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Research Article

Subclass of Multivalent Harmonic Functions with Missing Coefficients

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We have studied subclass of multivalent harmonic functions with missing coefficients in the open unit disc and obtained the basic properties such as coefficient characterization and distortion theorem, extreme points, and convolution.

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a simply connected complex domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. It was shown by Clunie and Sheil-Small [1] that such harmonic function can be represented by $f = h + \overline{g}$, where h and g are analytic in D. Also, a necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ (see also, [2–4]).

Denote by H the family of functions $f = h + \overline{g}$, which are harmonic univalent and sense-preserving in the open-unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = h(0) = f'_z(0) - 1 = 0$.

For $m \geq 1$, $0 \leq \beta < 1$, and $\gamma \geq 0$, let $R(m, \beta, \gamma)$ denote the class of all multivalent harmonic functions $f = h + \overline{g}$ with missing coefficients that are sense-preserving in U, and h, g are of the form

$$h(z) = z^m + \sum_{n=m+1}^{\infty} a_{n+1} z^{n+1}, \qquad g(z) = \sum_{n=m}^{\infty} b_{n+1} z^{n+1} \quad (m \geq 1;\ z \in U) \tag{1.1}$$

and satisfying the following condition:

$$\operatorname{Re}\left\{ \left(1+\gamma e^{i\phi}\right)\frac{zf'(z)}{z'f(z)} - \gamma m e^{i\phi} \right\} \geq m\beta \quad (m \geq 1;\ 0 \leq \beta < 1;\ \gamma \geq 0;\ \phi \text{ real}), \tag{1.2}$$

where

$$z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right), \qquad f'(z) = \frac{\partial}{\partial\theta} f\left(re^{i\theta}\right). \tag{1.3}$$

We note that:

(i) $R(m,\beta,1) = R(m,\beta)$ with $a_{m+1};\ b_m \neq 0$ (see Jahangiri et al. [5]);

(ii) $R(1,\beta,\gamma) = J_H(\alpha,\beta,\gamma)$ (see Kharinar and More [6]);

(iii) $R(1,\beta,1) = G_H(\beta)$ with $a_2;\ b_1 \neq 0$ (see Rosy et al. [7] and Ahuja and Jahangiri [2]).

Also, the subclass denoted by $T(m,\beta,\gamma)$ consists of harmonic functions $f = h + \overline{g}$, so that h and g are of the form

$$h(z) = z^m - \sum_{n=m+1}^{\infty} a_{n+1} z^{n+1}, \qquad g(z) = \sum_{n=m}^{\infty} b_{n+1} z^{n+1} \quad (a_{n+1};\ b_{n+1} \geq 0;\ m \geq 1;\ z \in U). \tag{1.4}$$

We note that:

(i) $T(m,\beta,1) = T(m,\beta)$ with $a_{m+1}; b_m \neq 0$ (see Jahangiri et al. [5]);

(ii) $T(1,\beta,\gamma) = \overline{J_H}(\alpha,\beta,\gamma)$ (see Kharinar and More [6]);

(iii) $T(1,\beta,1) = G_{\overline{H}}(\beta)$ with $a_2; b_1 \neq 0$ (see Rosy et al. [7] and Ahuja and Jahangiri [2]).

From Ahuja and Jahangiri [2] with slight modification and among other things proved, if $f = h + \overline{g}$ is of the form (1.1) and satisfies the coefficient condition

$$\sum_{n=m-1}^{\infty} \left[\frac{(n+1) - m\beta}{m(1-\beta)} |a_{n+1}| + \frac{(n+1) + m\beta}{m(1-\beta)} |b_{n+1}| \right] \leq 2 \quad (a_m = 1; a_{m+1} = b_m = 0), \tag{1.5}$$

then the harmonic function f is sense-preserving, harmonic multivalent with missing coefficients and starlike of order β $(0 \leq \beta < 1)$ in U. They also proved that the condition (1.5) is also necessary for the starlikeness of function $f = h + \overline{g}$ of the form (1.4).

In this paper, we obtain sufficient coefficient bounds for functions in the class $R(m,\beta,\gamma)$. These sufficient coefficient conditions are shown to be also necessary for functions in the class $T(m,\beta,\gamma)$. Basic properties such as distortion theorem, extreme points, and convolution for the class $T(m,\beta,\gamma)$ are also obtained.

2. Coefficient Characterization and Distortion Theorem

Unless otherwise mentioned, we assume throughout this paper that $m \geq 1, 0 \leq \beta < 1, \gamma \geq 0$, and ϕ is real. We begin with a sufficient condition for functions in the class $R(m,\beta,\gamma)$.

Theorem 2.1. *Let* $f = h + \overline{g}$ *be such that* h *and* g *are given by* (1.1). *Furthermore, let*

$$\sum_{n=m-1}^{\infty}\left[\frac{(1+\gamma)(n+1)-m(\gamma+\beta)}{m(1-\beta)}|a_{n+1}|+\frac{(1+\gamma)(n+1)+m(\gamma+\beta)}{m(1-\beta)}|b_{n+1}|\right]\leq 2, \tag{2.1}$$

where $a_m = 1$ *and* $a_{m+1} = b_m = 0$. *Then* f *is sense-preserving, harmonic multivalent in* U *and* $f \in R(m,\beta,\gamma)$.

Proof. To prove $f \in R(m,\beta,\gamma)$, by definition, we only need to show that the condition (2.1) holds for f. Substituting $h+\overline{g}$ for f in (1.2), it suffices to show that

$$\operatorname{Re}\left\{\frac{(1+\gamma e^{i\theta})\left(zh'(z)-\overline{zg'(z)}\right)-m(\beta+\gamma e^{i\theta})\left(h(z)+\overline{g(z)}\right)}{h(z)+\overline{g(z)}}\right\}\geq 0, \tag{2.2}$$

where $h'(z) = (\partial/\partial z)h(z)$ and $g'(z) = (\partial/\partial z)g(z)$. Substituting for $h,\ g, h'$, and g' in (2.2), and dividing by $m(1-\beta)z^m$, we obtain $\operatorname{Re}(A(z)/B(z)) \geq 0$, where

$$\begin{aligned}
A(z) = 1 &+ \sum_{n=m+1}^{\infty}\frac{(n+1)(1+\gamma e^{i\theta})-m(\beta+\gamma e^{i\theta})}{m(1-\beta)}a_{n+1}z^{n-m+1}\\
&-\left(\frac{\overline{z}}{z}\right)^{m}\sum_{n=m+1}^{\infty}\frac{(n+1)(1+\gamma e^{-i\theta})+m(\beta+\gamma e^{-i\theta})}{m(1-\beta)}b_{n+1}\overline{z}^{n-m+1},\\
B(z) = 1 &+ \sum_{n=m+1}^{\infty}a_{n+1}z^{n-m+1}+\left(\frac{\overline{z}}{z}\right)^{m}\sum_{n=m+1}^{\infty}b_{n+1}\overline{z}^{n-m+1}.
\end{aligned} \tag{2.3}$$

Using the fact that $\operatorname{Re}(w) \geq 0$ if and only if $|1+w| > |1-w|$ in U, it suffices to show that $|A(z)+B(z)| - |A(z)-B(z)| \geq 0$. Substituting for $A(z)$ and $B(z)$ gives

$$\begin{aligned}
&|A(z)+B(z)|-|A(z)-B(z)|\\
&\quad=\Bigg|2+\sum_{n=m+1}^{\infty}\frac{(n+1)(1+\gamma e^{i\theta})-m(-1+2\beta+\gamma e^{i\theta})}{m(1-\beta)}a_{n+1}z^{n-m+1}\\
&\qquad-\left(\frac{\overline{z}}{z}\right)^{m}\sum_{n=m}^{\infty}\frac{(n+1)(1+\gamma e^{-i\theta})+m(-1+2\beta+\gamma e^{-i\theta})}{m(1-\beta)}b_{n+1}\overline{z}^{n-m+1}\Bigg|\\
&\qquad-\Bigg|\sum_{n=m+1}^{\infty}\frac{(n+1)(1+\gamma e^{i\theta})-m(1+\gamma e^{i\theta})}{m(1-\beta)}a_{n+1}z^{n-m+1}\\
&\qquad-\left(\frac{\overline{z}}{z}\right)\sum_{n=m}^{\infty}\frac{(n+1)(1+\gamma e^{-i\theta})+m(1+\gamma e^{-i\theta})}{m(1-\beta)}b_{n+1}\overline{z}^{n-m+1}\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\geq 2-\sum_{n=m+1}^{\infty}\frac{(n+1)(1+\gamma)-m(2\beta+\gamma-1)}{m(1-\beta)}|a_{n+1}|\,|z|^{n-m+1}\\
&\quad-\sum_{n=m}^{\infty}\frac{(n+1)(1+\gamma)+m(2\beta+\gamma-1)}{m(1-\beta)}|b_{n+1}|\,|z|^{n-m+1}\\
&\quad-\sum_{n=m+1}^{\infty}\frac{(n+1)(1+\gamma)-m(1+\gamma)}{m(1-\beta)}|a_{n+1}|\,|z|^{n-m+1}\\
&\quad-\sum_{n=m}^{\infty}\frac{(n+1)(1+\gamma)+m(1+\gamma)}{m(1-\beta)}|b_{n+1}|\,|z|^{n-m+1}\\
&\geq 2\left\{1-\sum_{n=m+1}^{\infty}\frac{(n+1)(1+\gamma)-m(\beta+\gamma)}{m(1-\beta)}|a_{n+1}|\right.\\
&\qquad\left.-\sum_{n=m}^{\infty}\frac{(n+1)(1+\gamma)+m(\beta+\gamma)}{m(1-\beta)}|b_{n+1}|\right\}\\
&\geq 0 \quad \text{by (2.1)}
\end{aligned} \tag{2.4}$$

The harmonic functions

$$\begin{aligned}
f(z) = z^m &+ \sum_{n=m+1}^{\infty}\frac{m(1-\beta)}{(n+1)(1+\gamma)-m(\beta+\gamma)}x_n z^{n+1}\\
&+\sum_{n=m}^{\infty}\frac{m(1-\beta)}{(n+1)(1+\gamma)+m(\beta+\gamma)}\overline{y}_n\overline{z}^{n+1},
\end{aligned} \tag{2.5}$$

where $\sum_{n=m+1}^{\infty}|x_n|+\sum_{n=m}^{\infty}|y_n|=1$, show that the coefficient boundary given by (2.1) is sharp. The functions of the form (2.5) are in the class $R(m,\beta,\gamma)$ because

$$\begin{aligned}
&\sum_{n=m+1}^{\infty}\left[\frac{(1+\gamma)(n+1)-m(\beta+\gamma)}{m(1-\beta)}|a_{n+1}|+\sum_{n=m}^{\infty}\frac{(1+\gamma)(n+1)+m(\beta+\gamma)}{m(1-\beta)}|b_{n+1}|\right]\\
&\quad=\sum_{n=m+1}^{\infty}|x_n|+\sum_{n=m+1}^{\infty}|y_n|=1.
\end{aligned} \tag{2.6}$$

This completes the proof of Theorem 2.1. □

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f=h+\overline{g}$, where h and g are of the form (1.4).

Theorem 2.2. *Let $f = h+\overline{g}$ be such that h and g are given by* (1.4). *Then $f \in T(m,\beta,\gamma)$ if and only if*

$$\sum_{n=m-1}^{\infty}\left[\frac{(1+\gamma)(n+1)-m(\gamma+\beta)}{m(1-\beta)}a_{n+1}+\frac{(1+\gamma)(n+1)+m(\gamma+\beta)}{m(1-\beta)}b_{n+1}\right]\leq 2. \tag{2.7}$$

Proof. Since $R(m,\beta,\gamma)\subset T(m,\beta,\gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions f of the form (1.4), we notice that the condition $\operatorname{Re}\{(1+\gamma e^{i\theta})(zf'(z))/(z'f(z))-\gamma me^{i\theta}\}\geq m\beta$ is equivalent to

$$\operatorname{Re}\left\{\frac{(1+\gamma e^{i\theta})\left(zh'(z)-z\overline{g'(z)}\right)-m(\beta+\gamma e^{i\theta})\left(h(z)+\overline{g(z)}\right)}{h(z)+\overline{g(z)}}\right\}>0, \tag{2.8}$$

which implies that

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{[m(1+\gamma e^{i\theta})-m(\beta+\gamma e^{i\theta})]z^m-\sum_{n=m+1}^{\infty}[(1+\gamma e^{i\theta})(n+1)-m(\beta+\gamma e^{i\theta})]a_{n+1}z^{n+1}}{z^m-\sum_{n=m+1}^{\infty}a_{n+1}z^{n+1}+\sum_{n=m}^{\infty}b_{n+1}\overline{z}^{n+1}}\right.\\
&\qquad\left.-\frac{\sum_{n=m}^{\infty}[(1+\gamma e^{i\theta})(n+1)+m(\beta+\gamma e^{i\theta})]b_{n+1}\overline{z}^{n+1}}{z^m-\sum_{n=m+1}^{\infty}a_{n+1}z^{n+1}+\sum_{n=m}^{\infty}b_{n+1}\overline{z}^{n+1}}\right\}\\
&=\operatorname{Re}\left\{\frac{m(1-\beta)-\sum_{n=m+1}^{\infty}[(1+\gamma e^{i\theta})(n+1)-m(\beta+\gamma e^{i\theta})]a_{n+1}z^{n-m+1}}{1-\sum_{n=m+1}^{\infty}a_{n+1}z^{n-m+1}+\sum_{n=m}^{\infty}b_{n+1}\overline{z}^{n-m+1}}\right.\\
&\qquad\left.-\frac{\sum_{n=m}^{\infty}[(1+\gamma e^{i\theta})(n+1)+m(\beta+\gamma e^{i\theta})]b_{n+1}\overline{z}^{n-m+1}}{1-\sum_{n=m+1}^{\infty}a_{n+1}z^{n-m+1}+\sum_{n=m}^{\infty}b_{n+1}\overline{z}^{n-m+1}}\right\}>0.
\end{aligned} \tag{2.9}$$

Since $\operatorname{Re}(e^{i\theta})\leq|e^{i\theta}|=1$, the required condition is that (2.9) is equivalent to

$$\begin{aligned}
&\left\{\frac{1-\sum_{n=m+1}^{\infty}([(1+\gamma)(n+1)-m(\beta+\gamma)]/m(1-\beta))a_{n+1}r^{n-m+1}}{1-\sum_{n=m+1}^{\infty}a_{n+1}r^{n-m+1}+\sum_{n=m}^{\infty}b_{n+1}r^{n-m+1}}\right.\\
&\qquad\left.-\frac{\sum_{n=m}^{\infty}([(1+\gamma)(n+1)+m(\beta+\gamma)]/m(1-\beta))b_{n+1}r^{n-m+1}}{1-\sum_{n=m+1}^{\infty}a_{n+1}r^{n-m+1}+\sum_{n=m}^{\infty}b_{n+1}r^{n-m+1}}\right\}\geq 0.
\end{aligned} \tag{2.10}$$

If the condition (2.7) does not hold, then the numerator in (2.10) is negative for $z = r$ sufficiently close to 1. Hence there exists $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.10) is negative. This contradicts the required condition for $f \in T(m,\beta,\gamma)$, and so the proof of Theorem 2.2 is completed. □

Corollary 2.3. *The functions in the class $T(m,\beta,\gamma)$ are starlike of order $(\gamma+\beta)/(1+\gamma)$.*

Proof. The proof follows from (1.5), by putting (2.7) in the form

$$\sum_{n=m-1}^{\infty}\left[\frac{(n+1)-m((\gamma+\beta)/(1+\gamma))}{m(1-((\gamma+\beta)/(1+\gamma)))}a_{n+1}+\frac{(n+1)+m((\gamma+\beta)/(1+\gamma))}{m(1-((\gamma+\beta)/(1+\gamma)))}b_{n+1}\right]\leq 2. \quad (2.11)$$

□

Theorem 2.4. *Let $f\in T(m,\beta,\gamma)$. Then for $|z|=r<1$, we have*

$$\begin{aligned}\left|f(z)\right| &\leq (1+b_{m+1}r)r^{m}+\left\{\frac{m(1-\beta)}{m(1-\beta)+2(1+\gamma)}-\frac{m(1+2\gamma+\beta)+(1+\gamma)}{m(1-\beta)+2(1+\gamma)}b_{m+1}\right\}r^{m+2},\\ \left|f_m(z)\right| &\geq (1-b_{m+1}r)r^{m}-\left\{\frac{m(1-\beta)}{m(1-\beta)+2(1+\gamma)}-\frac{m(1+2\gamma+\beta)+(1+\gamma)}{m(1-\beta)+2(1+\gamma)}b_{m+1}\right\}r^{m+2}.\end{aligned} \quad (2.12)$$

Proof. We prove the left-hand-side inequality for $|f|$. The proof for the right-hand-side inequality can be done by using similar arguments.

Let $f\in T(m,\beta,\gamma)$, then we have

$$\begin{aligned}\left|f(z)\right| &= \left|z^{m}-\sum_{n=m+1}^{\infty}a_{n+1}z^{n+1}+\sum_{n=m}^{\infty}b_{n+1}z^{n+1}\right|\\ &\geq r^{m}-b_{m+1}r^{m+1}-\sum_{n=m+1}^{\infty}(a_{n+1}+b_{n+1})r^{m+2}\\ &\geq r^{m}-b_{m+1}r^{m+1}\\ &\quad -\frac{m(1-\beta)}{(1+\gamma)(m+2)-m(\gamma+\beta)}\sum_{n=m+1}^{\infty}\frac{(1+\gamma)(m+2)-m(\gamma+\beta)}{m(1-\beta)}(a_{n+1}+b_{n+1})r^{n+1}\\ &\geq r^{m}-b_{m+1}r^{m+1}\\ &\quad -\frac{m(1-\beta)}{(1+\gamma)(m+2)-m(\gamma+\beta)}\sum_{n=m+1}^{\infty}\left\{\frac{(1+\gamma)(n+1)-m(\gamma+\beta)}{m(1-\beta)}a_{n+1}\right.\\ &\qquad\left.+\frac{(1+\gamma)(n+1)+m(\gamma+\beta)}{m(1-\beta)}b_{n+1}\right\}r^{n+1}\end{aligned}$$

$$
\begin{aligned}
&\geq (1-b_{m+1}r)r^{m} \\
&\quad - \frac{m(1-\beta)}{(1+\gamma)(m+2)-m(\gamma+\beta)}\left\{1-\frac{(1+\gamma)(m+1)+m(\gamma+\beta)}{m(1-\beta)}b_{m+1}\right\}r^{m+2} \\
&\geq (1-b_{m+1}r)r^{m} \\
&\quad - \left\{\frac{m(1-\beta)}{m(1-\beta)+2(1+\gamma)}-\frac{m(1+2\gamma+\beta)+(1+\gamma)}{m(1-\beta)+2(1+\gamma)}b_{m+1}\right\}r^{m+2}.
\end{aligned}
\tag{2.13}
$$

This completes the proof of Theorem 2.4. □

The following covering result follows from the left-side inequality in Theorem 2.4.

Corollary 2.5. *Let $f \in T(m,\beta,\gamma)$, then the set*

$$
\left\{w : |w| < \frac{2(1+\gamma)}{m(1-\beta)+2(1+\gamma)} - \frac{(1+\gamma)-2m(\gamma+\beta)}{m(1-\beta)+2(1+\gamma)}b_{m+1}\right\}
\tag{2.14}
$$

is included in $f(U)$.

3. Extreme Points

Our next theorem is on the extreme points of convex hulls of the class $T(m,\beta,\gamma)$, denoted by $\operatorname{clco} T(m,\beta,\gamma)$.

Theorem 3.1. *Let $f = h + \overline{g}$ be such that h and g are given by* (1.4)*. Then $f \in \operatorname{clco} T(m,\beta,\gamma)$ if and only if f can be expressed as*

$$
f(z) = \sum_{n=m}^{\infty}\left[X_{n+1}h_{n+1}(z) + Y_{n+1}g_{n+1}(z)\right],
\tag{3.1}
$$

where

$$
\begin{gathered}
h_m(z) = z^m, \\
h_{n+1}(z) = z^m - \frac{m(1-\beta)}{(1+\gamma)(n+1)-m(\gamma+\beta)}z^{n+1} \quad (n = m+1,\ m+2,\ldots), \\
g_{n+1}(z) = z^m + \frac{m(1-\beta)}{(1+\gamma)(n+1)+m(\gamma+\beta)}\overline{z}^{n+1} \quad (n = m, m+1,\ m+2,\ldots), \\
X_{n+1} \geq 0, \quad Y_{n+1} \geq 0, \quad \sum_{n=m}^{\infty}\left[X_{n+1}+Y_{n+1}\right] = 1.
\end{gathered}
\tag{3.2}
$$

In particular, the extreme points of the class $T(m,\beta,\gamma)$ are $\{h_{n+1}\}$ and $\{g_{n+1}\}$, respectively.

Proof. For functions $f(z)$ of the form (3.1), we have

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty}[X_{n+1}+Y_{n+1}]z^m - \sum_{n=m}^{\infty}\frac{m(1-\beta)}{(1+\gamma)(n+1)-m(\gamma+\beta)}X_{n+1}z^{n+1} \\ &\quad + \sum_{n=m}^{\infty}\frac{m(1-\beta)}{(1+\gamma)(n+1)+m(\gamma+\beta)}Y_{n+1}\overline{z}^{n+1}. \end{aligned} \tag{3.3}$$

Then

$$\begin{aligned} &\sum_{n=m+1}^{\infty}\frac{(1+\gamma)(n+1)-m(\gamma+\beta)}{m(1-\beta)}\left(\frac{m(1-\beta)}{(1+\gamma)(n+1)-m(\gamma+\beta)}\right)X_{n+1} \\ &\quad + \sum_{n=m}^{\infty}\frac{(1+\gamma)(n+1)+m(\gamma+\beta)}{m(1-\beta)}\left(\frac{m(1-\beta)}{(1+\gamma)(n+1)+m(\gamma+\beta)}\right)Y_{n+1} \\ &\quad = \sum_{n=m+1}^{\infty}X_{n+1}+\sum_{n=m}^{\infty}Y_{n+1} = 1-X_m \le 1, \end{aligned} \tag{3.4}$$

and so $f(z)\in \operatorname{clco} T(m,\beta,\gamma)$. Conversely, suppose that $f(z)\in \operatorname{clco} T(m,\beta,\gamma)$. Set

$$\begin{aligned} X_{n+1} &= \frac{(1+\gamma)(n+1)-m(\gamma+\beta)}{m(1-\beta)}a_{n+1} \quad (n=m+1,\ldots), \\ Y_{n+1} &= \frac{(1+\gamma)(n+1)+m(\gamma+\beta)}{m(1-\beta)}b_{n+1} \quad (n=m,m+1,\ldots), \end{aligned} \tag{3.5}$$

then note that by Theorem 2.2, $0\le X_{n+1}\le 1 \ (n=m+1,\ldots)$ and $0\le Y_{n+1}\le 1 \ (n=m,m+1,\ldots)$. Consequently, we obtain

$$f(z)=\sum_{n=m}^{\infty}\left[X_{n+1}h_{n+1}(z)+Y_{n+1}g_{n+1}(z)\right]. \tag{3.6}$$

Using Theorem 2.2 it is easily seen that the class $T(m,\beta,\gamma)$ is convex and closed, and so $\operatorname{clco} T(m,\beta,\gamma)=T(m,\beta,\gamma)$. □

4. Convolution Result

For harmonic functions of the form

$$f(z)=z^m-\sum_{n=m+1}^{\infty}a_{n+1}z^{n+1}+\sum_{n=m}^{\infty}b_{n+1}\overline{z}^{n+1}, \tag{4.1}$$

$$G(z)=z^m-\sum_{n=m+1}^{\infty}A_{n+1}z^{n+1}+\sum_{n=m}^{\infty}B_{n+1}\overline{z}^{n+1}, \tag{4.2}$$

we define the convolution of two harmonic functions f and G as

$$\begin{aligned}(f * G)(z) &= f(z) * G(z)\\ &= z^m - \sum_{n=m+1}^{\infty} a_{n+1}A_{n+1}z^{n+1} + \sum_{n=m}^{\infty} b_{n+1}B_{n+1}\overline{z}^{n+1}.\end{aligned} \tag{4.3}$$

Using this definition, we show that the class $T(m,\beta,\gamma)$ is closed under convolution.

Theorem 4.1. *For $0 \leq \beta < 1$, let $f(z) \in T(m,\beta,\gamma)$ and $G(z) \in T(m,\beta,\gamma)$. Then $f(z) * G(z) \in T(m,\beta,\gamma)$.*

Proof. Let the functions $f(z)$ defined by (4.1) be in the class $T(m,\beta,\gamma)$, and let the functions $G(z)$ defined by (4.2) be in the class $T(m,\beta,\gamma)$. Obviously, the coefficients of f and G must satisfy a condition similar to the inequality (2.7). So for the coefficients of $f * G$ we can write

$$\begin{aligned}&\sum_{n=m-1}^{\infty} \frac{(1+\gamma)(n+1) - m(\gamma+\beta)}{m(1-\beta)} a_{n+1}A_{n+1} + \frac{(1+\gamma)(n+1) + m(\gamma+\beta)}{m(1-\beta)} b_{n+1}B_{n+1}\\ &\quad \leq \sum_{n=m-1}^{\infty} \left[\frac{(1+\gamma)(n+1) - m(\gamma+\beta)}{m(1-\beta)} a_{n+1} + \frac{(1+\gamma)(n+1) + m(\gamma+\beta)}{m(1-\beta)} b_{n+1}\right],\end{aligned} \tag{4.4}$$

where the right hand side of this inequality is bounded by 2 because $f \in T(m,\beta,\gamma)$. Then, $f(z) * G(z) \in T(m,\beta,\gamma)$. □

Finally, we show that $T(m,\beta,\gamma)$ is closed under convex combinations of its members.

Theorem 4.2. *The class $T(m,\beta,\gamma)$ is closed under convex combination.*

Proof. For $i = 1,2,3,\ldots.$ let $f_i \in T(m,\beta,\gamma)$, where the functions f_i are given by

$$f_i(z) = z^m - \sum_{n=m+1}^{\infty} a_{n+1,i}z^{n+1} + \sum_{n=m}^{\infty} b_{n+1,i}z^{n+1} \quad (a_{n+1,i}; b_{n+1,i} \geq 0; m \geq 1). \tag{4.5}$$

For $\sum_{i=1}^{\infty} t_i = 1;\ \ 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^m - \sum_{n=m+1}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{n+1,i}\right)z^{n+1} + \sum_{n=m}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{n+1,i}\right)z^{n+1} \tag{4.6}$$

Then by (2.7), we have

$$
\begin{aligned}
&\sum_{n=m-1}^{\infty}\left[\frac{(1+\gamma)(n+1)-m(\gamma+\beta)}{m(1-\beta)}\sum_{i=1}^{\infty}t_i a_{n+1,i}+\frac{(1+\gamma)(n+1)+m(\gamma+\beta)}{m(1-\beta)}\sum_{i=1}^{\infty}t_i b_{n+1,i}\right]\\
&\quad=\sum_{i=1}^{\infty}t_i\left\{\sum_{n=m-1}^{\infty}\left[\frac{(1+\gamma)(n+1)-m(\gamma+\beta)}{m(1-\beta)}a_{n+1,i}+\frac{(1+\gamma)(n+1)+m(\gamma+\beta)}{m(1-\beta)}b_{n+1,i}\right]\right\}\\
&\quad\leq 2\sum_{i=1}^{\infty}t_i=2.
\end{aligned}
\tag{4.7}
$$

This is the condition required by (2.7), and so $\sum_{i=1}^{\infty} t_i f_i(z) \in T(m,\beta,\gamma)$. This completes the proof of Theorem 4.2. □

Remark 4.3. Our results for $m = 1$ correct the results obtained by Kharinar and More [6].

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