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Research Article

Two-Point Boundary Value Problems for a Class of Second-Order Ordinary Differential Equations

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We study the general semilinear second-order ODE $u'' + g(t, u, u') = 0$ under different two-point boundary conditions. Using the method of upper and lower solutions, we obtain an existence result. Moreover, under a growth condition on g, we prove that the set of solutions of $u'' + g(t, u, u') = 0$ is homeomorphic to the two-dimensional real space.

1. Introduction

The Dirichlet problem for the semilinear second-order ODE

$$u'' + g(t, u, u') = 0 \tag{1.1}$$

has been studied by many authors from the pioneering work of Picard [1], who proved the existence of a solution by an application of the well-known method of successive approximations under a Lipschitz condition on g and a smallness condition on T. Sharper results were obtained by Hamel [2] in the special case of a forced pendulum equation (see also [3, 4]). The existence of periodic solutions for this case has been first considered by Duffing [5] in 1918. Variational methods have been also applied when $g = g(t, u)$ by Lichtenstein [6], who considered the functional

$$I(u) = \int_0^T \left(\frac{u'^2}{2} - G(t, u) \right) dt \tag{1.2}$$

with $G(t,u) = \int_0^u g(t,s)ds$. When g depends on u', the problem is nonvariational, and different techniques are required, for example, the shooting method introduced in 1905 by Severini [7] and the more general topological approach, which makes use of Leray-Schauder Degree theory. For an overview of the problem and further results, we refer the reader to [8]. A different kind of nonlinear boundary value PDE (quasilinear elliptic equations) was studied extensively in [9, 10].

This problem is recently studied in [11]. Also, this problem is generalized in [12–14]. Several much more general forms of the problem have been studied in [15–17] via lower and upper solution method. We will study the existence of solutions of (1.1) under Dirichlet, periodic, and nonlinear boundary conditions of the type

$$u'(0) = f_1(u(0)), \qquad u'(T) = f_2(u(T)), \tag{1.3}$$

where f_1 and f_2 are given continuous functions. Note that if $f_i(x) = a_i x + b_i$, then (1.3) corresponds to a particular case of Sturm-Liouville conditions and Neumann conditions when $a_1 = a_2 = 0$.

In the second section, we impose a growth condition on g in order to obtain unique solvability of the Dirichlet problem. Furthermore, we prove that the trace mapping

$$\mathrm{Tr} : \left\{ u \in H^2(0,T) : u'' + g(t,u,u') = 0 \right\} \longrightarrow \mathbb{R}^2 \tag{1.4}$$

given by $\mathrm{Tr}(u) = (u(0), u(T))$ is a homeomorphism, and we apply this result to obtain solutions for other boundary conditions in some specific cases.

In the third section, we construct solutions of the aforementioned problems by an iterative method based on the existence of an ordered couple (α, β) of a lower and an upper solution. This method has been successfully applied to different boundary value problems when g does not depend on u'. For general g, existence results have been obtained assuming that $|g(t,u,u')| \leq B(t,|u'|)$, where $B : [0,T] \times [0,+\infty) \to [0,+\infty)$ is a Bernstein function for some $N \geq \|\alpha'\|_\infty, \|\beta'\|_\infty$, namely,

(i) B is nondecreasing in u',

(ii) for $\alpha \leq u \leq \beta$, if $|u''(t)| \leq B(t,|u'(t)|)$ for any t, then $\|u'\|_\infty \leq N$ (see, e.g., [18]).

We will assume instead a Lipschitz condition with respect to u' and construct in each case a nonincreasing (resp., nondecreasing) sequence of upper (lower) solutions that converges to a solution of the problem.

2. A Growth Condition for g

For simplicity, let us assume that g is continuous. We may write it as

$$g(t,u,u') = r(t)u' + h(t,u,u'), \tag{2.1}$$

with r being continuous. We will assume that h satisfies a global Lipschitz condition on u', namely,

$$\left|\frac{h(t,u,x)-h(t,u,y)}{x-y}\right| \leq k < \frac{\pi}{T} \quad \text{for } x \neq y. \tag{2.2}$$

Remark 2.1. Without loss of generality, we may assume that $r \in W^{1,\infty}(0,T)$. Indeed, if not, we may multiply (1.1) for any positive p such that $\overline{r} := rp - p' \in W^{1,\infty}(0,T)$ in order to get the modified equation:

$$(pu')' + \overline{r}u' + ph(t,u,u') = 0. \tag{2.3}$$

Note that in this case the value π/T in (2.2) must be replaced by $\sqrt{\lambda_1}/\|p\|_\infty$, where λ_1 is the first eigenvalue of $(-pu')'$ for the Dirichlet conditions.

Furthermore, assume that h satisfies the following one-sided growth condition on u:

$$\frac{h(t,u,x)-h(t,v,x)}{u-v} \leq c, \tag{2.4}$$

for $u \neq v$, with

$$c + k\frac{\pi}{T} < \left(\frac{\pi}{T}\right)^2 + \frac{1}{2}\inf_{0\leq t\leq T} r'(t). \tag{2.5}$$

Under these assumptions, the set $\mathcal{S}$ of solutions of (1.1) is homeomorphic to $\mathbb{R}^2$. More precisely, one has the following.

Theorem 2.2. *Assume that* (2.2)–(2.4) *hold and let $a, b \in \mathbb{R}$. Then there exists a unique solution $u_{a,b}$ of* (1.1) *satisfying the nonhomogeneous Dirichlet condition:*

$$u_{a,b}(0) = a, \qquad u_{a,b}(T) = b. \tag{2.6}$$

Furthermore, the trace mapping $\mathrm{Tr} : (\mathcal{S}, \|\cdot\|_{H^2}) \to \mathbb{R}^2$ *given by* $\mathrm{Tr}(u) = (u(0), u(T))$ *is a homeomorphism.*

Proof. For any $\overline{u} \in H^1(0,T)$ let u be the unique solution of the linear problem:

$$\begin{aligned} u'' &= -\left[r\overline{u}' + h(t,\overline{u},\overline{u}')\right], \\ u(0) &= a, \qquad u(T) = b. \end{aligned} \tag{2.7}$$

It is immediate that the operator $A : H^1(0,T) \to H^1(0,T)$ given by $A(\overline{u}) = u$ is compact. Moreover, if $S_\sigma u := u'' + \sigma[ru' + h(t,u,u')]$ with $\sigma \in [0,1]$, a simple computation shows that the following a priori bound holds for any $u, v \in H^2(0,T)$ with $u - v \in H_0^1(0,T)$:

$$\left\|u' - v'\right\|_{L^2} \leq \mu\|S_\sigma u - S_\sigma v\|_{L^2}, \tag{2.8}$$

where

$$\mu = \begin{cases} \dfrac{1}{\pi/T - k}, & \text{if } c \leq \dfrac{1}{2}\inf r', \\ \dfrac{\pi/T}{\pi/T - k + ((1/2)\inf r' - c)T/\pi}, & \text{otherwise.} \end{cases} \tag{2.9}$$

Hence, if $u = \sigma A u$ (i.e., $S_\sigma(u) = 0$) for some $\sigma \in [0,1]$, setting $l_{a,b}(t) = ((b-a)/T)t + a$, we obtain

$$\left\| u' - \sigma l'_{a,b} \right\|_{L^2} \leq \mu \| S_\sigma(\sigma l_{a,b}) \|_{L^2} \leq M \tag{2.10}$$

for some fixed constant M. Thus, existence follows from Leray-Schauder Theorem. Uniqueness is an immediate consequence of (2.8) for $\sigma = 1$.

Hence, Tr is bijective, and its continuity is clear. On the other hand, if $(a,b) \to (a_0,b_0)$, applying (2.8) to $u = u_{a,b} - l_{a,b}$ and $v = u_{a_0,b_0} - l_{a_0,b_0}$, it is easy to see that $u_{a,b} \to u_{a_0,b_0}$ for the H^1-norm. As $u_{a,b}$ and u_{a_0,b_0} satisfy (1.1), we conclude that also $u''_{a,b} \to u''_{a_0,b_0}$ for the L^2-norm and the proof is complete. □

Remark 2.3. The proof of Theorem 2.2 still holds under more general assumptions for g. In fact, if g satisfies Caratheodory-type conditions, we may assume only that

$$\frac{h(t,u,x) - h(t,v,y)}{u - v} \leq c + k \left| \frac{x-y}{u-v} \right| \tag{2.11}$$

(for $u \neq v$ and c,k as before), which is not equivalent to (2.2)–(2.4) when h is noncontinuous. Thus, the result may be considered a slight extension of well-known results (see, e.g., [19], Corollary V.2).

As a simple consequence we have the following.

Corollary 2.4. *Assume that* (2.2) *and* (2.4) *hold. Further, assume that there exists a constant* $M > 0$ *such that*

$$\begin{gathered} h(t,u,0)\operatorname{sgn}(u) \geq 0 \quad \text{for } |u| \geq M, \\ \left\| \frac{h(\cdot,u,0)}{u} \right\|_{L^2} \leq \frac{\delta}{T^{1/2}\mu} \quad \text{for } |u| \geq M, \end{gathered} \tag{2.12}$$

where μ *is the constant defined by* (2.9) *and* $\delta < 1$. *Then* (1.1) *admits at least one T-periodic solution, which is unique if* $c < 0$ *in* (2.4).

Proof. With the notations of the previous theorem, let us consider the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(a) = u'_{a,a}(T) - u'_{a,a}(0). \tag{2.13}$$

From Theorem 2.2, φ is continuous, and it is clear that u is a periodic solution of the problem if and only if $u = u_{a,a}$ for some a with $\varphi(a) = 0$.

From (2.8), if $|a| \geq M$, we take $v = a$ and $\sigma = 1$. Observe that $S_1 u_{aa} = 0$. Therefore it follows that

$$\left\|u_{a,a}{}'\right\|_{L^2} \leq \mu \|h(\cdot, a, 0)\|_{L^2} \leq \frac{\delta}{T^{1/2}}|a|, \tag{2.14}$$

where in the last inequality we used (2.6). Hence $|u_{a,a} - a| \leq \int_0^T |u'_{a,a}| \leq \delta|a|$ or, equivalently:

$$(1-\delta)|a| \leq |u_{a,a}| \leq (1+\delta)|a|. \tag{2.15}$$

Let p be the unique solution (in distributional sense) of the following problem:

$$\left[p' - (r+\xi)p\right]' = 0, \quad p(0) = p(T) = 1, \tag{2.16}$$

where $\xi \in L^\infty(0,T)$ is given by

$$\xi(t) = \frac{h\left(t, u_{a,a}, u'_{a,a}\right) - h(t, u_{a,a}, 0)}{u'_{a,a}}, \tag{2.17}$$

with $\xi(t) = 0$ if $u'_{a,a}(t) = 0$. A simple computation shows that p is positive, and multiplying the equation by p, we obtain

$$\varphi(a) = \int_0^T \left(p u'_{a,a}\right)' = -\int_0^T p h(t, u_{a,a}, 0). \tag{2.18}$$

Hence by (2.12) $\varphi(a) \leq 0 \leq \varphi(-a)$ for $(1-\delta)a \geq M$ and existence follows from the continuity of φ. On the other hand, if u and v are periodic solutions of the problem, then

$$(u-v)'' + \left(r+\varphi\right)(u-v)' + h\left(t,u,v'\right) - h\left(t,v,v'\right) = 0, \tag{2.19}$$

where

$$\varphi(t) = \frac{h(t,u,u') - h(t,u,v')}{u'-v'} \in L^\infty(0,T). \tag{2.20}$$

Now take $p > 0$ as the unique solution of the problem $[p' - (r + \varphi)p]' = 0$, $p(0) = p(T) = 1$. Multiplying the previous equality by $p(u-v)$ and applying the boundary conditions for p, u, and v, we observe

$$\begin{aligned}
\int_0^T p(u-v)(u-v)'' &= -\int_0^T \left[p'(u-v) + p(u'-v')\right](u-v)' \\
&= -\int_0^T p'(u-v)(u-v)' + \int_0^T p(u'-v')^2, \\
\int_0^T p(r+\varphi)(u-v)(u-v)' &= -\int_0^T (p(r+\varphi))' \frac{(u-v)^2}{2} \\
&= -\int_0^T p'' \frac{(u-v)^2}{2} \\
&= \int_0^T (u-v)(u-v)'p'.
\end{aligned} \tag{2.21}$$

Thus we obtain

$$\begin{aligned}
0 &= \int_0^T p(u'-v')^2 - \int_0^T p\left[h(t,u,v') - h(t,v,v')\right](u-v) \\
&\geq \int_0^T p(u'-v')^2 - c\int_0^T p(u-v)^2.
\end{aligned} \tag{2.22}$$

If $c < 0$, we conclude that $u = v$. □

Remark 2.5. In the previous proof, note that the sign condition on h is only used for $(1-\delta)|a| \leq |u| \leq (1+\delta)|a|$. Thus, (2.12) may be replaced by the weaker condition

$$h(t,\cdot,0)|_{I_1} \geq 0 \geq h(t,\cdot,0)|_{I_2} \tag{2.23}$$

where $I_j = [a_j - \delta_j|a_j|, a_j + \delta_j|a_j|]$ for some $a_j \in \mathbb{R}$, $\delta_j < 1$ with

$$\left\| \frac{h(\cdot,u,0)}{u} \right\|_{L^2} \leq \frac{\delta_j}{T^{1/2}\mu} \quad \text{for } u \in I_j. \tag{2.24}$$

Remark 2.6. As a particular case of Corollary 2.4, we deduce the existence of T-periodic solutions under the following Landesman-Lazer type conditions (see, e.g., [20]):

$$\begin{aligned}
&\liminf_{|u|\to\infty} h(t,u,0)\,\mathrm{sgn}(u) \geq 0, \\
&\lim_{|u|\to\infty} \left\| \frac{h(\cdot,u,0)}{u} \right\|_{L^2} = 0.
\end{aligned} \tag{2.25}$$

As in the standard Duffing equation $u'' + h(u) = \theta(t)$, the asymptotic condition (2.6) can be dropped if the sign in (2.12) is reversed. More precisely, we have the following.

Corollary 2.7. *Assume that* (2.2) *and* (2.4) *hold. Further, assume that there exists a constant* $M > 0$ *such that*

$$h(t, u, 0)\,\mathrm{sgn}(u) < 0 \quad \textit{for } |u| \geq M. \tag{2.26}$$

Then (1.1) *is solvable under periodic or Sturm-Liouville conditions:*

$$u'(0) = a_1 u(0) + b_1, \qquad u'(T) = a_2 u(T) + b_2, \quad a_1 \geq 0 \geq a_2. \tag{2.27}$$

Furthermore, if $c < 0$ *in* (2.4)*, then the respective solutions are unique.*

Proof. For the periodic problem, define φ as in the previous corollary. For $a \geq M$, if $u_{a,a}(t_0) > a$ for some t_0, we may assume that t_0 is maximum, and hence

$$u''(t_0) = -g(t_0, u(t_0), 0) = -h(t_0, u(t_0), 0) > 0, \tag{2.28}$$

a contradiction. Thus, $u \leq a$, which implies that $\varphi(a) \geq 0$. In the same way, we deduce that $\varphi(a) \leq 0$ for $a \leq -M$. Uniqueness follows as in Corollary 2.4.

For (2.27) conditions, let us first note that if $\lambda > 0$, the linear problem

$$v'' + rv' - \lambda v = 0, \qquad v'(0) = a_1 v(0) + b_1, \qquad v'(T) = a_2 v(T) + b_2 \tag{2.29}$$

is uniquely solvable, and setting $w = u - v$ problem (1.1)–(2.27) is equivalent to

$$w'' + \overline{g}(t, w, w') = 0, \qquad w'(0) = a_1 w(0), \qquad w'(T) = a_2 w(0), \tag{2.30}$$

where $\overline{g}(t, w, w') := g(t, w + v, w' + v') - \lambda v$ satisfies the hypothesis. Hence, it suffices to consider only the homogeneous case $b_1 = b_2 = 0$. In the same way as before, define $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\varphi(a, b) = \left(u'_{a,b}(0), u'_{a,b}(T)\right). \tag{2.31}$$

For $a \geq M \geq |b|$, we obtain that $u'_{a,b}(0) \leq 0 \leq u'_{-a,b}(0)$, and for $b \geq M \geq |a|$, it holds that $u'_{a,b}(T) \geq 0 \geq u'_{a,-b}(T)$. By the generalized intermediate value theorem, we deduce that φ has a zero in $[-M, M] \times [-M, M]$. Uniqueness can be proved as in the periodic case. □

3. Iterative Sequences of Upper and Lower Solutions

In this section, we construct solutions of (1.1) under the mentioned two-point boundary conditions by an iterative method. As before, consider $g(t, u, u') = r(t)u' + h(t, u, u')$ where $r \in W^{1,\infty}(0, T)$ and h is globally Lipschitz on u' with constant $k < \pi/T$.

We will need the following auxiliary lemmas.

Lemma 3.1. *Assume that* (2.2) *holds and let* $\lambda > 0$ *be large enough. Then for any* $z, \theta \in C([0,T])$,

$$u'' + ru' + h(t, z, u') - \lambda u = \theta(t) \tag{3.1}$$

is uniquely solvable under Dirichlet, periodic, or (2.27) *conditions. Furthermore, the application* $T : C([0,T])^2 \to C([0,T])$ *given by* $T(z, \theta) = u$ *is compact.*

Proof. Taking $\lambda > k\pi/T - (\pi/T)^2 - (1/2)\inf r'$, existence and uniqueness follow as a particular case of Theorem 2.2 and Corollary 2.7 for

$$\overline{g}(t, u, u') = ru' + h(t, z, u') - \lambda u - \theta(t). \tag{3.2}$$

Let $(z, \theta) \to (z_0, \theta_0)$, and set $u = T(z, \theta)$, $u_0 = T(z_0, \theta_0)$. Then

$$(u - u_0)'' + (r + \varphi)(u - u_0)' - \lambda(u - u_0) = h(t, z, u_0') - h(t, z_0, u_0') + \theta - \theta_0, \tag{3.3}$$

where $\varphi(t) = h(t, z, u') - h(t, z, u_0')/u' - u_0'$. Hence, it suffices to prove that the following a priori bound holds for any w satisfying periodic or homogeneous Dirichlet or (2.27) conditions:

$$\|w\|_{H^1} \le c\left\|w'' + (r + \varphi)w' - \lambda w\right\|_{L^2}, \tag{3.4}$$

where the constant c depends only on k. For Dirichlet and (2.27) conditions, apply Cauchy-Schwartz inequality to the integral $-\int_0^T pLw \cdot w$ where $Lw = w'' + (r+\varphi)w' - \lambda w$ and $p = e^{\int_0^T (r+\varphi)}$, and observe that $0 < m \le p \le M$ for some m and M depending only on k (note that under homogeneous (2.27) conditions, it holds that $-pww'|_0^T = p(0)a_1 w(0)^2 - p(T)a_2 w(T)^2 \ge 0$). For periodic conditions, take p such that $p' = (r + \varphi)p - \overline{r}$ with $\overline{r}$ constant and $p(0) = p(T) = 1$ and the proof follows. □

Lemma 3.2. *Let* $\phi \in L^\infty(0, T)$ *and assume that* $w'' + \phi w' - \lambda w \ge 0$ *a.e. for* $\lambda > 0$. *Then* $w \le 0$, *provided that* w *satisfies one of the boundary conditions:*

(i) $w(0), w(T) \le 0$,

(ii) $w(T) - w(0) = 0 \ge w'(T) - w'(0)$,

(iii) $w'(0) - a_1 w(0) \ge 0 \ge w'(T) - a_2 w(T)$, $a_1 \ge 0 \ge a_2$.

Proof. For $w(0), w(T) \le 0$, the result is the well-known maximum principle for Dirichlet conditions. If (ii) holds and $w(0) = w(T) > 0$, as w cannot achieve a positive maximum on $(0, T)$, we have that $w'(0) = w'(T) = 0$ and $w, w' \ge 0$ over a maximal interval $(t_0, T]$. Taking $p = e^{\int_0^t \phi}$, we deduce that pw' is nondecreasing on $[t_0, T]$, a contradiction. If (iii) holds and, for example, $w(0) > 0$, restricting w up to its first zero if necessary, it suffices to consider only the case $w \ge 0$. As before, we get a contradiction from the fact that pw' is nondecreasing. The proof is similar if we assume that $w(T) > 0$. □

In order to prove the main result of this section, we recall that (α, β) is an ordered couple of a lower and an upper solution for (1.1) if $\alpha \le \beta$ and

$$\alpha'' + g(\cdot, \alpha, \alpha') \ge 0 \ge \beta'' + g(\cdot, \beta, \beta'), \tag{3.5}$$

under the following boundary conditions.

For the Dirichlet problem,

$$\alpha(0) \le a \le \beta(0), \qquad \alpha(T) \le b \le \beta(T). \tag{3.6}$$

For the periodic problem,

$$\alpha(T) - \alpha(0) = 0 = \beta(T) - \beta(0), \qquad \alpha'(T) - \alpha'(0) \le 0 \le \beta'(T) - \beta'(0). \tag{3.7}$$

For the problem (1.3),

$$\alpha'(0) - f_1(\alpha(0)) \ge 0 \ge \beta'(0) - f_1(\beta(0)), \qquad \alpha'(T) - f_2(\alpha(T)) \le 0 \le \beta'(T) - f_2(\beta(T)). \tag{3.8}$$

We make the following extra assumption.

There exists a constant $R > 0$ such that

$$\frac{h(t, u, \alpha') - h(t, v, \alpha')}{u - v} \le R, \qquad \frac{h(t, u, \beta') - h(t, v, \beta')}{u - v} \ge -R, \tag{3.9}$$

for any u, v such that $\alpha(t) \le u(t), v(t) \le \beta(t)$, and for (1.3): there exists a constant $R > 0$ such that

$$\begin{gathered} \frac{f_1(x) - f_1(y)}{x - y} \le R \quad \text{for } \alpha(0) \le x, y \le \beta(0), \\ \frac{f_2(x) - f_2(y)}{x - y} \ge -R \quad \text{for } \alpha(T) \le x, y \le \beta(T). \end{gathered} \tag{3.10}$$

Then we have the following.

Theorem 3.3. *Assume that there exists an ordered couple (α, β) of a lower and an upper solution for Dirichlet, periodic, or* (1.3) *conditions. Further, assume that* (2.2) *and* (3.9) *hold (and also* (3.10), *for the* (1.3) *case). Then the respective boundary value problem admits at least one solution u with $\alpha \le u \le \beta$.*

Remark 3.4. Observe that a Lipschitz condition is a particular case of a Nagumo condition. This result can also be obtained as a Corollary of Theorem 3.2 in [21].

Proof. For $\lambda \ge R$ large enough and $\overline{u} \in C([0, T])$ define $T\overline{u} = u$ to be the unique solution of the following problem:

$$u'' + ru' + h(t, \overline{u}, u') - \lambda u = -\lambda \overline{u} \tag{3.11}$$

satisfying, respectively, Dirichlet, periodic, or the Sturm-Liouville condition:

$$u'(0) - Ru(0) = f_1(\overline{u}(0)) - R\overline{u}(0), \qquad u'(T) + Ru(T) = f_2(\overline{u}(T)) + R\overline{u}(T). \tag{3.12}$$

Compactness of T follows easily from Lemma 3.1. Moreover, if $\overline{u} \le \beta$, then

$$\begin{aligned} u'' + ru' + h(t,\overline{u},u') + R\overline{u} - \lambda u &= (R-\lambda)\overline{u} \ge (R-\lambda)\beta \\ &\ge (R-\lambda)\beta + \beta'' + r\beta' + h(t,\beta,\beta'). \end{aligned} \tag{3.13}$$

Hence, setting

$$\varphi(t) = \frac{h(t,\overline{u},u') - h(t,\overline{u},\beta')}{u' - \beta'}, \tag{3.14}$$

we deduce that

$$(u-\beta)'' + (r+\varphi)(u-\beta)' - \lambda(u-\beta) \ge \left[h(t,\beta,\beta') + R\beta\right] - \left[h(t,\overline{u},\beta') + R\overline{u}\right] \ge 0. \tag{3.15}$$

For Dirichlet and periodic cases, it follows that $u \le \beta$. For (1.3), note that

$$\begin{aligned} u'(0) - Ru(0) &= f_1(\overline{u}(0)) - R\overline{u}(0) \ge f_1(\beta(0)) - R\beta(0)\ , \\ u'(T) + Ru(T) &= f_2(\overline{u}(T)) + R\overline{u}(T) \le f_2(\beta(T)) + R\beta(T). \end{aligned} \tag{3.16}$$

Hence,

$$(u-\beta)'(0) - R(u-\beta)(0) \ge 0 \ge (u-\beta)'(T) - R(u-\beta)(T), \tag{3.17}$$

and from Lemma 3.2, we also obtain that $u \le \beta$. In the same way, if $\overline{u} \ge \alpha$, we obtain that $u \ge \alpha$ and the proof follows from Schauder Fixed Point Theorem. □

Remark 3.5. Existence conditions in Corollary 2.7 are easily improved by applying Theorem 3.3. Indeed, under condition (2.26), it is immediate that $(-M, M)$ is an ordered couple of a lower and an upper solution.

Remark 3.6. In the context of the previous theorem, from Lemma 3.1, we deduce the existence of a constant K such that if $u = T\overline{u}$ for $\alpha \le \overline{u} \le \beta$, then

$$\|u'\|_{L^\infty} \le K. \tag{3.18}$$

Example 3.7. It is easy to see that the problem

$$\begin{gathered} u'' - u = 0, \\ u'(0) = u(0), \qquad u'(T) = u(T) + 1 \end{gathered} \tag{3.19}$$

has no solution, although $\alpha = -1$ is a lower solution. From the previous theorem, we deduce that no upper solution $\beta \geq -1$ exists. This can be proved directly from the following conditions:

$$\beta'' - \beta \leq 0, \qquad \beta'(0) \leq \beta(0), \qquad \beta'(T) \geq \beta(T) + 1. \tag{3.20}$$

Indeed, as no negative minimum exists, if $\beta(0), \beta(T) \geq 0$, we may take t_0 maximum such that $\beta' > 0$ over $(t_0, T]$. Hence, $(\beta')^2(T) - \beta^2(T) \leq (\beta')^2(t_0) - \beta^2(t_0) \leq 0$, a contradiction. On the other hand, if $\beta(0) < 0$, then $\beta' < 0$ on $[0, T)$ and $\beta'(T) = 0$, a contradiction since $\beta'' \leq \beta$. The case $\beta(0) \geq 0 > \beta(T)$ can be easily reduced to the previous one.

In order to construct solutions by iteration, we need a stronger assumption on h.

There exists a constant R such that

$$|h(t, u, x) - h(t, v, x)| \leq R|u - v| \tag{3.21}$$

for u, v such that $\alpha(t) \leq u(t), v(t) \leq \beta(t)$, and $x \in \mathbb{R}$.

Corollary 3.8. *Assume that there exists an ordered couple (α, β) of a lower and an upper solution for Dirichlet, periodic, or* (1.3) *conditions. Further, assume that* (2.2) *and* (3.21) *hold (and also* (3.10)*, for the* (1.3) *case). Set $\lambda \geq R$ large enough, and define the sequences $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ given by*

$$\underline{u}_0 = \alpha, \qquad \overline{u}_0 = \beta, \tag{3.22}$$

and $\overline{u}_{n+1}$, $\underline{u}_{n+1}$ are the (unique) solutions of the following problems:

$$\begin{aligned} \overline{u}''_{n+1} + r\overline{u}'_{n+1} + h\left(t, \overline{u}_n, \overline{u}'_{n+1}\right) - \lambda\overline{u}_{n+1} &= -\lambda\overline{u}_n, \\ \underline{u}''_{n+1} + r\underline{u}'_{n+1} + h\left(t, \underline{u}_n, \underline{u}'_{n+1}\right) - \lambda\underline{u}_{n+1} &= -\lambda\underline{u}_n, \end{aligned} \tag{3.23}$$

under the respective boundary conditions. Then $(\underline{u}_n\ \overline{u}_n)$ is an ordered couple of a lower and an upper solution. Furthermore, $\{\overline{u}_n\}$ (resp., $\{\underline{u}_n\}$) is nonincreasing (nondecreasing) and converges to a solution of the problem.

Remark 3.9. Observe that this is also a classical result that can be found in the works of Adje [22] or Cabada [23], for example.

Proof. From the previous theorem, we know that $\alpha \leq \overline{u}_1 \leq \beta$. Moreover,

$$\overline{u}''_1 + r\overline{u}'_1 + h\left(t, \overline{u}_1, \overline{u}'_1\right) = (\lambda - R)\left(\overline{u}_1 - \beta\right) + \left[h\left(t, \overline{u}_1, \overline{u}'_1\right) + R\overline{u}_1\right] - \left[h\left(t, \beta, \overline{u}'_1\right) + R\beta\right] \leq 0. \tag{3.24}$$

Hence, $\overline{u}_1$ is an upper solution of the problem. Inductively, it follows that $\overline{u}_n$ is an upper solution for every n, with $\alpha \leq \overline{u}_{n+1} \leq \overline{u}_n$. Hence $\overline{u}_n$ converges pointwise to a function $\overline{u}$.

From

$$\overline{u}''_{n+1} + r\overline{u}'_{n+1} + h\left(t, \overline{u}_n, \overline{u}'_{n+1}\right) \longrightarrow 0 \tag{3.25}$$

pointwise. Moreover, by Lemma 3.1, we know that $\{\overline{u}_n\}$ is bounded in $H^1(0,T)$; hence in $H^2(0,T)$, it follows easily that

$$\overline{u}'' + r\overline{u}' + h(t,\overline{u},\overline{u}') = 0. \tag{3.26}$$

Thus, $\overline{u}$ is a solution of the problem. The proof for $\underline{u}_n$ is analogous. Moreover, if we assume that $\underline{u}_n \leq \overline{u}_n$, it is immediate that $\underline{u}_{n+1} \leq \overline{u}_{n+1}$. □

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