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Research Article

On Subspaces of an Almost φ-Lagrange Space

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We discuss the subspaces of an almost φ-Lagrange space (APL space in short). We obtain the induced nonlinear connection, coefficients of coupling, coefficients of induced tangent and induced normal connections, the Gauss-Weingarten formulae, and the Gauss-Codazzi equations for a subspace of an APL-space. Some consequences of the Gauss-Weingarten formulae have also been discussed.

1. Introduction

The credit for introducing the geometry of Lagrange spaces and their subspaces goes to the famous Romanian geometer Miron [1]. He developed the theory of subspaces of a Lagrange space together with Bejancu [2]. Miron and Anastasiei [3] and Sakaguchi [4] studied the subspaces of generalized Lagrange spaces (GL spaces in short). Antonelli and Hrimiuc [5, 6] introduced the concept of φ-Lagrangians and studied φ-Lagrange manifolds. Generalizing the notion of a φ-Lagrange manifold, the present authors recently studied the geometry of an almost φ-Lagrange space (APL space briefly) and obtained the fundamental entities related to such space [7]. This paper is devoted to the subspaces of an APL space.

Let $F^n = (M, F(x,y))$ be an n-dimensional Finsler space and $\varphi : \mathbb{R}^+ \to \mathbb{R}$ a smooth function. If the function φ has the following properties:

(a) $\varphi'(t) \neq 0$,

(b) $\varphi'(t) + \varphi''(t) \neq 0$, for every $t \in \operatorname{Im}(F^2)$,

then the Lagrangian given by

$$L(x,y) = \varphi\left(F^2\right) + A_i(x)y^i + U(x), \tag{1.1}$$

where $A_i(x)$ is a covector and $U(x)$ is a smooth function, is a regular Lagrangian [7]. The space $L^n = (M, L(x,y))$ is a Lagrange space. The present authors [7] called such space as an almost φ-Lagrange space (shortly APL space) associated to the Finsler space F^n. An APL space reduces to a φ-Lagrange space if and only if $A_i(x) = 0$ and $U(x) = 0$. We take

$$g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2, \qquad a_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L, \quad \text{where } \dot{\partial}_i \equiv \frac{\partial}{\partial y^i}. \tag{1.2}$$

We indicate all the geometrical objects related to F^n by putting a small circle "$\circ$" over them. Equations (1.2), in view of (1.1), provide the following expressions for a_{ij} and its inverse (cf. [7]):

$$a_{ij} = \varphi' \cdot \left(g_{ij} + \frac{2\varphi''}{\varphi'} \overset{\circ}{y}_i \overset{\circ}{y}_j \right), \qquad a^{ij} = \frac{1}{\varphi'} \left(g^{ij} - \frac{2\varphi''}{\varphi' + 2F^2\varphi''} y^i\, y^j \right), \tag{1.3}$$

where $g_{ij} y^j = \overset{\circ}{y}_i$.

Let $\check{M}$ be a smooth manifold of dimension m, $1 < m < n$, immersed in M by immersion $i : \check{M} \to M$. The immersion i induces an immersion $T_i : T\check{M} \to TM$ making the following diagram commutative:

$$\begin{array}{ccc} T\check{M} & \xrightarrow{T_i} & TM \\ \check{\pi} \downarrow & & \downarrow \pi \\ \check{M} & \underset{i}{\longrightarrow} & M. \end{array} \tag{1.4}$$

Let (u^α, v^α) (throughout the paper, the Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to m) be local coordinates on $T\check{M}$. The restriction of the Lagrangian L on $T\check{M}$ is $L(u,v) = L(x(u), y(u,v))$. Let $a_{\alpha\beta} = (1/2)(\partial^2 \check{L}/\partial u^\alpha \partial u^\beta)$. Then, we have (cf. [8]) $a_{\alpha\beta} = B^i_\alpha B^j_\beta a_{ij}$ where $B^i_\alpha(u) = \partial x^i/\partial u^\alpha$ are the projection factors. The pair $\check{L}^m = (\check{M}, \check{L}(u,v))$ is also a Lagrange space, called the subspace of L^n. For the natural bases $(\partial/\partial x^i, \partial/\partial y^i)$ on TM and $(\partial/\partial u^\alpha, \partial/\partial v^\alpha)$ on $T\check{M}$, we have [8]

$$\frac{\partial}{\partial u^\alpha} = B^i_\alpha \frac{\partial}{\partial x^i} + B^i_{0\alpha} \frac{\partial}{\partial y^i}, \qquad \frac{\partial}{\partial v^\alpha} = B^i_\alpha \frac{\partial}{\partial y^i}, \tag{1.5}$$

where $B^i_{0\alpha} = B^i_{\beta\alpha} v^\beta$, $B^i_{\beta\alpha} = \partial^2 x^i/\partial u^\alpha \partial u^\beta$.

For the bases (dx^i, dy^i) and (du^α, dv^α), we have

$$dx^i = B^i_\alpha du^\alpha, \qquad dy^i = B^i_\alpha dv^\alpha + B^i_{0\alpha} du^\alpha. \tag{1.6}$$

Since (B^i_α) are m linearly independent vector fields tangent to $\check{M}$, a vector field $\xi^i(x,y)$ is normal to $\check{M}$ along $T\check{M}$ if on $T\check{M}$, we have

$$a_{ij}\big(x(u), y(u,v)\big) B^i_\alpha \xi^j = 0, \quad \forall \alpha = 1, 2, \ldots, m. \tag{1.7}$$

There are, at least locally, $(n-m)$ unit vector fields $B^i_a(u,v)$ $(a=m+1,m+2,\ldots,n)$ normal to $\check{M}$ and mutually orthonormal, that is,

$$a_{ij}B^i_\alpha B^j_b = 0, \quad a_{ij}B^i_a B^j_b = \delta_{ab}, \quad (a,b=m+1,m+2,\ldots,n). \tag{1.8}$$

Thus, at every point $(u,v)\in T\check{M}$, we have a moving frame $\mathfrak{R}=((u,v),B^i_\alpha(u,v),B^i_a(u,v))$. Using (1.3) in the first expression of (1.8) and keeping $\overset{\circ}{y}_i B^i_a = 0$ (this fact is clear from $g_{ij}y^iB^j_a = 0$) in view, we observe that B^i_a's are normal to $\check{M}$ with respect to L^n if and only if they are so with respect to F^n. The dual frame of $\mathfrak{R}$ is $\mathfrak{R}^*=((u,v),B^\alpha_i(u,v),B^a_i(u,v))$ with the following duality conditions:

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \qquad B^i_a B^\beta_i = 0, \qquad B^i_\alpha B^b_i = 0, \qquad B^i_a B^b_i = \delta^b_a, \qquad B^i_a B^a_j + B^i_\alpha B^\alpha_j = \delta^i_j. \tag{1.9}$$

We will make use of the following results due to the present authors [7], during further discussion.

Theorem 1.1 (cf. [7]). *The canonical nonlinear connection of an APL space L^n has the local coefficients given by*

$$N^i_j = \overset{\circ}{N}{}^i_j - V^i_j, \tag{1.10}$$

where $V^i_j = (1/2)F^i_j - S^{ir}_j(2F_{rk}y^k + \partial_r U)$,

$$S^{ir}_j = \frac{1}{2\varphi'}\overset{\circ}{C}{}^i_{qj}g^{qr} + \frac{1}{2}\frac{\varphi''}{\varphi'^2}g^{ir}\overset{\circ}{y}_j + \frac{\varphi''\left(\delta^r_j y^i + \delta^i_j y^r\right)}{2\varphi'(\varphi'+2F^2\varphi'')} + \frac{\varphi'^2\varphi''' - 2\varphi''^3F^2 - 4\varphi'\varphi''^2}{2\varphi'^2\left(\varphi'+2F^2\varphi''\right)^2}y^i\overset{\circ}{y}_j y^r,$$
$$F_{rk}(x) = \frac{1}{2}(\partial_r A_k - \partial_k A_r), \qquad F^i_j = a^{ik}F_{kj}. \tag{1.11}$$

Theorem 1.2 (cf. [7]). *The coefficients of the canonical metrical d-connection $C\Gamma(N)$ of an APL space L^n are given by*

$$C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \frac{\varphi''}{\varphi'}\left(\delta^i_j\overset{\circ}{y}_k + \delta^i_k\overset{\circ}{y}_j\right) + \frac{\varphi''}{\varphi'+2F^2\varphi''}g_{jk}y^i + \frac{2\left(\varphi'''\varphi' - 2\varphi''^2\right)}{\varphi'\left(\varphi'+2F^2\varphi''\right)}y^i\overset{\circ}{y}_j\overset{\circ}{y}_k, \tag{1.12}$$

$$L^i_{jk} = \overset{\circ}{L}{}^i_{jk} + V^r_k C^i_{jr} + V^r_j C^i_{kr} + V^r_p a^{ip}C_{rkj}. \tag{1.13}$$

For basic notations related to a Finsler space, a Lagrange space, and their subspaces, we refer to the books [8, 9].

2. Induced Nonlinear Connection

Let $\check{N} = (\check{N}^{\alpha}_{\beta}(u,v))$ be a nonlinear connection for $\check{L}^m = (\check{M}, \check{L}(u,v))$. The adapted basis of $T_{(u,v)}T\check{M}$ induced by $\check{N}$ is $(\delta/\delta u^{\alpha} = \delta_{\alpha}, \partial/\partial v^{\alpha} = \dot{\partial}_{\alpha})$, where

$$\delta_{\alpha} = \partial_{\alpha} - \check{N}^{\beta}_{\alpha}\dot{\partial}_{\beta}. \tag{2.1}$$

The dual basis (cobasis) of the adapted basis $(\delta_{\alpha}, \dot{\partial}_{\alpha})$ is $(du^{\alpha}, \delta v^{\alpha} = dv^{\alpha} + \check{N}^{\alpha}_{\beta}du^{\beta})$.

Definition 2.1 (cf. [8]). A nonlinear connection $\check{N} = (\check{N}^{\alpha}_{\beta}(u,v))$ of $\check{L}^m$ is said to be induced by the canonical nonlinear connection N if the following equation holds good:

$$\delta v^{\alpha} = B^{\alpha}_{i}\delta y^{i}. \tag{2.2}$$

The local coefficients of the induced nonlinear connection $\check{N} = (\check{N}^{\alpha}_{\beta}(u,v))$ for the subspace $\check{L}^m = (\check{M}, \check{L}(u,v))$ of a Lagrange space $L^n = (M, L(x,y))$ are given by (cf. [8])

$$\check{N}^{\alpha}_{\beta} = B^{\alpha}_{i}\left(N^{i}_{j}B^{j}_{\beta} + B^{i}_{0\beta}\right), \tag{2.3}$$

N^{i}_{j} being the local coefficients of canonical nonlinear connection N of the Lagrange space $L^n = (M, L(x,y))$. Now using (1.10) in (2.3), we get

$$\check{N}^{\alpha}_{\beta} = B^{\alpha}_{i}\left(\overset{\circ}{N}{}^{i}_{j}B^{j}_{\beta} + B^{i}_{0\beta}\right) - B^{\alpha}_{i}V^{i}_{j}B^{j}_{\beta}. \tag{2.4}$$

If we take $\overset{\circ}{\check{N}}{}^{\alpha}_{\beta} = B^{\alpha}_{i}(\overset{\circ}{N}{}^{i}_{j}B^{j}_{\beta} + B^{i}_{0\beta})$, it follows from (2.4) that

$$\check{N}^{\alpha}_{\beta} = \overset{\circ}{\check{N}}{}^{\alpha}_{\beta} - B^{\alpha}_{i}V^{i}_{j}B^{j}_{\beta}. \tag{2.5}$$

Thus, we have the following.

Theorem 2.2. *The local coefficients of the induced nonlinear connection $\check{N}$ of the subspace $\check{L}^m$ of an APL space L^n are given by* (2.5).

In view of (2.5), (2.1) takes the following form, for the subspace $\check{L}^m$ of an APL space L^n:

$$\delta_{\beta} = \overset{\circ}{\delta}_{\beta} + B^{\alpha}_{p}V^{p}_{j}B^{j}_{\beta}\dot{\partial}_{\alpha}, \tag{2.6}$$

where $\overset{\circ}{\delta}_{\beta} = \partial_{\beta} - \overset{\circ}{\check{N}}{}^{\alpha}_{\beta}\dot{\partial}_{\alpha}$.

We can put $(dx^i, \delta y^i)$ as (cf. [8])

$$dx^i = B^i_\alpha du^\alpha, \qquad \delta y^i = B^i_\alpha \delta y^\alpha + B^i_a H^a_\alpha du^\alpha, \tag{2.7}$$

where

$$H^a_\alpha = B^a_i \left(N^i_j B^j_\alpha + B^i_{0\alpha} \right). \tag{2.8}$$

Using (1.10) in (2.8) and simplifying, we get

$$H^a_\alpha = B^a_i \left(\overset{\circ}{N}{}^i_j B^j_\alpha + B^i_{0\alpha} \right) - B^a_i V^i_j B^j_\alpha. \tag{2.9}$$

Taking $\overset{\circ}{H}{}^a_\alpha = B^a_i (\overset{\circ}{N}{}^i_j B^j_\alpha + B^i_{0\alpha})$, in (2.9), it follows that

$$H^a_\alpha = \overset{\circ}{H}{}^a_\alpha - B^a_i V^i_j B^j_\alpha. \tag{2.10}$$

Now, $dx^i = B^i_\alpha du^\alpha$, $\delta y^i = B^i_\alpha \delta y^\alpha$ if and only if $H^a_\alpha = 0$, that is, if and only if $\overset{\circ}{H}{}^a_\alpha = B^a_i V^i_j B^j_\alpha$. Thus, we have the following.

Theorem 2.3. *The adapted cobasis $(dx^i, \delta y^i)$ of the basis $(\partial/\partial x^i, \partial/\partial y^i)$ induced by the nonlinear connection N of an APL space L^n is of the form $dx^i = B^i_\alpha du^\alpha, \delta y^i = B^i_\alpha \delta y^\alpha$ if and only if $\overset{\circ}{H}{}^a_\alpha = B^a_i V^i_j B^j_\alpha$.*

Definition 2.4 (cf. [8]). Let $D = D\Gamma(N)$ be the canonical metrical d-connection of L^n. An operator $\check{D}$ is said to be a coupling of D with $\check{N}$ if

$$\check{D} X^i = X^i_{|\alpha} du^\alpha + X^i|_\alpha \delta v^\alpha, \tag{2.11}$$

where $X^i_{|\alpha} = \delta_\alpha X^i + X^j \check{L}^i_{j\alpha}$, $X^i|_\alpha = \dot{\partial}_\alpha X^i + X^j \check{C}^i_{j\alpha}$.

The coefficients $(\check{L}^i_{j\alpha}, \check{C}^i_{j\alpha})$ of coupling $\check{D}$ of D with $\check{N}$ are given by

$$\check{L}^i_{j\alpha} = L^i_{jk} B^k_\alpha + C^i_{jk} B^k_a H^a_\alpha, \tag{2.12}$$

$$\check{C}^i_{j\alpha} = C^i_{jk} B^k_\alpha. \tag{2.13}$$

Using (1.12) and (1.13) in (2.12), we have

$$\begin{aligned}\check{L}^i_{j\beta} &= \left(\overset{\circ}{L}{}^i_{jk} + V^r_k C^i_{jr} + V^r_j C^i_{kr} + V^r_p a^{ip} C_{rkj}\right) B^k_\beta \\ &\quad + \left[\overset{\circ}{C}{}^i_{jk} + \frac{\varphi''}{\varphi'}\left(\delta^i_j \overset{\circ}{y}_k + \delta^i_k \overset{\circ}{y}_j\right) + \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i \right. \\ &\quad \left. + \frac{2\left(\varphi'''\varphi' - 2\varphi''^2\right)}{\varphi'\left(\varphi' + 2F^2\varphi''\right)} y^i \overset{\circ}{y}_j \overset{\circ}{y}_k\right] B^k_a H^a_\beta.\end{aligned} \tag{2.14}$$

In view of (2.10) and $\overset{\circ}{y}_i B^i_a = 0$, (2.14) becomes

$$\begin{aligned}\check{L}^i_{j\beta} &= \left(\overset{\circ}{L}{}^i_{jk} B^k_\beta + \overset{\circ}{C}{}^i_{jk} B^k_a \overset{\circ}{H}{}^a_\beta\right) + \left(V^r_k C^i_{jr} + V^r_j C^i_{kr} + V^r_p a^{ip} C_{rkj} - \overset{\circ}{C}{}^i_{jr} B^r_b B^b_p V^p_k\right) B^k_\beta \\ &\quad + \left(\frac{\varphi''}{\varphi'} \overset{\circ}{y}_j \delta^i_k + \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i\right) B^k_a H^a_\beta,\end{aligned} \tag{2.15}$$

that is,

$$\begin{aligned}\check{L}^i_{j\beta} &= \overset{\circ}{\check{L}}{}^i_{j\beta} + \left(V^r_k C^i_{jr} + V^r_j C^i_{kr} + V^r_p a^{ip} C_{rkj} - \overset{\circ}{C}{}^i_{jr} B^r_b B^b_p V^p_k\right) B^k_\beta \\ &\quad + \left(\frac{\varphi''}{\varphi'} \overset{\circ}{y}_j \delta^i_k + \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i\right) B^k_a H^a_\beta,\end{aligned} \tag{2.16}$$

where $\overset{\circ}{\check{L}}{}^i_{j\beta} = \overset{\circ}{L}{}^i_{jk} B^k_\beta + \overset{\circ}{C}{}^i_{jk} B^k_a \overset{\circ}{H}{}^a_\beta$.

Using (1.12) in (2.13), we find that

$$\begin{aligned}\check{C}^i_{j\beta} &= \overset{\circ}{C}{}^i_{jk} B^k_\beta + \left(\frac{\varphi''}{\varphi'}\left(\delta^i_j \overset{\circ}{y}_k + \delta^i_k \overset{\circ}{y}_j\right) + \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i \right. \\ &\quad \left. + \frac{2\left(\varphi'''\varphi' - 2\varphi''^2\right)}{\varphi'\left(\varphi' + 2F^2\varphi''\right)} y^i \overset{\circ}{y}_j \overset{\circ}{y}_k\right) B^k_\beta,\end{aligned} \tag{2.17}$$

that is,

$$\begin{aligned}\check{C}^i_{j\beta} &= \overset{\circ}{\check{C}}{}^i_{j\beta} + \left(\frac{\varphi''}{\varphi'}\left(\delta^i_j \overset{\circ}{y}_k + \delta^i_k \overset{\circ}{y}_j\right) + \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i \right. \\ &\quad \left. + \frac{2\left(\varphi'''\varphi' - 2\varphi''^2\right)}{\varphi'\left(\varphi' + 2F^2\varphi''\right)} y^i \overset{\circ}{y}_j \overset{\circ}{y}_k\right) B^k_\beta,\end{aligned} \tag{2.18}$$

where $\overset{\circ}{\check{C}}{}^{i}_{j\beta} = \overset{\circ}{C}{}^{i}_{jk} B^k_\beta$. Thus, we have the following.

Theorem 2.5. *The coefficients of coupling for the subspace $\check{L}^m$ of an APL space L^n are given by* (2.16) *and* (2.18).

Definition 2.6 (cf. [8]). An operator D^T given by

$$D^T X^\alpha = X^\alpha_{|\beta} du^\beta + X^\alpha|_\beta \delta v^\beta, \tag{2.19}$$

where $X^\alpha_{|\beta} = \delta_\beta X^\alpha + X^\gamma L^\alpha_{\gamma\beta}$, $X^\alpha|_\beta = \dot{\partial}_\beta X^\alpha + X^\gamma C^\alpha_{\gamma\beta}$, is called the induced tangent connection by D. This defines an N-linear connection for $\check{L}^m$.

The coefficients $(L^\alpha_{\gamma\beta}, C^\alpha_{\gamma\beta})$ of D^T are given by

$$L^\alpha_{\beta\gamma} = B^\alpha_i \left(B^i_{\beta\gamma} + B^j_\beta \check{L}^i_{j\gamma} \right), \tag{2.20}$$

$$C^\alpha_{\beta\gamma} = B^\alpha_i B^j_\beta \check{C}^i_{j\gamma}. \tag{2.21}$$

Using (2.16) in (2.20), we get

$$\begin{aligned} L^\alpha_{\beta\gamma} = B^\alpha_i B^i_{\beta\gamma} + B^j_\beta B^\alpha_i \Bigg[\overset{\circ}{\check{L}}{}^{i}_{j\gamma} + \left(V^r_k C^i_{jr} + V^r_j C^i_{kr} + V^r_p a^{ip} C_{rkj} - \overset{\circ}{C}{}^{i}_{jr} B^r_b B^b_p V^p_k \right) B^k_\gamma \\ + \left(\frac{\varphi''}{\varphi'} \overset{\circ}{y}_j \delta^i_k + \frac{\varphi''}{\varphi' + 2F^2 \varphi''} g_{jk} y^i \right) B^k_a H^a_\gamma \Bigg], \end{aligned} \tag{2.22}$$

that is,

$$\begin{aligned} L^\alpha_{\beta\gamma} = B^\alpha_i \left(B^i_{\beta\gamma} + \overset{\circ}{\check{L}}{}^{i}_{j\gamma} B^j_\beta \right) + B^\alpha_i B^j_\beta \Bigg[\left(V^r_k C^i_{jr} + V^r_j C^i_{kr} + V^r_p a^{ip} C_{rkj} - \overset{\circ}{C}{}^{i}_{jr} B^r_b B^b_p V^p_k \right) B^k_\gamma \\ + \left(\frac{\varphi''}{\varphi'} \overset{\circ}{y}_j \delta^i_k + \frac{\varphi''}{\varphi' + 2F^2 \varphi''} g_{jk} y^i \right) B^k_a H^a_\gamma \Bigg]. \end{aligned} \tag{2.23}$$

If we take $\overset{\circ}{L}{}^{\alpha}_{\beta\gamma} = B^\alpha_i (B^i_{\beta\gamma} + \overset{\circ}{\check{L}}{}^{i}_{j\gamma} B^j_\beta)$, the last expression gives

$$\begin{aligned} L^\alpha_{\beta\gamma} = \overset{\circ}{L}{}^{\alpha}_{\beta\gamma} + B^\alpha_i B^j_\beta \Bigg[\left(V^r_k C^i_{jr} + V^r_j C^i_{kr} + V^r_p a^{ip} C_{rkj} - \overset{\circ}{C}{}^{i}_{jr} B^r_b B^b_p V^p_k \right) B^k_\gamma \\ + \left(\frac{\varphi''}{\varphi'} \overset{\circ}{y}_j \delta^i_k + \frac{\varphi''}{\varphi' + 2F^2 \varphi''} g_{jk} y^i \right) B^k_a H^a_\gamma \Bigg]. \end{aligned} \tag{2.24}$$

Next, using (2.18) in (2.21), we obtain

$$C^{\alpha}_{\beta\gamma} = B^{\alpha}_{i} B^{j}_{\beta} \overset{\circ}{\check{C}}{}^{i}_{j\gamma} + \left(\frac{\varphi''}{\varphi'} \left(\delta^{i}_{j} \overset{\circ}{y}_{k} + \delta^{i}_{k} \overset{\circ}{y}_{j} \right) + \frac{\varphi''}{\varphi' + 2F^{2}\varphi''} g_{jk} y^{i} + \frac{2(\varphi'''\varphi' - 2\varphi''^{2})}{\varphi'(\varphi' + 2F^{2}\varphi'')} y^{i} \overset{\circ}{y}_{j} \overset{\circ}{y}_{k} \right) B^{k}_{\gamma} B^{\alpha}_{i} B^{j}_{\beta}. \tag{2.25}$$

If we take $\overset{\circ}{C}{}^{\alpha}_{\beta\gamma} = B^{\alpha}_{i} B^{j}_{\beta} \overset{\circ}{\check{C}}{}^{i}_{j\gamma}$, (2.25) becomes

$$C^{\alpha}_{\beta\gamma} = \overset{\circ}{C}{}^{\alpha}_{\beta\gamma} + \left(\frac{\varphi''}{\varphi'} \left(\delta^{i}_{j} \overset{\circ}{y}_{k} + \delta^{i}_{k} \overset{\circ}{y}_{j} \right) + \frac{\varphi''}{\varphi' + 2F^{2}\varphi''} g_{jk} y^{i} + \frac{2(\varphi'''\varphi' - 2\varphi''^{2})}{\varphi'(\varphi' + 2F^{2}\varphi'')} y^{i} \overset{\circ}{y}_{j} \overset{\circ}{y}_{k} \right) B^{k}_{\gamma} B^{\alpha}_{i} B^{j}_{\beta}. \tag{2.26}$$

Thus, we have the following.

Theorem 2.7. *The coefficients of the induced tangent connection D^T for the subspace $\check{L}^m$ of an APL space are given by* (2.24) *and* (2.26).

Remarks. The torsion $T^{\alpha}_{\beta\gamma} = L^{\alpha}_{\beta\gamma} - L^{\alpha}_{\gamma\beta}$ does not vanish, in general, while $S^{\alpha}_{\beta\gamma} = C^{\alpha}_{\beta\gamma} - C^{\alpha}_{\gamma\beta} = 0$. These facts may be observed from (2.24) and (2.26).

Definition 2.8 (cf. [8]). An operator $D^{\perp}$ given by

$$D^{\perp} X^{a} = X^{a}_{|\alpha} du^{\alpha} + X^{a}|_{\alpha} \delta v^{\alpha}, \tag{2.27}$$

where $X^{a}_{|\alpha} = \delta_{\alpha} X^{a} + X^{b} L^{a}_{b\alpha}$, $X^{a}|_{\alpha} = \dot{\partial}_{\alpha} X^{a} + X^{b} C^{a}_{b\alpha}$, is called the induced normal connection by D.

The coefficients $(L^{a}_{b\gamma}, C^{a}_{b\gamma})$ of $D^{\perp}$ are given by

$$L^{a}_{b\gamma} = B^{a}_{i} \left(\delta_{\gamma} B^{i}_{b} + B^{j}_{b} \check{L}^{i}_{j\gamma} \right), \tag{2.28}$$

$$C^{a}_{b\gamma} = B^{a}_{i} \left(\dot{\partial}_{\gamma} B^{i}_{b} + B^{j}_{b} \check{C}^{i}_{j\gamma} \right). \tag{2.29}$$

Using (2.6) and (2.16) in (2.28), we find

$$\begin{aligned} L^{a}_{b\gamma} = {} & B^{a}_{i} \overset{\circ}{\delta}_{\gamma} B^{i}_{b} + B^{a}_{i} B^{\alpha}_{p} V^{p}_{j} B^{j}_{\gamma} \dot{\partial}_{\alpha} B^{i}_{b} \\ & + B^{j}_{b} B^{a}_{i} \Bigg[\overset{\circ}{\check{L}}{}^{i}_{j\gamma} + \left(V^{r}_{k} C^{i}_{jr} + V^{r}_{j} C^{i}_{kr} + V^{r}_{p} a^{ip} C_{rkj} - \overset{\circ}{C}{}^{i}_{jr} B^{r}_{c} B^{c}_{p} V^{p}_{k} \right) B^{k}_{\gamma} \\ & + \left(\frac{\varphi''}{\varphi'} \overset{\circ}{y}_{j} \delta^{i}_{k} + \frac{\varphi''}{\varphi' + 2F^{2}\varphi''} g_{jk} y^{i} \right) B^{k}_{c} H^{c}_{\gamma} \Bigg]. \end{aligned} \tag{2.30}$$

Taking $\overset{\circ}{L}{}^a_{b\gamma} = B^a_i(\overset{\circ}{\delta}_\gamma B^i_b + B^j_b \overset{\circ}{\check{L}}{}^i_{j\gamma})$ and using $\overset{\circ}{y}_j B^j_b = 0$, (2.30) reduces to

$$L^a_{b\gamma} = \overset{\circ}{L}{}^a_{b\gamma} + B^a_i B^\alpha_p V^p_j B^j_\gamma \dot{\partial}_\alpha B^i_b + \left(V^r_k C^i_{jr} + V^r_j C^i_{kr} + V^r_p a^{ip} C_{rkj} - \overset{\circ}{C}{}^i_{jr} B^r_b B^b_p V^p_k \right) B^a_i B^j_b B^k_\gamma$$
$$+ \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i B^k_c H^c_\gamma B^a_i B^j_b. \tag{2.31}$$

Next, using (2.18) in (2.29), we have

$$C^a_{b\gamma} = B^a_i \left(\dot{\partial}_\gamma B^i_b + B^j_b \overset{\circ}{\check{C}}{}^i_{j\gamma} \right) + \left[\frac{\varphi''}{\varphi'} \left(\delta^i_j \overset{\circ}{y}_k + \delta^i_k \overset{\circ}{y}_j \right) + \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i \right.$$
$$\left. + \frac{2(\varphi'''\varphi' - 2\varphi''^2)}{\varphi'(\varphi' + 2F^2\varphi'')} y^i \overset{\circ}{y}_j \overset{\circ}{y}_k \right] B^k_\gamma B^a_i B^j_b. \tag{2.32}$$

Taking $\overset{\circ}{C}{}^a_{b\gamma} = B^a_i(\dot{\partial}_\gamma B^i_b + B^j_b \overset{\circ}{\check{C}}{}^i_{j\gamma})$ and using (1.9) and $\overset{\circ}{y}_j B^j_b = 0$, the last equation yields

$$C^a_{b\gamma} = \overset{\circ}{C}{}^a_{b\gamma} + \frac{\varphi''}{\varphi'} \delta^a_b \overset{\circ}{y}_k B^k_\gamma + \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i B^k_\gamma B^a_i B^j_b. \tag{2.33}$$

Thus, we have the following.

Theorem 2.9. *The coefficients of induced normal connection $D^\perp$ for the subspace $\check{L}^m$ of an APL space L^n are given by* (2.31) *and* (2.33).

Definition 2.10 (cf. [8]). The (mixed) derivative of a mixed d-tensor field $T^{i\cdots\alpha\cdots a}_{j\cdots\beta\cdots b}$ is given by

$$\nabla T^{i\cdots\alpha\cdots a}_{j\cdots\beta\cdots b} = \left(\delta_\eta T^{i\cdots\alpha\cdots a}_{j\cdots\beta\cdots b} + T^{k\cdots\alpha\cdots a}_{j\cdots\beta\cdots b} \check{L}^i_{k\eta} + \cdots + T^{i\cdots\gamma\cdots a}_{j\cdots\beta\cdots b} L^\alpha_{\gamma\eta} + \cdots + T^{i\cdots\alpha\cdots c}_{j\cdots\beta\cdots b} L^a_{c\eta} \right.$$
$$\left. - T^{i\cdots\alpha\cdots a}_{k\cdots\beta\cdots b} \check{L}^k_{j\eta} - \cdots - T^{i\cdots\alpha\cdots a}_{j\cdots\gamma\cdots b} L^\gamma_{\beta\eta} - \cdots - T^{i\cdots\alpha\cdots a}_{j\cdots\beta\cdots c} \check{L}^c_{b\eta} \right) du^\eta$$
$$+ \left(\dot{\partial}_\eta T^{i\cdots\alpha\cdots a}_{j\cdots\beta\cdots b} + T^{k\cdots\alpha\cdots a}_{j\cdots\beta\cdots b} \check{C}^i_{k\eta} + \cdots + T^{i\cdots\gamma\cdots a}_{j\cdots\beta\cdots b} C^\alpha_{\gamma\eta} + \cdots + T^{i\cdots\alpha\cdots c}_{j\cdots\beta\cdots b} C^a_{c\eta} \right.$$
$$\left. - T^{i\cdots\alpha\cdots a}_{k\cdots\beta\cdots b} \check{C}^k_{j\eta} - \cdots - T^{i\cdots\alpha\cdots a}_{j\cdots\gamma\cdots b} C^\gamma_{\beta\eta} - \cdots - T^{i\cdots\alpha\cdots a}_{j\cdots\beta\cdots c} \check{C}^c_{b\eta} \right) \delta v^\eta. \tag{2.34}$$

The connection 1-forms,

$$\check{\omega}^i_j =: \check{L}^i_{j\alpha} du^\alpha + \check{C}^i_{j\alpha} \delta v^\alpha, \tag{2.35}$$

$$\omega^\alpha_\beta =: L^\alpha_{\beta\gamma} du^\gamma + C^\alpha_{\beta\gamma} \delta v^\gamma, \tag{2.36}$$

$$\omega^a_b =: L^a_{b\gamma} du^\gamma + C^a_{b\gamma} \delta v^\gamma, \tag{2.37}$$

are called the connection 1-forms of ∇. We have the following structure equations of ∇.

Theorem 2.11 (cf. [8]). *The structure equations of ∇ are as follows:*

$$\begin{aligned}
&d(du^{\alpha}) - du^{\beta} \wedge \omega^{\alpha}_{\beta} = -\Omega^{\alpha},\\
&d(\delta u^{\alpha}) - \delta u^{\beta} \wedge \omega^{\alpha}_{\beta} = -\dot{\Omega}^{\alpha},\\
&d\breve{\omega}^{i}_{j} - \breve{\omega}^{h}_{j} \wedge \breve{\omega}^{i}_{h} = -\breve{\Omega}^{i}_{j},\\
&d\omega^{\alpha}_{\beta} - \omega^{\gamma}_{\beta} \wedge \omega^{\alpha}_{\gamma} = -\Omega^{\alpha}_{\beta},\\
&d\omega^{a}_{b} - \omega^{c}_{b} \wedge \omega^{a}_{c} = -\Omega^{a}_{b},
\end{aligned} \tag{2.38}$$

where the 2-forms of torsions $\Omega^{\alpha}, \dot{\Omega}^{\alpha}$ are given by

$$\begin{aligned}
\Omega^{\alpha} &= \frac{1}{2} T^{\alpha}_{\beta\gamma} du^{\beta} \wedge du^{\gamma} + C^{\alpha}_{\beta\gamma} du^{\beta} \wedge \delta v^{\gamma},\\
\dot{\Omega}^{\alpha} &= \frac{1}{2} R^{\alpha}_{\beta\gamma} du^{\beta} \wedge du^{\gamma} + P^{\alpha}_{\beta\gamma} du^{\beta} \wedge \delta v^{\gamma},
\end{aligned} \tag{2.39}$$

with $P^{\alpha}_{\beta\gamma} = \dot{\partial}_{\gamma} \check{N}^{\alpha}_{\beta} - L^{\alpha}_{\beta\gamma}$, and the 2-forms of curvature $\breve{\Omega}^{i}_{j}, \Omega^{\alpha}_{\beta}$ and Ω^{a}_{b}, are given by

$$\begin{aligned}
\breve{\Omega}^{i}_{j} &= \frac{1}{2} \breve{R}^{i}_{j\alpha\beta} du^{\alpha} \wedge du^{\beta} + \breve{P}^{i}_{j\alpha\beta} du^{\alpha} \wedge \delta v^{\beta} + \frac{1}{2} \breve{S}^{i}_{j\alpha\beta} \delta v^{\alpha} \wedge \delta v^{\beta},\\
\Omega^{\alpha}_{\beta} &= \frac{1}{2} R^{\alpha}_{\beta\gamma\delta} du^{\gamma} \wedge du^{\delta} + P^{\alpha}_{\beta\gamma\delta} du^{\gamma} \wedge \delta v^{\delta} + \frac{1}{2} S^{\alpha}_{\beta\gamma\delta} \delta v^{\gamma} \wedge \delta v^{\delta},\\
\Omega^{a}_{b} &= \frac{1}{2} R^{a}_{b\alpha\beta} du^{\alpha} \wedge du^{\beta} + P^{a}_{b\alpha\beta} du^{\alpha} \wedge \delta v^{\beta} + \frac{1}{2} S^{a}_{b\alpha\beta} \delta v^{\alpha} \wedge \delta v^{\beta}.
\end{aligned} \tag{2.40}$$

We will use the following notations in Section 4:

$$\text{(a) } \breve{\Omega}_{ij} = \breve{\Omega}^{h}_{i} a_{hj}, \qquad \text{(b) } \Omega_{\alpha\beta} = \Omega^{\gamma}_{\alpha} a_{\gamma\beta}, \qquad \text{(c) } \Omega_{ab} = \Omega^{c}_{b} \delta_{ac}. \tag{2.41}$$

3. The Gauss-Weingarten Formulae

The Gauss-Weingarten formulae for the subspace $\check{L}^{m} = (\check{M}, \check{L}(u,v))$ of a Lagrange space L^{n} are given by (cf. [8])

$$\nabla B^{i}_{\alpha} = B^{i}_{a} \Pi^{a}_{\alpha}, \qquad \nabla B^{i}_{a} = -B^{i}_{\beta} \Pi^{\beta}_{a}, \tag{3.1}$$

where

$$\Pi^a_\alpha = H^a_{\alpha\beta} du^\beta + K^a_{\alpha\beta}\delta v^\beta,$$
$$\Pi^\beta_a = g^{\beta\gamma}\delta_{ab}\Pi^b_\gamma, \tag{3.2}$$

$$\text{(a)}\ H^a_{\alpha\beta} = B^a_i\left(\delta_\beta B^i_\alpha + B^j_\alpha \check{L}^i_{j\beta}\right), \qquad \text{(b)}\ K^a_{\alpha\beta} = B^a_i B^j_\alpha \check{C}^i_{j\beta}. \tag{3.3}$$

Using (2.6) and (2.16) in (3.3)(a), we have

$$\begin{aligned} H^a_{\alpha\beta} = {} & B^a_i\left(\mathring{\delta}_\beta B^i_\alpha + B^j_\alpha \mathring{\check{L}}{}^i_{j\beta}\right) + B^a_i B^\gamma_p V^p_j B^j_\beta B^i_{\alpha\gamma} \\ & + \left(V^r_k C^i_{jr} + V^r_j C^i_{kr} + V^r_p a^{ip} C_{rkj} - \mathring{C}{}^i_{jr} B^r_b B^b_p V^p_k\right) B^a_i B^j_\alpha B^k_\beta \\ & + \left(\frac{\varphi''}{\varphi'}\mathring{y}_j \delta^i_k + \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i\right) B^k_b H^b_\beta B^a_i B^j_\alpha. \end{aligned} \tag{3.4}$$

If we take $\mathring{H}{}^a_{\alpha\beta} = B^a_i(\mathring{\delta}_\beta B^i_\alpha + B^j_\alpha \mathring{\check{L}}{}^i_{j\beta})$, the last expression provides

$$\begin{aligned} H^a_{\alpha\beta} = {} & \mathring{H}{}^a_{\alpha\beta} + B^a_i B^\gamma_p V^p_j B^j_\beta B^i_{\alpha\gamma} + \left(V^r_k C^i_{jr} + V^r_j C^i_{kr} + V^r_p a^{ip} C_{rkj} - \mathring{C}{}^i_{jr} B^r_b B^b_p V^p_k\right) B^a_i B^j_\alpha B^k_\beta \\ & + \left(\frac{\varphi''}{\varphi'}\mathring{y}_j \delta^i_k + \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i\right) B^k_b H^b_\beta B^a_i B^j_\alpha. \end{aligned} \tag{3.5}$$

Next, using (2.18) in (3.3)(b) and keeping (1.9) in view, we find

$$K^a_{\alpha\beta} = \mathring{K}{}^a_{\alpha\beta} + \left(\frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i + \frac{2(\varphi'''\varphi' - \varphi''^2)}{\varphi'(\varphi' + 2F^2\varphi'')} y^i \mathring{y}_j \mathring{y}_k\right) B^a_i B^j_\alpha B^k_\beta, \tag{3.6}$$

where $\mathring{K}{}^a_{\alpha\beta} = B^a_i B^j_\alpha \mathring{\check{C}}{}^i_{j\beta}$. Thus, we have the following.

Theorem 3.1. *The following Gauss-Weingarten formulae for the subspace $\check{L}^m$ of an APL space hold:*

$$\nabla B^i_\alpha = B^i_a \Pi^a_\alpha, \qquad \nabla B^i_a = -B^i_\beta \Pi^\beta_a, \tag{3.7}$$

where

$$\Pi^a_\alpha = H^a_{\alpha\beta} du^\beta + K^a_{\alpha\beta}\delta v^\beta, \qquad \Pi^\beta_a = g^{\beta\gamma}\delta_{ab}\Pi^b_\gamma,$$

$$\begin{aligned} H^a_{\alpha\beta} &= \overset{\circ}{H}{}^a_{\alpha\beta} + B^a_i B^\gamma_p V^p_j B^j_\beta B^i_{\alpha\gamma} + \left(V^r_k C^i_{jr} + V^r_j C^i_{kr} + V^r_p a^{ip} C_{rkj} - \overset{\circ}{C}{}^i_{jr} B^r_b B^b_p V^p_k \right) B^a_i B^j_\alpha B^k_\beta \\ &\quad + \left(\frac{\varphi''}{\varphi'} \overset{\circ}{y}_j \delta^i_k + \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i \right) B^k_b H^b_\beta B^a_i B^j_\alpha, \\ K^a_{\alpha\beta} &= \overset{\circ}{K}{}^a_{\alpha\beta} + \left(\frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i + \frac{2(\varphi'''\varphi' - \varphi''^2)}{\varphi'(\varphi' + 2F^2\varphi'')} y^i \overset{\circ}{y}_j \overset{\circ}{y}_k \right) B^a_i B^j_\alpha B^k_\beta. \end{aligned} \tag{3.8}$$

Remark 3.2. $H^a_{\alpha\beta}$ and $K^a_{\alpha\beta}$ given, respectively, by (3.5) and (3.6) are called the second fundamental d-tensor fields of immersion i.

The following consequences of Theorem 3.1 are straightforward.

Corollary 3.3. *In a subspace $\check{L}^m$ of an APL space, we have the following:*

$$\begin{aligned} &(a)\ \nabla a_{\alpha\beta} = 0, \\ &(b)\ \nabla B^i_\alpha = 0, \end{aligned} \tag{3.9}$$

if and only if

$$\begin{aligned} \overset{\circ}{H}{}^a_{\alpha\beta} &= -\Bigg[B^a_i B^\gamma_p V^p_j B^j_\beta B^i_{\alpha\gamma} + \left(V^r_k C^i_{jr} + V^r_j C^i_{kr} + V^r_p a^{ip} C_{rkj} - \overset{\circ}{C}{}^i_{jr} B^r_b B^b_p V^p_k \right) B^a_i B^j_\alpha B^k_\beta \\ &\quad + \left(\frac{\varphi''}{\varphi'} \overset{\circ}{y}_j \delta^i_k + \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i \right) B^k_b H^b_\beta B^a_i B^j_\alpha \Bigg], \\ \overset{\circ}{K}{}^a_{\alpha\beta} &= -\left(\frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i + \frac{2\left(\varphi'''\varphi' - \varphi''^2\right)}{\varphi'(\varphi' + 2F^2\varphi'')} y^i \overset{\circ}{y}_j \overset{\circ}{y}_k \right) B^a_i B^j_\alpha B^k_\beta. \end{aligned} \tag{3.10}$$

4. The Gauss-Codazzi Equations

The Gauss-Codazzi Equations for the subspace $\check{L}^m = (\check{M}, \check{L}(u,v))$ of a Lagrange space L^n are given by (cf. [8])

$$B^i_\alpha B^j_\beta \check{\Omega}_{ij} - \Omega_{\alpha\beta} = \Pi_{\beta a} \wedge \Pi^a_\alpha, \tag{4.1}$$

$$B^i_a B^j_b \check{\Omega}_{ij} - \Omega_{ab} = \Pi_{\gamma b} \wedge \Pi^\gamma_a, \tag{4.2}$$

$$-B^i_\alpha B^j_a \check{\Omega}_{ij} = \delta_{ab}\left(d\Pi^b_\alpha + \Pi^b_\beta \wedge \omega^\beta_\alpha - \Pi^c_\alpha \wedge \omega^b_c \right), \tag{4.3}$$

where

$$\text{(a) } \Pi_{\alpha a} = g_{\alpha\beta}\Pi_{a}^{\beta}, \qquad \text{(b) } \Pi_{\gamma b} = \delta_{bc}\Pi_{\gamma}^{c}. \tag{4.4}$$

Using (1.3) in (2.41)(a), we find that

$$\check{\Omega}_{ij} = \varphi'\check{\Omega}_{i}^{h}g_{hj} + 2\varphi''\check{\Omega}_{i}^{h}\mathring{y}_{h}\mathring{y}_{j}. \tag{4.5}$$

Applying $a_{\gamma\beta} = B_{\gamma}^{i}B_{\beta}^{j}a_{ij}$ in (2.41)(b), we have $\Omega_{\alpha\beta} = B_{\gamma}^{i}B_{\beta}^{j}\Omega_{\alpha}^{\gamma}a_{ij}$, which in view of (1.3) becomes

$$\Omega_{\alpha\beta} = \varphi' g_{ij}B_{\gamma}^{i}B_{\beta}^{j}\Omega_{\alpha}^{\gamma} + 2\varphi''\mathring{y}_{i}\mathring{y}_{j}B_{\gamma}^{i}B_{\beta}^{j}\Omega_{\alpha}^{\gamma}, \tag{4.6}$$

that is,

$$\Omega_{\alpha\beta} = \varphi' g_{\gamma\beta}\Omega_{\alpha}^{\gamma} + 2\varphi''\mathring{y}_{i}\mathring{y}_{j}B_{\gamma}^{i}B_{\beta}^{j}\Omega_{\alpha}^{\gamma}. \tag{4.7}$$

For the subspace $\check{L}^{m}$ of an APL space, (4.4)(a) is of the form $\Pi_{\alpha a} = a_{\alpha\beta}\Pi_{a}^{\beta}$, which in view of $a_{\alpha\beta} = B_{\alpha}^{i}B_{\beta}^{j}a_{ij}$ and (1.3) becomes $\Pi_{\alpha a} = \varphi' B_{\alpha}^{i}B_{\beta}^{j}a_{ij}\Pi_{a}^{\beta} + 2\varphi''\mathring{y}_{i}\mathring{y}_{j}B_{\alpha}^{i}B_{\beta}^{j}\Pi_{a}^{\beta}$, that is,

$$\Pi_{\alpha a} = \varphi' g_{\alpha\beta}\Pi_{a}^{\beta} + 2\varphi''\mathring{y}_{i}\mathring{y}_{j}B_{\alpha}^{i}B_{\beta}^{j}\Pi_{a}^{\beta}. \tag{4.8}$$

Thus, we have the following.

Theorem 4.1. *The Gauss-Codazzi equations for a Lagrange subspace $\check{L}^{m}$ of an APL space are given by* (4.1)–(4.3) *with $\Pi_{\alpha a}$, $\Pi_{\gamma b}$, $\check{\Omega}_{ij}$, $\Omega_{\alpha\beta}$, and ω_{c}^{b}, respectively, given by* (4.8), (4.4)*(b),* (4.5), (4.7), *and* (2.37).

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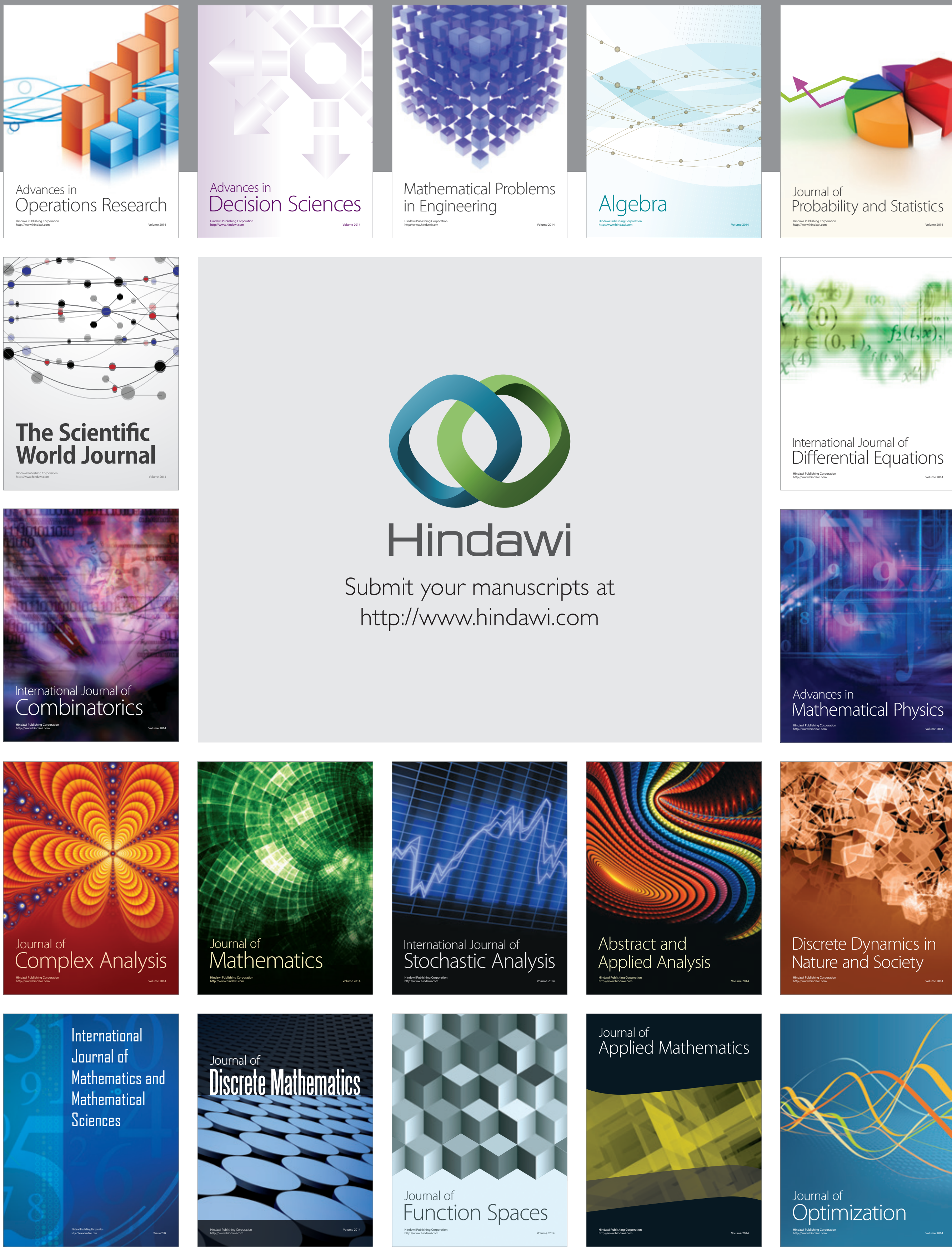
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