}{9}u_p\left(\frac{1}{4}\right) + \frac{1}{2}u_p\left(\frac{1}{2}\right), \quad p = 1,2,3,\ldots \\ u^{(i)}(0) = u^{(j)}(1) = 0, \quad 1\le i\le 3, 0\le j\le 5, \end{gathered} \tag{4.1}$$

where $n = 10, k = 4$, and

$$H(u_p) = \begin{cases} \dfrac{2}{3}\times 10^5 u_p + \dfrac{N}{\|u_0\|}\times 10^{12}u_p^2, & u_p \in \left[0, \dfrac{\|u_0\|}{N}\right], \\ \left(\dfrac{2}{3}\times 10^5 + 1\times 10^{12}\right)u_p, & u_p \in \left[\dfrac{\|u_0\|}{N}, 1 + r_3(\|u_0\|) + \dfrac{\|u_0\|}{N}\right], \\ \dfrac{2}{3}\times 10^5 u_p + \sqrt{1 + r_3(\|u_0\|) \ + \dfrac{\|u_0\|}{N}} & \\ \times 10^{12}\sqrt{u_p}, & u_p \in \left[1 + r_3(\|u_0\|) + \dfrac{\|u_0\|}{N}, +\infty\right), \end{cases} \tag{4.2}$$

where $r_3(\cdot)$ is as defined in (3.10). Then the problem (4.1) can be regarded as a BVP of the form (1.1) in E. In this situation, $u = (u_1,\ldots,u_P,\ldots,u_N) \in R^N$, $f = (f_1,\ldots,f_P,\ldots,f_N)$, $\theta = (0,0,\ldots,0) \in R^N$ and $a_1 = 1/9$, $\xi_1 = 1/4$, $a_2 = \xi_2 = 1/2$, $g = (g_1,\ldots,g_P,\ldots,g_N)$, $h = (h_1,\ldots,h_P,\ldots,h_N)$, in which

$$\begin{gathered} f_p(t,u_1,u_2,\ldots,u_N) = \frac{1+t}{\sqrt{t(1-t)}}\left(\sin^2 u_{p+1} + \ln\left(1+u_p^2\right) + \sqrt{t(1-t)}H(u_p)\right), \\ p(t) = \frac{t^4(1-t)^6}{9}, \qquad m(t) = \frac{t^3(1-t)^5}{4}, \qquad q(s) = \frac{s^6(1-s)^4}{3!\cdot 5!}. \end{gathered} \tag{4.3}$$

Obviously, $f : (0,1) \times E \to E$ is continuous. Taking $\tau = 1/4$. By direct account, we have

$$\sum_{i=1}^{m-2} a_i \Phi(\xi_i) = \frac{1}{9}\Phi\left(\frac{1}{4}\right) + \frac{1}{2}\Phi\left(\frac{1}{2}\right) \approx 0.219 < 1. \tag{4.4}$$

Now we show that (C_1)–(C_5) hold.

First, it is easy to see that $\|f(t,u)\| \le g(t)\|h(u)\|, t \in (0,1), u \in E$, with

$$g_p(t) = \frac{2}{\sqrt{t(1-t)}}, \qquad h_p(u) = \sin^2 u_{p+1} + \ln\left(1 + u_p^2\right) + \frac{1}{2}H(u_p), \tag{4.5}$$

and $\int_0^1 g_p(s)ds = 2\pi < +\infty$.

For sufficiently large number $R > 0$, take $P = \{u = (u_1, \ldots, u_p, \ldots, u_N) \in R^N : u_p \ge 0, \|u\| \le R, p = 1, 2, \ldots, N\}$.

Then $h : P \to P$ is continuous and bounded. Therefore, (C_1) is satisfied. In addition, (C_2) and (C_3) are automatically satisfied in N-dimensional Euclidean space R^N.

As far as (C_4) is concerned, we get

$$M^* = 2\pi \max_{s\in[0,1]} \frac{s^6(1-s)^4}{3!5!} \cdot \left(1 + (1-0.219)^{-1} \times \left(\frac{1}{9} + \frac{1}{2}\right)\right) \approx 1.86 \times 10^{-5},$$
$$c_1 = \overline{\lim_{\|u\|\to\infty}} \frac{\|h(u)\|}{\|u\|} = \frac{1}{3} \times 10^5, \qquad c_2 = \overline{\lim_{\|u\|\to\infty}} \frac{\|h(u)\|}{\|u\|} = \frac{1}{3} \times 10^5, \qquad d_i = M^* c_i \approx 0.62 < 1. \tag{4.6}$$

Then (C_4) is satisfied.

On the other hand, P is a solid cone, for all $u_0 \in \mathrm{Int}(P)$. For all $t \in I_\tau = [1/4, 3/4]$, we have

$$f_p(t,u) \ge 1 \times 10^{12}(1+t)u_p \ge l(t)u_0^{(p)}, \quad \text{for } u \ge u_0, \tag{4.7}$$

where $u_0 = (u_0^{(1)}, \ldots, u_0^{(p)}, \ldots, u_0^{(N)}), l(t) = 1 \times 10^{12}(1+t)$, and

$$l_0 = \min_{t\in[1/4,3/4]} p(t) \int_{1/4}^{3/4} q(s)l(s)ds \approx 7.055 > 1. \tag{4.8}$$

So (C_5) is also satisfied. For example, if $u_0 = \{1, 1, \ldots, 1\}$, we let $\beta = 1$, then $r_3(\|u_0\|) + (\|u_0\|)/N + 1 \approx 9.745 \times 10^7(2N+1)$. By Theorem 3.1, the problem (4.1) has at least two positive solutions in E.

Example 4.2. Consider the following boundary value problems in l^∞:

$$
\begin{gathered}
(-1)^3 u_p^{(9)}(t) = \frac{\pi}{\sqrt{t}}\left(\ln\left(1+\frac{u_p^2}{p}\right)+\frac{t(1-t)\sqrt{u_p}}{p}\left|\sin u_{p+1}\right| + H(u_p)\right), \quad 0<t<1, \\
u_p(0) = \frac{1}{4}u_p\left(\frac{3}{4}\right), \quad p=1,2,3,\dots \\
u^{(i)}(0) = u^{(j)}(1) = 0, \quad 1\le i\le 5, 0\le j\le 2,
\end{gathered}
\tag{4.9}
$$

where $n=9, k=6$, and

$$
H(u_p) = \begin{cases}
8\times 10^8 u_p, & u_p \in \left[0, \dfrac{1}{16}\times 10^{-8}\right], \\
100\sqrt[4]{u_p}, & u_p \in \left[\dfrac{1}{16}\times 10^{-8}, 1\times 10^4\right], \\
8\times 10^8 u_p + \left(10 - 8\times 10^{10}\right)\sqrt{u_p}, & u_p \in \left[1\times 10^4, +\infty\right).
\end{cases}
\tag{4.10}
$$

Let $P = \{u = (u_1,\dots,u_p,\dots) \in l^\infty : u_p \ge 0, p = 1,2,\dots\}, \|u\| = \sup_{p\ge 1}|u_p| < +\infty$. Then l^∞ is a Banach space. Thus the problem (4.9) can be regarded as a BVP of the form (1.1) in l^∞. In this situation, $u = (u_1,\dots,u_P,\dots) \in l^\infty, f = (f_1,\dots,f_P,\dots), \theta = (0,0,\dots) \in l^\infty$ and $a_1 = 1/4, \xi = 3/4, g = (g_1,\dots,g_P,\dots,), h = (h_1,\dots,h_P,\dots,)$, in which

$$
\begin{gathered}
f_p(t,u_1,u_2,\dots,) = \frac{\pi}{\sqrt{t}}\left(\ln\left(1+\frac{u_p^2}{p}\right)+\frac{t(1-t)\sqrt{u_p}}{p}\left|\sin u_{p+1}\right| + H(u_p)\right), \\
p(t) = \frac{t^6(1-t)^3}{8}, \qquad m(t) = \frac{t^5(1-t)^2}{3}, \qquad q(s) = \frac{s^3(1-s)^6}{5!\cdot 2!}.
\end{gathered}
\tag{4.11}
$$

Obviously, $f : (0,1)\times E \to E$ is continuous taking $\tau = 1/8$. By direct account, we have

$$
\begin{gathered}
\sum_{i=1}^{m-2} a_i \Phi(\xi_i) = \frac{1}{4}\Phi\left(\frac{3}{4}\right) \approx 0.0804 < 1, \quad M^* = 1.099\times 10^{-4}, \\
\rho^*(\tau) = \min\left\{\rho(\tau), \min_{[1/8,7/8]}\Phi(t)\right\} \approx \min\{0.6471, 0.0674\} = 0.0674.
\end{gathered}
\tag{4.12}
$$

Let

$$g_p(t) = \frac{\pi}{\sqrt{t}}, \qquad h_p(u) = \ln\left(\left(1 + \frac{u_p^2}{p}\right) + \frac{\sqrt{u_p}}{2p}\left|\sin u_{p+1}\right| + H(u_p)\right), \tag{4.13}$$

and $\int_0^1 g_p(s)ds = 2\pi < +\infty$,

$$M_0 = \left(\min_{t\in[1/8,7/8]} p(t) \cdot \rho^*(\tau) \int_{1/8}^{7/8} q(s)ds\right)^{-1} \approx 7.0811 \times 10^8. \tag{4.14}$$

Similar to the proofs of Example 4.1, we can show that (C_1)–(C_3) hold. Taking $R_0 = 10^{8/3}$, then $\sup_{u\in P_{R_0}}\{\|h(u)\|\} < M^* R_0$, which implies (C_6), holds.

For $u \in P$, choose $\varphi \in P^*$ with $\varphi(u) = \sum_{p=1}^{+\infty} u_p/p^2$. Obviously, $\varphi(u) > 0$ for $u > \theta$, and

$$\begin{aligned}
&\lim_{\|u\|\to+\infty} \min_{t\in[1/8,7/8]} \frac{\varphi(f(t,u))}{\varphi(u)} \\
&\quad = \lim_{\|u\|\to+\infty} \min_{t\in[1/8,7/8]} \frac{\pi}{\sqrt{t}} \cdot \frac{\sum_{p=1}^{+\infty} 1/p^2 \left\{\ln\left(1 + u_p^2/p\right) + \sqrt{u_p}/2p\left|\sin u_{p+1}\right| + H(u_p)\right\}}{\sum_{p=1}^{+\infty} u_p/p^2} \\
&\quad > 8 \times 10^8 > M_0,
\end{aligned} \tag{4.15}$$

which implies (C_7) is satisfied. Similarly, we can show that (C_8) holds. By Theorem 3.3, the problem (4.9) has at least two positive solutions in E.

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