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Research Article

Generalization of a Quadratic Transformation Formula due to Gauss

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The aim of this research paper is to obtain explicit expressions of $(1-x)^{-2a}$ ${}_2F_1\left[\begin{matrix} a, & b \\ 2b+j \end{matrix};-4x/(1-x)^2\right]$ for $j=0, \pm1, \pm2$. For $j=0$, we have the well-known transformation formula due to Gauss. The results are derived with the help of generalized Watson's theorem. Some known results obtained earlier follow special cases of our main findings.

1. Introduction

The generalized hypergeometric function with p numerator and q denominator parameters is defined by [1, page 73, equation (2)]

$$\begin{aligned} {}_pF_q\left[\begin{matrix} \alpha_1, & \ldots, & \alpha_p \\ \beta_1, & \ldots, & \beta_q, \end{matrix};\ z\right] &= {}_pF_q[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z] \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \end{aligned} \tag{1.1}$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined, for any complex number α, by

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1), & n \in \mathbb{N}, \\ 1, & n = 0. \end{cases} \tag{1.2}$$

Using the fundamental property $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $(\alpha)_n$ can be written in the form

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \tag{1.3}$$

where $\Gamma(\cdot)$ is the well known Gamma function.

The special case of (1.1) for $p = 2$ and $q = 1$, namely

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a, \ b \\ \\ c \end{matrix} \ ; \ z\right] &= 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2c(c+1)} z^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \end{aligned} \tag{1.4}$$

was systematically studied by Gauss [2] in 1812.

The series (1.1) is of great importance to mathematicians and physicists. All the elements a, b, and c (similarly for (1.1)) in (1.4) are called the parameters of the series and z is called the variable of the series. All four quantities a, b, c, and z may be real or complex with one exception that the denominator parameter c should not be zero or a negative integer. Also it can easily been seen that if any one of the numerator parameters a or b or both is a negative integer, the series terminates that is, reduces to a polynomial.

The series (1.4) is known as Gauss series or the ordinary hypergeometric series and may be regarded as a generalization of the elementary geometric series. In fact (1.4) reduces to the elementary geometric series in two cases, when $a = c$ and $b = 1$ and also when $b = c$ and $a = 1$.

For convergence (including absolute convergence) we refer the reader to the standard texts [3] and [1].

It is interesting to mention here that in (1.4), if we replace z by z/b and let $b \to \infty$, then since $((b)_n z^n)/b^n \to z^n$ we arrive at the following series:

$$\begin{aligned} {}_1F_1\left[\begin{matrix} a \\ \\ c \end{matrix} \ ; \ z\right] &= 1 + \frac{a}{1 \cdot c} z + \frac{a(a+1)}{1 \cdot 2c(c+1)} z^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}, \end{aligned} \tag{1.5}$$

which is called the Kummer's series or the confluent hypergeometric series.

Gauss's hypergeometric function ${}_2F_1$ and its confluent case ${}_1F_1$ form the core of the special functions and include, as their special cases, most of the commonly elementary functions.

It should be remarked here that whenever hypergeometric and generalized hypergeometric functions reduce to gamma functions, the results are very important from an application point of view. Only a few summation theorems for the series ${}_2F_1$ and ${}_3F_2$ are available.

In this context, it is well known that the classical summation theorems such as of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple, and Saalschütz for the series ${}_3F_2$ play an important rule in the theory of hypergeometric and generalized series.

Several formulae were given by Gauss [2] and Kummer [4] expressing the product of the hypergeometric series as a hypergeometric series, such as $e^{-x}{}_1F_1(x)$ as a series of the type ${}_1F_1(-x)$ and $(1+x)^{-p}{}_2F_1[4x/(1+x)^2]$ as a series of the type ${}_2F_1(x)$. In 1927, Whipple [5] has obtained a formula expressing $(1-x)^{-p}\ {}_3F_2[-4x/(1-x)^2]$ as a series of the type ${}_3F_2$.

By employing the above mentioned classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$, Bailey [6] in his well known, interesting and popular research paper made a systematic study and obtained a large number of such formulas.

Gauss [2] obtained the following quadratic transformation formula, namely

$$(1-x)^{-2a}{}_2F_1\left[\begin{matrix} a,\ b \\ \\ 2b \end{matrix}\ ;\ -\frac{4x}{(1-x)^2}\right] = {}_2F_1\left[\begin{matrix} a, & a-b+\frac{1}{2} \\ \\ b+\frac{1}{2} & \end{matrix}\ ;\ x^2\right], \tag{1.6}$$

which is also contained in [7, entry (8.1.1.41), page 573].

Berndt [8] pointed out that the result (1.6) is precisely (5) of Erdèlyi treatise [9, page 111], and is the Entry 3 of the Chapter 11 of Ramanujan's Notebooks [8, page 50] (of course, by replacing x by $-x$).

Bailey [6] established the result (1.6) with the help of the following classical Watson's summation theorem [3], namely

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} a, & b,\ c \\ \\ \frac{1}{2}(a+b+1), & 2c \end{matrix}\ ;\ 1\right] \\ &\quad = \frac{\Gamma(1/2)\Gamma(c+1/2)\Gamma(c-(1/2)a-(1/2)b+1/2)\Gamma((1/2)a+(1/2)b+1/2)}{\Gamma((1/2)a+1/2)\Gamma((1/2)b+1/2)\Gamma(c-(1/2)a+1/2)\Gamma(c-(1/2)b+1/2)}, \end{aligned} \tag{1.7}$$

provided that $\mathrm{Re}(2c-a-b) > -1$.

The proof of (1.7) when one of the parameters a or b is a negative integer was given in Watson [10]. Subsequently, it was established more generally in the nonterminating case by Whipple [5]. The standard proof of the nonterminating case was given in Bailey's tract [3] by employing the fundamental transformation due to Thomae combined with the classical Dixon's theorem of the sum of a ${}_3F_2$. For a very recent proof of (1.7), see [11].

It is not out of place to mention here that in (1.6), if we replace x by x/a and let $a \to \infty$, then after a little simplification, we get the following well-known Kummer's second theorem [4, page 140] [12, page 132], namely

$$e^{-x/2}{}_1F_1\left[\begin{matrix} b \\ 2b \end{matrix} ; x\right] = {}_0F_1\left[\begin{matrix} - \\ b+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right], \tag{1.8}$$

which also appeared as Entry 7 of the chapter 11 of Ramanujan's Notebooks [8, page 50] (of course, by replacing x by $x/2$).

Very recently, Kim et al. [13] have obtained sixty six results closely related to (1.8) out of which four results are given here. These are

$$e^{-x/2}{}_1F_1(\alpha;\ 2\alpha+1;x) = {}_0F_1\left(-;\alpha+\frac{1}{2};\frac{x^2}{16}\right) - \frac{x}{2(2\alpha+1)}{}_0F_1\left(-;\alpha+\frac{3}{2};\frac{x^2}{16}\right), \tag{1.9}$$

$$e^{-x/2}{}_1F_1(\alpha;\ 2\alpha-1;x) = {}_0F_1\left(-;\alpha-\frac{1}{2};\frac{x^2}{16}\right) + \frac{x}{2(2\alpha-1)}{}_0F_1\left(-;\alpha+\frac{1}{2};\frac{x^2}{16}\right), \tag{1.10}$$

$$\begin{aligned} e^{-x/2}{}_1F_1(\alpha;\ 2\alpha+2;x) &= {}_0F_1\left(-;\alpha+\frac{3}{2};\frac{x^2}{16}\right) - \frac{x}{2(\alpha+1)}{}_0F_1\left(-;\alpha+\frac{3}{2};\frac{x^2}{16}\right) \\ &\quad + \frac{x^2}{4(\alpha+1)(2\alpha+3)}{}_0F_1\left(-;\alpha+\frac{5}{2};\frac{x^2}{16}\right), \end{aligned} \tag{1.11}$$

$$\begin{aligned} e^{-x/2}{}_1F_1(\alpha;\ 2\alpha-2;x) &= {}_0F_1\left(-;\alpha-\frac{1}{2};\frac{x^2}{16}\right) + \frac{x}{2(\alpha-1)}{}_0F_1\left(-;\alpha-\frac{3}{2};\frac{x^2}{16}\right) \\ &\quad + \frac{x^2}{4(\alpha-1)(2\alpha-1)}{}_0F_1\left(-;\alpha+\frac{1}{2};\frac{x^2}{16}\right). \end{aligned} \tag{1.12}$$

We remark in passing that the results (1.9) and (1.10) are also recorded in [14].

Recently, a good progress has been made in generalizing the classical Watson's theorem (1.7) on the sum of a ${}_3F_2$. In 1992, Lavoie et al. [15] have obtained explicit expressions of

$${}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c+j & \end{matrix} ; 1\right] \quad \text{for } i,j = 0,\pm1,\pm2. \tag{1.13}$$

For $i = j = 0$, we get Watson's theorem (1.7). In the same paper [15], they have also obtained a large number of very interesting limiting and special cases of their main findings.

In [16], a summation formula for (1.7) with fixed j and arbitrary $i(i,j \in \mathbb{Z})$ was given. This result generalizes the classical Watson's summation theorem with the case $i = j = 0$.

Table 1

j	-2	-1	0	1	2
D_j	$1-((4n(a+n))/((c-1)(2c-2a-3)))$	1	1	1	$1-((4n(a+n))/((c+1)(2c-2a+1)))$
E_j	$-1/(c-1)$	$-1/(2c-1)$	0	$1/(2c+1)$	$1/(c+1)$

For the a recent generalization of Watson's summation theorems and other classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$ in the most general case, see [17].

The aim of this research paper is to obtain the explicit expressions of

$$(1-x)^{-2a}{}_2F_1\left[\begin{matrix} a, \quad b \\ \\ 2b+j \end{matrix} ;\ -\frac{4x}{(1-x)^2}\right] \quad \text{for } j=0,\pm1,\pm2. \tag{1.14}$$

In order to derive our main results, we shall require the following.

(1) The following special cases of (1.13) for $i=0$, recorded in [15]:

$${}_3F_2\left[\begin{matrix} -2n, \quad 2a+2n, \quad c \\ \\ a+\dfrac{1}{2}, \quad 2c+j \end{matrix} ;\ 1\right] = D_j \frac{(1/2)_n\left(a-c+(1/2)-[j/2]\right)_n}{(a+(1/2))_n\left(c+(1/2)+[j/2]\right)_n}, \tag{1.15}$$

$${}_3F_2\left[\begin{matrix} -2n-1, \quad 2a+2n+1, \quad c \\ \\ a+\dfrac{1}{2}, \quad 2c+j \end{matrix} ;\ 1\right] = E_j \frac{(3/2)_n\left(a-c+(3/2)-[(j+1)/2]\right)_n}{(a+(1/2))_n\left(c+(1/2)+[(j+1)/2]\right)_n} \tag{1.16}$$

each for $j=0,\pm1,\pm2$. Also, as usual, $[x]$ denotes the greatest integer less than or equal to x, and its modulus is defined by $|x|$. The coefficients D_j and E_j are given in Table 1.

(2) The known identities [1, page 22, lemma 5; page 58, equation 1; page 52, equation 2; page 58, equation 3]

$$(\lambda)_{2n} = 2^{2n}\left(\frac{1}{2}\lambda\right)_n\left(\frac{1}{2}\lambda+\frac{1}{2}\right)_n \quad (n\in\mathbb{N}\cup\{0\}), \tag{1.17}$$

$$(1-x)^{-a} = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}x^n, \tag{1.18}$$

$$(\lambda)_{n-k} = \frac{(-1)^k(\lambda)_n}{(1-\lambda-n)_k} \quad (0\le k\le n;\ n\in\mathbb{N}\cup\{0\}), \tag{1.19}$$

$$(n-k)! = \frac{(-1)^k n!}{(-n)_k} \quad (0\le k\le n;\ n\in\mathbb{N}\cup\{0\}). \tag{1.20}$$

2. Main Transformation Formulae

The generalization of the quadratic transformation (1.6) due to Gauss to be established is

$$
\begin{aligned}
&(1-x)^{-2a}{}_2F_1\left[\begin{matrix} a, \quad b \\ \\ 2b+j \end{matrix} ; -\frac{4x}{(1-x)^2}\right] \\
&\quad = \sum_{n=0}^{\infty} D_j \frac{(a)_n \left(a-b+(1/2)-[j/2]\right)_n}{\left(b+(1/2)+[j/2]\right)_n n!} x^{2n} \\
&\qquad + 2a\sum_{n=0}^{\infty} E_j \frac{(a+1)_n \left(a-b+(3/2)-[(j+1)/2]\right)_n}{\left(b+(1/2)+[(j+1)/2]\right)_n n!} x^{2n+1} \quad \text{for } j = 0, \pm 1, \pm 2.
\end{aligned}
\tag{2.1}
$$

Also, as usual, $[x]$ represents the greatest integer less than or equal to x, and its modulus is denoted by $|x|$. The coefficients D_j and E_j are given in Table 1.

2.1. Derivation

In order to derive our main transformation (2.1), we proceed as follows.

Proof. Denoting the left-hand side of (2.1) by S, we have

$$
S = (1-x)^{-2a}{}_2F_1\left[\begin{matrix} a, \quad b \\ \\ 2b+j \end{matrix} ; -\frac{4x}{(1-x)^2}\right]. \tag{2.2}
$$

Expressing ${}_2F_1$ as a series and after a little simplification

$$
= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (-1)^k 2^{2k} x^k}{k!(2b+j)_k} (1-x)^{-(2a+2k)}. \tag{2.3}
$$

Using Binomial theorem (1.18), we have

$$
= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (-1)^k 2^{2k} x^k}{k!(2b+j)^k} \sum_{n=0}^{\infty} \frac{(2a+2k)_n}{n!} x^n, \tag{2.4}
$$

which on simplification gives

$$
= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(2a+2k)_n (a)_k (b)_k (-1)^k 2^{2k}}{(2b+j)_k \, n!k!} x^{n+k}. \tag{2.5}
$$

Changing n to $n-k$ and using the result [1, page 57, lemma 11]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k), \tag{2.6}$$

we have

$$\begin{aligned}S&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(2a+2k)_{n-k}(a)_k(b)_k(-1)^k2^{2k}}{(2b+j)_k\,(n-k)!k!}x^n\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\Gamma(2a+n+k)}{\Gamma(2a+2k)}\frac{(a)_k(b)_k(-1)^k2^{2k}}{(2b+j)_k\,(n-k)!k!}x^n.\end{aligned} \tag{2.7}$$

Using (1.20) and after a little algebra

$$=\sum_{n=0}^{\infty}\frac{(2a)_n}{n!}x^n\sum_{k=0}^{n}\frac{(-n)_k(2a+n)_k(b)_k}{(a+(1/2))_k(2b+j)_kk!}. \tag{2.8}$$

Summing up the inner series, we have

$$S=\sum_{n=0}^{\infty}\frac{(2a)_n}{n!}x^n{}_3F_2\left[\begin{matrix}-n, & 2a+n, & b\\ & & & ;\ 1\\ a+\dfrac{1}{2}, & 2b+j & \end{matrix}\right], \tag{2.9}$$

separating into even and odd powers of x, we have

$$\begin{aligned}S&=\sum_{n=0}^{\infty}\frac{(2a)_{2n}}{(2n)!}x^{2n}{}_3F_2\left[\begin{matrix}-2n, & 2a+2n, & b\\ & & & ;\ 1\\ a+\dfrac{1}{2}, & 2b+j & \end{matrix}\right]\\&+\sum_{n=0}^{\infty}\frac{(2a)_{2n+1}}{(2n+1)!}x^{2n+1}{}_3F_2\left[\begin{matrix}-2n-1, & 2a+2n+1, & b\\ & & & ;\ 1\\ a+\dfrac{1}{2}, & 2b+j & \end{matrix}\right].\end{aligned} \tag{2.10}$$

Finally, using (1.17), (1.15), and (1.16) and after a little algebra, we easily arrive at the right-hand side of (2.1).

This completes the proof of (2.1). □

3. Special Cases

In (2.1), if we put $j = 0, \pm 1, \pm 2$, we get, after summing up the series in terms of generalized hypergeometric function, the following interesting results:

(i) For $j = 0$,

$$(1-x)^{-2a} {}_2F_1\left[\begin{matrix} a, \ b \\ \\ 2b \end{matrix} ; -\frac{4x}{(1-x)^2}\right] = {}_2F_1\left[\begin{matrix} a, \quad a-b+\frac{1}{2} \\ \\ b+\frac{1}{2} \end{matrix} ; x^2\right]. \tag{3.1}$$

(ii) For $j = 1$,

$$\begin{aligned} &(1-x)^{-2a} {}_2F_1\left[\begin{matrix} a, \quad b \\ \\ 2b+1 \end{matrix} ; -\frac{4x}{(1-x)^2}\right] \\ &= {}_2F_1\left[\begin{matrix} a, \quad a-b+\frac{1}{2} \\ \\ b+\frac{1}{2} \end{matrix} ; x^2\right] + \frac{2ax}{(2b+1)} {}_2F_1\left[\begin{matrix} a+1, \ a-b+\frac{1}{2} \\ \\ b+\frac{3}{2} \end{matrix} ; x^2\right]. \end{aligned} \tag{3.2}$$

(iii) For $j = -1$,

$$\begin{aligned} &(1-x)^{-2a} {}_2F_1\left[\begin{matrix} a, \quad b \\ \\ 2b-1 \end{matrix} ; -\frac{4x}{(1-x)^2}\right] \\ &= {}_2F_1\left[\begin{matrix} a, \quad a-b+\frac{3}{2} \\ \\ b-\frac{1}{2} \end{matrix} ; x^2\right] - \frac{2ax}{(2b-1)} {}_2F_1\left[\begin{matrix} a+1, \ a-b+\frac{3}{2} \\ \\ b+\frac{1}{2} \end{matrix} ; x^2\right]. \end{aligned} \tag{3.3}$$

(iv) For $j = 2$,

$$(1-x)^{-2a}{}_2F_1\left[\begin{matrix} a, \quad b \\ \\ 2b+2 \end{matrix}\;;\; -\frac{4x}{(1-x)^2}\right]$$

$$= {}_2F_1\left[\begin{matrix} a, \quad a-b+\dfrac{1}{2} \\ \\ b+\dfrac{3}{2} \end{matrix}\;;\; x^2\right] + \frac{2ax}{(b+1)}{}_2F_1\left[\begin{matrix} a+1, \; a-b+\dfrac{1}{2} \\ \\ b+\dfrac{3}{2} \end{matrix}\;;\; x^2\right] \tag{3.4}$$

$$+ \frac{4a(a+1)}{(b+1)(2b+3)}\, x^2{}_2F_1\left[\begin{matrix} a+2, \; a-b+\dfrac{1}{2} \\ \\ b+\dfrac{5}{2} \end{matrix}\;;\; x^2\right].$$

(v) For $j = -2$,

$$(1-x)^{-2a}{}_2F_1\left[\begin{matrix} a, \quad b \\ \\ 2b-2 \end{matrix}\;;\; -\frac{4x}{(1-x)^2}\right]$$

$$= {}_2F_1\left[\begin{matrix} a, \quad a-b+\dfrac{3}{2} \\ \\ b-\dfrac{1}{2} \end{matrix}\;;\; x^2\right] - \frac{2ax}{(b-1)}{}_2F_1\left[\begin{matrix} a+1, \; a-b+\dfrac{5}{2} \\ \\ b-\dfrac{1}{2} \end{matrix}\;;\; x^2\right] \tag{3.5}$$

$$+ \frac{4a(a+1)}{(b-1)(2b-1)}\, x^2{}_2F_1\left[\begin{matrix} a+2, \; a-b+\dfrac{5}{2} \\ \\ b+\dfrac{1}{2} \end{matrix}\;;\; x^2\right].$$

Clearly, the result (3.1) is the well-known quadratic transformation due to Gauss (1.6) and the results (3.2) to (3.5) are closely related to (3.1).

Remark 3.1. The results (3.2) and (3.3) are also recorded in [18].

In (3.1), (3.2), and (3.4), if we take $b = 1/2$, we get the following results:

(1) For $j = 0$,

$$(1-x)^{-2a} {}_2F_1\left[\begin{matrix} a, & \frac{1}{2} \\ 1 & \end{matrix} ; -\frac{4x}{(1-x)^2}\right] = {}_2F_1\left[\begin{matrix} a, & a \\ 1 & \end{matrix} ; x^2\right]. \tag{3.6}$$

(2) For $j = 1$,

$$\begin{aligned} &(1-x)^{-2a} {}_2F_1\left[\begin{matrix} a, & \frac{1}{2} \\ 2 & \end{matrix} ; -\frac{4x}{(1-x)^2}\right] \\ &\quad = {}_2F_1\left[\begin{matrix} a, & a \\ 1 & \end{matrix} ; x^2\right] + ax\,{}_2F_1\left[\begin{matrix} a+1, & a \\ 2 & \end{matrix} ; x^2\right]. \end{aligned} \tag{3.7}$$

(3) For $j = 2$,

$$\begin{aligned} &(1-x)^{-2a} {}_2F_1\left[\begin{matrix} a, & \frac{1}{2} \\ 3 & \end{matrix} ; -\frac{4x}{(1-x)^2}\right] \\ &\quad = {}_2F_1\left[\begin{matrix} a, & a \\ 2 & \end{matrix} ; x^2\right] + \frac{4ax}{3}\,{}_2F_1\left[\begin{matrix} a+1, & a \\ 2 & \end{matrix} ; x^2\right] \\ &\qquad + \frac{2a(a+1)}{3} x^2\,{}_2F_1\left[\begin{matrix} a+2, & a \\ 3 & \end{matrix} ; x^2\right]. \end{aligned} \tag{3.8}$$

We remark in passing that the result (3.6) is the Entry 5 of Chapter 11 in Ramanujan's Notebooks [8, page 50] (with x replaced by $-x$), and the results (3.7) and (3.8) are closely related to (3.6).

3.1. Limiting Cases

In the special cases (3.1) to (3.5), if we replace x by x/b and let $b \to \infty$, we get, after a little simplification, the known results (1.8) and (1.9) to (1.12), respectively.

4. Application

In this section, we shall first establish the following result, which is given as Entry 4 in the Ramanujan's Notebooks [8, page 50]:

$$
{}_2F_1\left[\begin{matrix}\frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2} \\ b+\frac{1}{2} & \end{matrix};\frac{4x}{(1+x)^2}\right]=(1+x)^a{}_2F_1\left[\begin{matrix}a, & a-b+\frac{1}{2} \\ b+\frac{1}{2} & \end{matrix};x\right], \tag{4.1}
$$

by employing (3.1).

Proof. In order to prove (4.1), we require the following result due to Kummer [4]:

$$
{}_2F_1\left[\begin{matrix}a, & b \\ 2b & \end{matrix};\frac{2x}{1+x}\right]=(1+x)^a{}_2F_1\left[\begin{matrix}\frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2} \\ b+\frac{1}{2} & \end{matrix};x^2\right]. \tag{4.2}
$$

Equation (4.2) is a well-known quadratic transformation recorded in Erdèlyi et al. [9, equation 4, page 111] and also recorded as an Entry 2 in the Ramanujan's Notebooks [8, page 50]. In (4.2), if we replace x by $2\sqrt{x}/(1+x)$, then we have

$$
{}_2F_1\left[\begin{matrix}a, & b \\ 2b & \end{matrix};\frac{4\sqrt{x}}{(1+\sqrt{x})^2}\right]=\frac{(1+\sqrt{x})^{2a}}{(1+x)^a}{}_2F_1\left[\begin{matrix}\frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2} \\ b+\frac{1}{2} & \end{matrix};\frac{4x}{(1+x)^2}\right]. \tag{4.3}
$$

Transposing the above equation, we have

$$
{}_2F_1\left[\begin{matrix}\frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2} \\ b+\frac{1}{2} & \end{matrix};\frac{4x}{(1+x)^2}\right]=\frac{(1+x)^a}{(1+\sqrt{x})^{2a}}{}_2F_1\left[\begin{matrix}a, & b \\ 2b & \end{matrix};\frac{4\sqrt{x}}{(1+\sqrt{x})^2}\right]. \tag{4.4}
$$

Now, in (3.1) first replacing x by $-x$ and then replacing x by $\sqrt{x}$ and using on the right-hand side of (4.4), we get

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2} \\ & \\ b+\frac{1}{2} & \end{matrix} ; \frac{4x}{(1+x)^2}\right] = (1+x)^a {}_2F_1\left[\begin{matrix} a, & a-b+\frac{1}{2} \\ & \\ b+\frac{1}{2} & \end{matrix} ; x\right]. \tag{4.5} $$

This completes the proof of (4.1). □

Remark 4.1. (1) The result (4.1) can also be established by employing Gauss's summation theorem.

(2) For generalization of (4.2), see a recent paper by Kim et al. [13].

In our next application, we would like to mention here that in 1996, there was an open problem posed by Baillon and Bruck [19, equation (5.17)] who needed to verify the following hypergeometric identity:

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} -m, & \frac{1}{2} \\ & \\ 2 & \end{matrix} ; 4z(1-z)\right] &= (m+1)(1-z)z^{2m-1} {}_2F_1\left[\begin{matrix} -m, & -m \\ & \\ 2 & \end{matrix} ; \left(\frac{1-z}{z}\right)^2\right] \\ &\quad + (2z-1)z^{2m-1} {}_2F_1\left[\begin{matrix} -m, & -m \\ & \\ 1 & \end{matrix} ; \left(\frac{1-z}{z}\right)^2\right], \end{aligned} \tag{4.6} $$

in order to derive a quantitative form of the Ishikawa-tdelstin-ó Brain asymptotic regularity theorem. Using Zeilberger's algorithm [20], Baillon and Bruck [19] gave a computer proof of this identity which is the key to the integral representation [19, equation (2.1)] of their main theorem.

Soon after, Paule [21] gave the proof of (4.6) by using classical hypergeometric machinery by means of contiguous functions relations and Gauss's quadratic transformation (3.1).

Our objective of this section is to obtain first three results from (3.1), (3.2), and (3.4) and then establish again three new results out of which one will be the natural generalization of the Baillon-Bruck identity (4.6).

For this, in our results (3.1), (3.2), and (3.4), if we replace x by $(z-1)/z$, we get after a little simplification the following results:

$$ {}_2F_1\left[\begin{matrix} a, \ b \\ 2b \end{matrix} ; 4z(1-z)\right] = z^{-2a}{}_2F_1\left[\begin{matrix} a, \quad a-b+\frac{1}{2} \\ b+\frac{1}{2} \end{matrix} ; \left(\frac{1-z}{z}\right)^2\right], \tag{4.7} $$

$$ \begin{aligned} {}_2F_1&\left[\begin{matrix} a, \quad b \\ 2b+1 \end{matrix} ; 4z(1-z)\right] \\ &= z^{-2a}\left\{ {}_2F_1\left[\begin{matrix} a, \quad a-b+\frac{1}{2} \\ b+\frac{1}{2} \end{matrix} ; \left(\frac{1-z}{z}\right)^2\right] \right. \\ &\qquad \left. -\frac{2a}{2b+1}\left(\frac{1-z}{z}\right){}_2F_1\left[\begin{matrix} a+1, \ a-b+\frac{1}{2} \\ b+\frac{3}{2} \end{matrix} ; \left(\frac{1-z}{z}\right)^2\right]\right\}, \end{aligned} \tag{4.8} $$

$$ \begin{aligned} {}_2F_1&\left[\begin{matrix} a, \quad b \\ 2b+2 \end{matrix} ; 4z(1-z)\right] \\ &= z^{-2a}\left\{ {}_2F_1\left[\begin{matrix} a, \quad a-b+\frac{1}{2} \\ b+\frac{1}{2} \end{matrix} ; \left(\frac{1-z}{z}\right)^2\right] \right. \\ &\qquad -\frac{2a}{2b+1}\left(\frac{1-z}{z}\right){}_2F_1\left[\begin{matrix} a+1, \ a-b+\frac{1}{2} \\ b+\frac{3}{2} \end{matrix} ; \left(\frac{1-z}{z}\right)^2\right] \\ &\qquad \left. +\frac{4a(a+1)}{(b+1)(2b+3)}\left(\frac{1-z}{z}\right)^2{}_2F_1\left[\begin{matrix} a+2, \ a-b+\frac{1}{2} \\ b+\frac{5}{2} \end{matrix} ; \left(\frac{1-z}{z}\right)^2\right]\right\}. \end{aligned} \tag{4.9} $$

Finally, in (4.7), (4.8), and (4.9) if we take $a = -m$ and $b = 1/2$, we get the following very interesting results:

$$
{}_2F_1\left[\begin{matrix}\frac{1}{2}, & -m \\ & & ;4z(1-z) \\ 1 & \end{matrix}\right] = z^{2m}{}_2F_1\left[\begin{matrix}-m, & -m \\ & & ;\left(\frac{1-z}{z}\right)^2 \\ 1 & \end{matrix}\right], \tag{4.10}
$$

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix}\frac{1}{2}, & -m \\ & & ;4z(1-z) \\ 1 & \end{matrix}\right] = z^{2m}\Bigg\{&{}_2F_1\left[\begin{matrix}-m, & -m \\ & & ;\left(\frac{1-z}{z}\right)^2 \\ 1 & \end{matrix}\right] \\
&+m\left(\frac{1-z}{z}\right){}_2F_1\left[\begin{matrix}-m, & -m+1 \\ & & ;\left(\frac{1-z}{z}\right)^2 \\ 2 & \end{matrix}\right]\Bigg\},
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix}\frac{1}{2}, & -m \\ & & ;4z(1-z) \\ 2 & \end{matrix}\right] \\
&= z^{2m}\Bigg\{{}_2F_1\left[\begin{matrix}-m, & -m \\ & & ;\left(\frac{1-z}{z}\right)^2 \\ 1 & \end{matrix}\right] \\
&\quad + \frac{4m}{3}\left(\frac{1-z}{z}\right){}_2F_1\left[\begin{matrix}-m, & -m+1 \\ & & ;\left(\frac{1-z}{z}\right)^2 \\ 2 & \end{matrix}\right] \\
&\quad - \frac{2m(1-m)}{3}\left(\frac{1-z}{z}\right)^2{}_2F_1\left[\begin{matrix}-m, & -m+2 \\ & & ;\left(\frac{1-z}{z}\right)^2 \\ 3 & \end{matrix}\right]\Bigg\}.
\end{aligned} \tag{4.12}
$$

Equation (4.7) is a natural generalization of Baillon-Bruck result (4.6). The result (4.11) is an alternate form of the Baillon-Bruck result (4.6). Its exact form can be obtained from (4.11) by using the contiguous function relation

$$
\frac{ab}{c(c-1)}z\,{}_2F_1\left[\begin{matrix}a+1, & b+1 \\ & & ;z \\ c+1 & \end{matrix}\right] = {}_2F_1\left[\begin{matrix}a, & b \\ & & ;z \\ c+1 & \end{matrix}\right] - {}_2F_1\left[\begin{matrix}a, & b \\ & & ;z \\ c & \end{matrix}\right] \tag{4.13}
$$

with $a = b = -m$ and $c = 1$.

We conclude this section by remarking that the result (4.7) is also recorded in [22] by Rathie and Kim who obtained it by other means and the results (4.8) and (4.9) are believed to be new.

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