Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 843562, 13 pages
doi:10.1155/2012/843562

*Review Article*

# On Summability of Spectral Expansions Corresponding to the Sturm-Liouville Operator

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Received 26 March 2012; Revised 23 May 2012; Accepted 27 May 2012

Academic Editor: H. Srivastava



We study the completeness property and the basis property of the root function system of the Sturm-Liouville operator defined on the segment [0, 1]. All possible types of two-point boundary conditions are considered.

## 1. Introduction

The spectral theory of two-point differential operators was begun by Birkhoff in his two papers [1, 2] of 1908 where he introduced regular boundary conditions for the first time. It was continued by Tamarkin [3, 4] and Stone [5, 6]. Afterwards their investigations were developed in many directions. There is an enormous literature related to the spectral theory outlined above, and we refer to [7–18] and their extensive reference lists for this activity.

The present communication is a brief survey of results in the spectral theory of the Sturm-Liouville operator:

$$Lu = u'' - q(x)u, \tag{1.1}$$

with two-point boundary conditions

$$B_i(u) = a_{i1}u'(0) + a_{i2}u'(1) + a_{i3}u(0) + a_{i4}u(1) = 0, \tag{1.2}$$

where the $B_i(u)$ $(i = 1, 2)$ are linearly independent forms with arbitrary complex-valued coefficients and $q(x)$ is an arbitrary complex-valued function of class $L_1(0, 1)$.

Our main focus is on the non-self-adjoint case. We will study the completeness property and the basis property of the root function system of operator (1.1), (1.2). The convergence of spectral expansions is investigated only in classical sense; that is, the question about the summability of divergent series by a generalized method is not considered.

## 2. Preliminaries

Let us present briefly the main definitions and facts which will be used in what follows.

Let $\mathfrak{B}$ be a Banach space with the norm $\|\cdot\|_{\mathfrak{B}}$, and let $\mathfrak{B}^*$ be its dual with the norm $\|\cdot\|_{\mathfrak{B}^*}$.

A system of elements $\{e_n\}_{n=1}^{\infty}$ is said to be closed in $\mathfrak{B}$ if the linear span of this system is everywhere dense in $\mathfrak{B}$; that is, any element of the space $\mathfrak{B}$ can be approximated by a linear combination of elements of this system with any accuracy in the norm of the space $\mathfrak{B}$.

A system of elements $\{e_n\}_{n=1}^{\infty}$ is said to be minimal in $\mathfrak{B}$ if none of its elements belongs to the closure of the linear span of the other elements of this system.

**Theorem 2.1** (see [19]). *A system $\{e_n\}_{n=1}^{\infty}$ is minimal if and only if there exists a biorthogonal system dual to it, that is, a system of linear functionals $\{g_n\}_{n=1}^{\infty}$ from $\mathfrak{B}^*$ such that $(e_n, g_k) = \delta_{nk}$ for all $n, k \in \mathbb{N}$. Moreover, if the initial system is simultaneously closed and minimal in $\mathfrak{B}$, then the system biorthogonally dual to it is uniquely defined.*

We say that a system $\{e_n\}_{n=1}^{\infty}$ is uniformly minimal in $\mathfrak{B}$, if there exists $\gamma > 0$ such that for all $n \in \mathbb{N}$,

$$\operatorname{dist}(e_n, E_{(n)}) > \gamma \|e_n\|_{\mathfrak{B}}, \tag{2.1}$$

where $E_{(n)}$ is the closure of the linear span of all elements $e_l$ with serial numbers $l \neq n$.

**Theorem 2.2** (see [19]). *A closed and minimal system $\{e_n\}_{n=1}^{\infty}$ is uniformly minimal in $\mathfrak{B}$ if and only if:*

$$\sup_{n\geq 1}\left(\|e_n\|_{\mathfrak{B}} \cdot \|g_n\|_{\mathfrak{B}^*}\right) < \infty. \tag{2.2}$$

A system $\{e_n\}_{n=1}^{\infty}$ forms a basis of the space $\mathfrak{B}$ if, for any element $f \in \mathfrak{B}$, there exists a unique expansion of it in the elements of the system, that is, the series $\sum_{n=1}^{\infty} c_n e_n$ convergent to $f$ in the norm of the space $\mathfrak{B}$. Any basis is a closed and minimal system in $\mathfrak{B}$, and, therefore, we can uniquely find its biorthogonal dual system $\{g_n\}_{n=1}^{\infty}$, and hence the expansion of any element of $f$ with respect to the basis $\{e_n\}_{n=1}^{\infty}$ coincides with its biorthogonal expansion, that is, $c_n = (f, g_n)$ for all $n \in \mathbb{N}$.

Any basis in $\mathfrak{B}$ is a uniformly minimal system, and, therefore, (2.2) holds. However, it is well known that a closed and uniformly minimal system may not form a basis in $\mathfrak{B}$.

A system biorthogonally dual to a basis in a reflexive Banach space $\mathfrak{B}$ itself forms a basis in $\mathfrak{B}^*$.

A basis $\{e_n\}_{n=1}^{\infty}$ in the space $\mathfrak{B}$ is said to be an unconditional basis, if it remains a basis for any permutation for its elements.

In a Hilbert space $H$, along with the concept of an unconditional basis, we have the close concept of a Riesz basis. A system $\{e_n\}_{n=1}^{\infty}$ is called a Riesz basis of the space $H$ if there exists a bounded invertible operator $U$ such that the system $\{Ue_n\}_{n=1}^{\infty}$ forms an orthonormal basis in $H$.

**Theorem 2.3** (see [20]). *A system $\{e_n\}_{n=1}^{\infty}$ forms a Riesz basis of the space $H$ if and only if it is an unconditional basis almost normalized in $H$, that is,*

$$0 < \inf \|e_n\|_H \leq \sup \|e_n\|_H < \infty. \tag{2.3}$$

A system $\{e_n\}_{n=1}^{\infty}$ is said to be complete in $H$ if the equality $(f, e_n) = 0$ for all $n \in \mathbb{N}$ implies $f = 0$. In a Hilbert space, the properties of completeness and closeness of a system are equivalent.

We consider the operator $L$ as a linear operator on $L_2(0,1)$ defined by (1.1) with the domain $D(L) = \{u \in L_2(0,1) \mid u(x), u'(x)$ being absolutely continuous on $[0,1]$, $u'' - q(u)u \in L_2(0,1)$, $B_i(u) = 0$ $(i = 1,2)\}$.

By *an eigenfunction* of the operator $L$ corresponding to an eigenvalue $\lambda \in \mathbb{C}$, we mean any function $\overset{0}{u}(x) \in D(L)$ $(\overset{0}{u}(x) \not\equiv 0)$ which satisfies the equation:

$$L\overset{0}{u} + \lambda \overset{0}{u} = 0 \tag{2.4}$$

almost everywhere on $[0,1]$.

By *an associated function* of the operator $L$ of order $p$ $(p = 1,2,\ldots)$ corresponding to the same eigenvalue $\lambda$ and the eigenfunction $\overset{0}{u}(x)$, we mean any function $\overset{p}{u}(x) \in D(L)$ which satisfies the equation:

$$L\overset{p}{u} + \lambda \overset{p}{u} = \overset{p-1}{u} \tag{2.5}$$

almost everywhere on $[0,1]$. One can also say that an eigenfunction $\overset{0}{u}(x)$ is an associated function of zeroth order. The set of all eigen- and associated functions (or root functions) corresponding to the same eigenvalue $\lambda$ together with the function $u(x) \equiv 0$ forms a root linear manifold. This manifold is called a root subspace if its dimension is finite.

Let the set of the eigenvalues of the operator $L$ be countable and all root linear manifolds root subspaces. Let us choose a basis in each root subspace. Any system $\{u_n(x)\}$ obtained as the union of chosen bases of all the root subspaces is called *a system of eigen- and associated functions* (or root function system) of the operator $L$.

The main purpose of this paper is to study the basis property of the root function system of the operator $L$. Before starting our investigation, we must verify completeness of the root function system in $L_2(0,1)$.

It is convenient to write conditions (1.2) in the matrix form:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \tag{2.6}$$

and denote the matrix composed of the $i$th and $j$th columns of $A$ $(1 \leq i < j \leq 4)$ by $A(ij)$; we set $A_{ij} = \det A(ij)$.

Denote by $c(x,\lambda), s(x,\lambda)$ the fundamental system of solutions to (1.1) with the initial conditions $c(0,\lambda) = s'(0,\lambda) = 1$, $c'(0,\lambda) = s(0,\lambda) = 0$. The eigenvalues of problem (1.1), (1.2) are the roots of the characteristic determinant:

$$\Delta(\lambda) = \begin{vmatrix} B_1(c(x,\lambda)) & B_1(s(x,\lambda)) \\ B_2(c(x,\lambda)) & B_2(s(x,\lambda)) \end{vmatrix}. \tag{2.7}$$

Simple calculations show that

$$\Delta(\lambda) = -A_{13} - A_{24} + A_{34}s(1,\lambda) - A_{23}s'(1,\lambda) - A_{14}c(1,\lambda) - A_{12}c'(1,\lambda). \tag{2.8}$$

It is easily seen that if $q(x) \equiv 0$ then the characteristic determinant $\Delta_0(\lambda)$ of the corresponding problem (1.1), (1.2) has the form:

$$\Delta_0(\lambda) = -A_{13} - A_{24} + A_{34} \sin\frac{\sqrt{\lambda}}{\sqrt{\lambda}} - (A_{23} + A_{14}) \cos\sqrt{\lambda} + A_{12}\sqrt{\lambda}\sin\sqrt{\lambda}. \tag{2.9}$$

Boundary conditions (1.2) are called *nondegenerate* if they satisfy one of the following relations:

(1) $A_{12} \neq 0$,

(2) $A_{12} = 0$, $A_{14} + A_{23} \neq 0$,

(3) $A_{12} = 0$, $A_{14} + A_{23} = 0$, $A_{34} \neq 0$.

Evidently, boundary conditions (1.2) are nondegenerate if and only if $\Delta_0(\lambda) \neq \text{const}$.

Notice that for any nondegenerate boundary conditions an asymptotic representation for the characteristic determinant $\Delta(\lambda)$ as $|\lambda| \to \infty$ one can find in [10].

**Theorem 2.4** (see [10]). *For any nondegenerate conditions, the spectrum of problem* (1.1), (1.2) *consists of a countable set* $\{\lambda_n\}$ *of eigenvalues with only one limit point* $\infty$, *and the dimensions of the corresponding root subspaces are bounded by one constant. The system* $\{u_n(x)\}$ *of eigen- and associated functions is complete and minimal in* $L_2(0,1)$; *hence, it has a biorthogonal dual system* $\{v_n(x)\}$.

For convenience, we introduce numbers $\mu_n$, where $\mu_n$ is the square root of $\lambda_n$ with non-negative real part.

It is known that nondegenerate conditions can be divided into three classes:

(1) strengthened regular conditions;

(2) regular but not strengthened regular conditions;

(3) irregular conditions.

The definitions are given in [8]. These three cases should be considered separately.

## 3. Strengthened Regular Conditions

Let boundary conditions (1.2) belong to class (1). According to [8], this is equivalent to the fulfillment one of the following conditions:

$$A_{12} \neq 0; \quad A_{12} = 0, \quad A_{14} + A_{23} \neq 0, \quad A_{14} + A_{23} \neq \mp (A_{13} + A_{24}); \\ A_{12} = 0, \quad A_{14} + A_{23} = 0, \quad A_{13} + A_{24} = 0, \quad A_{13} = A_{24}, \quad A_{34} \neq 0. \tag{3.1}$$

It is well known that all but finitely many eigenvalues $\lambda_n$ are simple (in other words, they are asymptotically simple), and the number of associated functions is finite. Moreover, the $\lambda_n$ is separated in the sense that there exists a constant $c_0 > 0$ such that for any sufficiently large different numbers $\lambda_k$ and $\lambda_m$, we have

$$\left|\mu_k - \mu_m\right| \geq c_0. \tag{3.2}$$

**Theorem 3.1.** *The system of root functions $\{u_n(x)\}$ forms a Riesz basis in $L_2(0,1)$.*

This statement was proved in [21, 22] and [9, Chapter XIX].

Class (1) contains many types of boundary conditions, for example, the Dirichlet boundary conditions $u(0) = u(1) = 0$, the Newmann boundary conditions $u'(0) = u'(1) = 0$, the Dirichlet-Newmann boundary conditions $u(0) = u'(1) = 0$ and others.

## 4. Regular but Not Strengthened Regular Conditions

Let boundary conditions belong to class (2). According to [8], this is equivalent to the fulfillment of the conditions

$$A_{12} = 0, \quad A_{14} + A_{23} \neq 0, \quad A_{14} + A_{23} = (-1)^{\theta+1}(A_{13} + A_{24}), \tag{4.1}$$

where $\theta = 0, 1$. It is well known [10] that the eigenvalues of problem (1.1), (1.2) form two series:

$$\lambda_0 = \mu_0^2, \qquad \lambda_{n,j} = (2\pi n + o(1))^2 \tag{4.2}$$

(if $\theta = 0$) and

$$\lambda_{n,j} = ((2n-1)\pi + o(1))^2 \tag{4.3}$$

(if $\theta = 1$). Here, in both cases, $j = 1, 2$ and $n = 1, 2, \ldots$. We denote $\mu_{n,j} = \sqrt{\lambda_{n,j}} = (2n-\theta)\pi + o(1)$. It follows from [8] that asymptotic formulas (4.2) and (4.3) can be refined. Specifically,

$$\mu_{n,j} = (2n-\theta)\pi + O\left(n^{-1/2}\right). \tag{4.4}$$

Obviously, $|\mu_{n,1} - \mu_{n,2}| = O(n^{-1/2})$; that is, $\mu_{n,1}$ and $\mu_{n,2}$ become infinitely close to each other as $n \to \infty$. If $\mu_{n,1} = \mu_{n,2}$ for all $n$, except, possibly, a finite set, then the spectrum of problem (1.1), (1.2) is called *asymptotically multiple*. If the set of multiple eigenvalues is finite, then the spectrum of problem (1.1), (1.2) is called *asymptotically simple*.

There exist numerous examples when the number of multiple eigenvalues is finite or infinite, and the total number of associated functions is finite or infinite also. We see that separation condition (3.2) never holds. Depending on the particular form of the boundary conditions and the potential $q(x)$, the system of root functions may have or may not have the basis property [17, 22, 23], and even for fixed boundary conditions, this property may appear or disappear under arbitrary small variations of the coefficient $q(x)$ in the corresponding metric [24]. Thus, the considered case is much more complicated than the previous one, so we will study it in detail.

For any problem (1.1), (1.2) let $Q$ denote the set of potentials $q(x)$ from the class $L_1(0,1)$ such that the system of root functions forms a Riesz basis in $L_2(0,1)$, $\overline{Q} = L_1(0,1) \setminus Q$.

To analyze this class of problems, it is reasonable [12] to divide conditions (1.2) satisfying (4.1) into four types:

(I) $A_{14} = A_{23},\ A_{34} = 0$;

(II) $A_{14} = A_{23},\ A_{34} \neq 0$;

(III) $A_{14} \neq A_{23},\ A_{34} = 0$;

(IV) $A_{14} \neq A_{23},\ A_{34} \neq 0$.

The eigenvalue problem for (1.1) with boundary conditions of type I, II, III, or IV is called the problem of type I, II, III, or IV, respectively.

At first we consider the problems of type I. It was shown in [12] that any boundary conditions of type I are equivalent to the boundary conditions specified by the matrix:

$$A = \begin{pmatrix} 1 & (-1)^{\theta+1} & 0 & 0 \\ 0 & 0 & 1 & (-1)^{\theta+1} \end{pmatrix}, \tag{4.5}$$

that is, to periodic or antiperiodic boundary conditions. These boundary conditions are self-adjoint.

We set $\alpha_n = \int_0^1 q(x)e^{2\pi inx}dx,\ \beta_n = \int_0^1 q(x)e^{-2\pi inx}dx$.

**Theorem 4.1** (see [25]). *Suppose that $q(x) \in W_1^m[0,1]$, where $m = 0,1,\ldots$, and $q^{(l)}(0) = q^{(l)}(1)$ for $l = \overline{0,m-1}$. If there exists an $N$ such that*

$$|\alpha_{2n-\theta}| > c_0 n^{-m-1}, \qquad 0 < c_1 < \frac{|\alpha_{2n-\theta}|}{|\beta_{2n-\theta}|} < c_2, \tag{4.6}$$

*($c_0 > 0$) for all $n > N$, then the system of functions $\{u_n(x)\}$ is a Riesz basis in $L_2(0,1)$.*

*If there exists a sequence of $n_k$ $(k = 1,2,\ldots)$ such that $|\alpha_{2n_k-\theta}| > c_0 n_k^{-m-1}$, $|\beta_{2n_k-\theta}| > c_0 n_k^{-m-1}$, and*

$$\lim_{k\to\infty}\left(\frac{|\alpha_{2n_k-\theta}|}{|\beta_{2n_k-\theta}|} + \frac{|\beta_{2n_k-\theta}|}{|\alpha_{2n_k-\theta}|}\right) = \infty, \tag{4.7}$$

*then the system of functions $\{u_n(x)\}$ is not a basis in $L_2(0,1)$.*

It is easy to verify that if

$$q(x) = \sum_{n=1}^{\infty} q_n \left( \frac{e^{2\pi i n x}}{n^{m+\varepsilon_1}} + \frac{e^{-2\pi i n x}}{n^{m+\varepsilon_2}} \right), \tag{4.8}$$

where $0 < \varepsilon_1 < \varepsilon_2 < 1$, $q_n = 1$ for $n = 2^p - \theta$, and $q_n = 0$ for $n \neq 2^p - \theta$, then the system of functions $\{u_n(x)\}$ is not a basis in $L_2(0,1)$.

The following theorem is an easy corollary to Theorem 4.1.

**Theorem 4.2** (see [25]). *The sets $Q$ and $\overline{Q}$ are everywhere dense in $L_1(0,1)$.*

Theorem 4.1 was generalized in [26]. Recently, (see [27–29] and their extensive reference lists) by a number of authors, a very nice theory of the problems of type I was built. In particular, in papers [28, 29] a criterion to have a Riesz basis property was established. The criterion is formulated in terms of periodic (resp., antiperiodic) and Dirichlet eigenvalues. Also in [30], it was established the criterion for these boundary value problems to have a Riesz basis property in terms of a potential $q(x)$ provided that it is a special trigonometric polynomial. The later criterion has an advantage since it is given in terms of the coefficients of the potential.

Let us consider the problems of type II. It was also established in [12] that any boundary conditions of type II are equivalent to the boundary conditions specified by the matrix:

$$A = \begin{pmatrix} 1 & -1 & 0 & a_{14} \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 1 & 1 & 0 & a_{14} \\ 0 & 0 & 1 & 1 \end{pmatrix}, \tag{4.9}$$

where $a_{14} \neq 0$ in both cases. If $a_{14}$ is a real number and $q(x)$ is a real function, then the corresponding boundary value problem is self-adjoint.

**Theorem 4.3** (see [31]). *If $A_{14} = A_{23}$ and $A_{34} \neq 0$, then the system $\{u_n(x)\}$ forms a Riesz basis in $L_2(0,1)$, and the spectrum is asymptotically simple.*

Denote by $\{v_n(x)\}$ the biorthogonal dual system. The key point in the proof of Theorem 4.3 is obtaining the estimate:

$$\max_{(x,\xi)\in[0,1]\times[0,1]} \left| u_n(x)\overline{v_n(\xi)} \right| \le C, \tag{4.10}$$

which is valid for any number $n$. It follows from (4.10) and [32] that the system $\{u_n(x)\}$ forms a Riesz basis in $L_2(0,1)$.

A comprehensive description of boundary conditions of types III and IV was given in [12]. In particular, it is known that all of them are non-self-adjoint.

At first we consider the problems of type III. According to [12], any boundary conditions of type III are equivalent to the boundary conditions determined by the matrix:

$$A = \begin{pmatrix} 1 & a_{12} & 0 & 0 \\ 0 & 0 & 1 & a_{24} \end{pmatrix}, \tag{4.11}$$

where either $a_{12} = \mp 1$, $a_{24} \neq 1$, and $a_{24} \neq -1$; $a_{24} = \mp 1$, $a_{12} \neq 1$, and $a_{12} \neq -1$;

$$A = \begin{pmatrix} 1 & \mp 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mp 1 \end{pmatrix}. \tag{4.12}$$

The sign is always upper if $\theta = 0$ and lower if $\theta = 1$.

Let us consider the problems of type IV. According to [12], any boundary conditions of type IV are equivalent to the boundary conditions determined by the matrix:

$$A = \begin{pmatrix} 1 & a_{12} & 0 & a_{14} \\ 0 & 0 & 1 & a_{24} \end{pmatrix}, \tag{4.13}$$

where either $a_{12} = \mp 1$, $a_{24} \neq 1$, $a_{24} \neq -1$, and $a_{14} \neq 0$; $a_{24} = \mp 1$, $a_{12} \neq 1$, $a_{12} \neq -1$, and $a_{14} \neq 0$;

$$A = \begin{pmatrix} 1 & \mp 1 & a_{13} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{where } a_{13} \neq 0, \tag{4.14}$$

or

$$A = \begin{pmatrix} 0 & 1 & 0 & a_{14} \\ 0 & 0 & 1 & \mp 1 \end{pmatrix}, \quad \text{where } a_{14} \neq 0. \tag{4.15}$$

The sign is always upper if $\theta = 0$ and lower if $\theta = 1$.

**Theorem 4.4** (see [31]). *If $A_{14} \neq A_{23}$, then the system of root functions $\{u_n(x)\}$ of problem* (1.1), (1.2) *is a Riesz basis in $L_2(0,1)$ if and only if the spectrum is asymptotically multiple.*

Thus, we have established that for problems of types III and IV, the question about the basis property for the system of eigen- and associated functions is reduced to the question about asymptotic multiplicity of the spectrum. The presence of this property depends essentially on the particular form of the boundary conditions and the function $q(x)$.

**Theorem 4.5** (see [33, 34]). *If $A_{14} \neq A_{23}$, then, for any function $q(x) \in L_2(0,1)$ and any $\varepsilon > 0$, there exists a function $\tilde{q}(x) \in L_2(0,1)$ such that $\|q(x) - \tilde{q}(x)\|_{L_2(0,1)} < \varepsilon$ and problem* (1.1), (1.2) *with the potential $\tilde{q}(x)$ has an asymptotically multiple spectrum.*

For $A_{14} = A_{23}$ and $A_{34} = 0$, a similar proposition was deduced in [35].

Theorems 4.2, 4.3, and 4.5 and the results of [36] imply that the whole class of regular but not strengthened regular boundary conditions splits into two subclasses (a) and (b). Subclass (a) coincides with the second type of boundary conditions and is characterized by the fact that the system of root functions of problem (1.1), (1.2) with boundary conditions from this subclass forms a Riesz basis in $L_2(0,1)$ for any potential $q(x) \in L_1(0,1)$; that is, $Q = L_1(0,1)$, $\overline{Q} = \emptyset$. We will see below that boundary conditions from the subclass (a) are the only boundary conditions (in addition to strengthened regular ones) that ensure the Riesz basis property of the system of root functions for any potential $q(x) \in L_1(0,1)$.

Subclass (b) contains the remaining regular but not strengthened regular boundary conditions. An entirely different situation takes place in this case. For any problem with boundary conditions from this subclass, the sets $Q$ and $\overline{Q}$ are dense everywhere in $L_1(0,1)$.

For problems of types III and IV with an arbitrary potential $q(x) \in L_1(0,1)$ the following theorem is valid.

**Theorem 4.6** (see [34]). *Each root subspace contains one eigenfunction and, possibly, associated functions.*

By Theorem 4.5, for problems of types III and IV, the set of potentials $q(x)$ that ensure an asymptotically multiple spectrum is dense everywhere in $L_1(0,1)$. Therefore, it follows from Theorem 4.3 that we have discovered a new wide class of eigenvalue problems for the Sturm-Liouville operator that have an infinite number of associated functions.

## 5. Irregular Conditions

Let boundary conditions (1.2) belong to class (3). According to [8, 12], this is equivalent to the fulfillment one of the following conditions:

$$\begin{gathered} A_{12}=0, \quad A_{14}+A_{23}=0, \quad A_{13}+A_{24}=0, \quad A_{13}\neq A_{24}, \quad A_{34}\neq 0; \\ A_{12}=0, \quad A_{14}+A_{23}=0, \quad A_{13}+A_{24}\neq 0, \quad A_{34}\neq 0. \end{gathered} \tag{5.1}$$

According to [12], any boundary conditions of the considered class are equivalent to the boundary conditions determined by the matrix:

$$A=\begin{pmatrix} 1 & \pm 1 & 0 & b_0 \\ 0 & 0 & 1 & \mp 1 \end{pmatrix}, \quad \text{where } b_0 \neq 0, \tag{5.2}$$

or

$$A=\begin{pmatrix} 1 & b_1 & 0 & b_0 \\ 0 & 0 & 1 & -b_1 \end{pmatrix}, \quad \text{where } b_1 \neq \pm 1,\ b_0 \neq 0, \tag{5.3}$$

or

$$A=\begin{pmatrix} 0 & 1 & a_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{where } a_0 \neq 0. \tag{5.4}$$

In case (3), as well as in case (1), all but finitely many eigenvalues $\lambda_n$ are simple, the number of associated functions is finite, and separation condition (3.2) holds. However, the system $\{u_n(x)\}$ never forms even a usual basis in $L_2(0,1)$, because $\|u_n\|_{L_2(0,1)}\|v_n\|_{L_2(0,1)} \to \infty$ as $n \to \infty$. Here $\{v_n(x)\}$ is the biorthogonal dual system. This case was investigated in [5, 6, 37].

## 6. Degenerate Conditions

Let boundary conditions (1.2) be degenerate. According to [10, 12], this is equivalent to the fulfillment of the following conditions:

$$A_{12}=0, \quad A_{14}+A_{23}=0, \quad A_{34}=0. \tag{6.1}$$

According to [12], any boundary conditions of the considered class are equivalent to the boundary conditions determined by the matrix:

$$A = \begin{pmatrix} 1 & b_1 & 0 & 0 \\ 0 & 0 & 1 & -b_1 \end{pmatrix}, \quad \text{or} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.2}$$

If in the first case $b_1 = 0$ then for any potential $q(x)$, we have the initial value problem (the Cauchy problem) which has no eigenvalues. The same situation takes place in the second case.

Further we will consider the first case if $b_1 \neq 0$. Then the boundary conditions can be written in more visual form:

$$u'(0) + b_1 u'(1) = 0, \qquad u(0) - b_1 u(1) = 0. \tag{6.3}$$

Simple calculations show that if $b_1 = \pm 1$ and $q(x) \equiv 0$ then any $\lambda \in \mathbb{C}$ is an eigenvalue of infinite multiplicity. This abnormal example illustrates the difficulty of investigation of problems with boundary conditions of the considered class.

If $q(x) = \varepsilon(x-1/2)$ $(\varepsilon \neq 0)$ then [38, 39] the root function system is complete in $L_2(0,1)$. Let $q(x) = 0$ if $0 < \varepsilon < |x-1/2| \leq 1/2$ and $q(x) = \varepsilon(x-1/2)$ if $|x-1/2| \leq \varepsilon$. One can calculate that the characteristic determinant $\Delta(\lambda) \not\equiv 0$. This, together with [38, 39], implies that the root function system is not complete in $L_2(0,1)$. We see that depending on the potential $q(x)$ the system of root functions may have or may not have the completeness property, moreover, this property may appear or disappear under arbitrary small variations of the coefficient $q(x)$ in the corresponding metric even for fixed boundary conditions.

Notice, that the most general results on completeness of the root function system of problem (1.1), (6.3) were obtained in [39]. The main result of the mentioned paper is:

**Theorem 6.1** (see [39]). *If $q(x) \in C^k[0,1]$ for some $k = 0, 1, \ldots$ and $q^{(k)}(0) \neq (-1)^k q^{(k)}(1)$, then the system of root functions is complete in $L_2(0,1)$.*

Recently, it was proved in [40] that the root function system never forms an unconditional basis in $L_2(0,1)$ if multiplicities of the eigenvalues are uniformly bounded by some constant. Moreover, under the condition mentioned above, it was established there that if the eigen- and associated function system of general ordinary differential operator with two-point boundary conditions forms an unconditional basis then the boundary conditions are regular. Article [40] was published in 2006. At that time, it was unknown whether there exists a potential $q(x)$ providing unbounded growth of multiplicities of the eigenvalues. However, in 2010 in [41] an example of a potential $q(x)$ for which the characteristic determinant has the roots of arbitrary high multiplicity was constructed. Hence, the corresponding root function system $\{u_n(x)\}$ contains associated functions of arbitrary high order.

Denote by $\lambda_n$ $(n = 1, 2, \ldots)$ the eigenvalues of operator (1.1) with boundary conditions (6.3). Let $h(\lambda_n)$ denote multiplicity of the corresponding eigenvalue.

**Theorem 6.2** (see [42]). *If*

$$\underline{\lim}_{n \to \infty} h(\lambda_n) < \infty \tag{6.4}$$

*then the system $\{u_n(x)\}$ is not a basis in $L_2(0,1)$.*

**Table** 1

| Case | Class | Conditions on the $A_{ij}$ | Completeness | Basis property |
|---|---|---|---|---|
| (1) | SR | $A_{12} \neq 0$; $A_{12} = 0$, $A_{14}+A_{23} \neq 0$, $A_{14}+A_{23} \neq \mp(A_{13}+A_{24})$; $A_{12} = 0$, $A_{14} + A_{23} = 0$, $A_{13} + A_{24} = 0$, $A_{13} = A_{24}$, $A_{34} \neq 0$ | Yes | Yes |
| (2a) | WR | $A_{12} = 0$, $A_{14} + A_{23} \neq 0$, $A_{14} + A_{23} = \mp(A_{13} + A_{24})$, $A_{14} = A_{23}$, $A_{34} \neq 0$ | Yes | Yes |
| (2b) | WR | $A_{12} = 0$, $A_{14} + A_{23} \neq 0$, $A_{14} + A_{23} = \mp(A_{13} + A_{24})$, $A_{14} = A_{23}$, $A_{34} = 0$; $A_{12} = 0$, $A_{14}+A_{23} \neq 0$, $A_{14}+A_{23} = \mp(A_{13} + A_{24})$, $A_{14} \neq A_{23}$ | Yes | Yes/No |
| (3) | IR | $A_{12} = 0$, $A_{14} + A_{23} = 0$, $A_{13} + A_{24} = 0$, $A_{13} \neq A_{24}$, $A_{34} \neq 0$; $A_{12} = 0$, $A_{14} + A_{23} = 0$, $A_{13} + A_{24} \neq 0$, $A_{34} \neq 0$ | Yes | No |
| (4) | DEG | $A_{12} = 0$, $A_{14} + A_{23} = 0$, $A_{34} = 0$ | Yes/No | ?/No |

The following assertion is a trivial corollary of Theorem 6.2.

*If the system $\{u_n(x)\}$ is a basis in $L_2(0,1)$ then*

$$\lim_{n\to\infty} h(\lambda_n) = \infty. \tag{6.5}$$

Clearly, since Theorem 6.2 contains supplementary condition (6.4), it does not give the definitive solution of the basis property problem. If this condition does not hold then the mentioned problem has not been solved. Moreover, it is unknown whether there exists a potential $q(x)$ such that

$$\lim_{n\to\infty} h(\lambda_n) = \infty. \tag{6.6}$$

## 7. Conclusion

In this section, we present Table 1 summarizing the spectral properties, outlined in the introduction for operator (1.1), (1.2). The second column indicates classification for the case depending on the type of boundary conditions (SR: strengthened regular, WR: weakly regular—regular, but not strengthened regular, IR: irregular, DEG: degenerate). YES/NO means that the indicated property may appear or disappear under variation of the coefficient $q(x)$; ?/NO means that it has been proved that for a subset of potentials $q(x) \in L_2(0,1)$ the property does not take place, and an example when the property holds is unknown, thus, the definitive solution has not been received.

## Acknowledgment

This work was supported by the Russian Foundation for Basic Research, Project no. 10-01-411.

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