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A CHARACTERIZATION OF B^*-ALGEBRAS

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Abstract. A characterization of B^*–algebras amongst all Banach algebras with bounded approximate identities is obtained.

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1. Introduction.

We recall that an approximate identity in a Banach algebra A is a net $\{e_\alpha : \alpha \in I\}$ in A where I is a directed set such that $\lim\limits_\alpha e_\alpha x = x = \lim\limits_\alpha x e_\alpha$ for every x in A. If there is a finite constant M such that $\|e_\alpha\| \leq M$ for all α, then the approximate identity is said to be bounded.

Let A be a Banach algebra. For each x in A, let

$$D_A(x) = \{f \in A' : \|f\| = 1 = f(x)\}.$$

By a corollary of the Hahn–Banach theorem, $D_A(x)$ is non-empty. We denote $S(A) = \{x \in A : \|x\| = 1\}$.

For each $a \in A$, we call the set $V_A(a) = \{f(ax) : f \in D_A(x),\ x \in S(A)\}$ the *spatial numerical range* of a.

We recall [5] that the relative numerical range of a in A with respect to $x \in A$, is defined as

$$\overset{\circ}{V}_x(A,a) = \{f(ax) : f \in D_A(x)\}.$$

Thus we see that $V_A(a) = \bigcup \left\{ \overset{\circ}{V}_x(A,a) : x \in S(A) \right\}$, which is a bounded subset of the complex numbers bounded by $\|a\|$.

If A has an approximate identity of norm less than or equal to one then A can embedded, isometrically and isomorphically, in a unital Banach algebra A^+ in such a way that for each a in A

$$V(A^+, a) = \overline{co}\, V_A(a),$$

where $V(A^+, a) = \{f(a) : f \in (A^+)',\ \|f\| = 1 = f(a) = \|a\|\}$. For details see [4], Theorem 2.3.

An element h of a Banach algebra A is said to be Hermitian if $V_A(a) \subset R$. We denote by $H(A)$ the set of all Hermitian elements of A. A B^*-algebra is a Banach algebra A with an involution, $a \to a^*$ satisfying the following conditions:

(1) $(a+b)^* = a^* + b^*$;

(2) $(ab)^* = b^*a^*$;

(3) $(\alpha a)^* = \bar{\alpha}a^*$;

(4) $a^{**} = a$; and

(5) $|a^*a| = |a|^2$

for all a, b in A and α in C.

An element a in a B^*-algebra is said to be self-adjoint if $a = a^*$. The set of all self adjoint elements will be denoted by $S(A)$. Each element $a \in A$ can be written uniquely in the form $a = h + ik$ where $h, k \in S(A)$. Some of the well known properties of $S(A)$ are the following:

a) The set $S(A)$ is a real partially ordered Banach space,

b) each of its elements has real spectrum,

c) if $h, k \in S(A)$ then $i(hk - kh) \in S(A)$, and

d) for each $h \in S(A)$, the spectral radius $\rho(h) = \|h\|$.

It is clear that the set of Hermitian elements, $H(A)$, of a Banach algebra with a bounded approximate identity of norm less than or equal to one has many of the properties of $S(A)$ in a B^*-algebra.

In this note we prove that in an arbitrary B^*-algebra A, $H(A) = S(A)$ in Theorem 2.1. This results mimics a result by Bohnenblust and Karlin [2].

In [8], Vidav has shown that a unital Banach algebra A with the following conditions:

(1) $A = H(A) + iH(A)$;

(2) for each h in $H(A)$ there exists h_1, h_2 in $H(A)$ such that $h_1 + ih_2 = h^2$ and $h_1h_2 = h_2h_1$

is a B^*-algebra with Vidav-involution. Combining the results of Vidav [8], Berkson [1], and Glickfeld [6] we obtain the result that if A is a unital Banach algebra such that $A = H(A) + iH(A)$ then A is a B^*-algebra under the Vidav-involution. Here, we extend this result to the nonunital case in the form of Lemma 3.1.

Finally, combining the results of Theorem 2.1 and Lemma 3.1 we have a characterization of B^*-algebras with bounded approximate identities.

2. Some Results.

We now prove the following theorem.

Theorem 2.1 *Let A be a B^*-algebra with a bounded approximate identity of norm less than or equal to one. An element of A is Hermitian if and only if it is self-adjoint.*

Proof. *Case 1.* Suppose that A has a unit element 1. Let $f \in D_A(1)$. Then it is known that such a functional has the property that $f(h^*) = \overline{f(h)}$, for every h in A. Thus if h is a self-adjoint element of A, $f(h) = f(h^*) = \overline{f(h)}$ and hence $f(h)$ is real for all f in $D_A(1)$. Hence, $S(A) \subseteq H(A)$.

Case 2. If A has no identity element then it will have an approximate identity of norm less than or equal to one. Also, with the involution defined by $(a, \alpha)^* = (a^*, \bar{\alpha})$ for $(a, \alpha) \in A^+$, and by Theorem 2.3 in [4], A^+ becomes a unital B^*- algebra containing as a sub-B^*-algebra, ([3], 1.3.8).

Let h be a self-adjoint element of A. Then $(h, 0)$ is self-adjoint and hence Hermitian in the unital B^*-algebra A^+. Hence $h \in H(A)$. We have therefore for any B^*-algebra, $S(A) \subseteq H(A)$.

Suppose conversely that $h \in H(A)$. Then for h_1 and h_2 in $S(A)$, $h = h_1 + ih_2$. This implies that $\nu(h_2) = 0$ (where $\nu(x) = \sup\{|\lambda| : \lambda \in V_A(x)\}$ and is called numerical radius of x in A) and hence $h_2 = 0$. Thus $h = h_1$ so that h is self-adjoint. That is $H(A) \subseteq S(A)$ and hence the theorem.

Remark 2.1 The above theorem shows that in a B^*-algebra the Hermitian elements generate the whole algebra in the sense that each element a may be written in the form $a = h_1 + ih_2$ with h_1 and h_2 in $H(A)$. In an arbitrary Banach algebra A this is not true. We therefore consider the set $J(A) = H(A) + iH(A)$. Since $H(A)$ is a real space it follows that $J(A)$ is a complex linear space. If A has no unit element then by Theorem 2.3, [4], $J(A) \times C = J(A^+)$. We define a map $a \to a^*$ from $J(A)$ into itself by

$$(h_1 + ih_2)^* = h_1 - ih_2, \text{ for all } h_1,\ h_2 \in H(A).$$

The linear map $a \to a^*$ is known as the Vidav-involution on $J(A)$.

Remark 2.2 If A has no unit element then it is a simple matter to verify that the Vidav-involution on $J(A^+)$ is an extension of the Vidav-involution on $J(A)$. The space $J(A)$ is a complex Banach space and $a \to a^*$ is a continuous linear involution on $J(A)$. In general, the Banach space $J(A)$ is not an algebra, and if $J(A)$ is an algebra under some conditions, then the Vidav-involution has the additional property

$$(ab)^* = a^*b^*, \text{ for all } a,\ b \in J(A).$$

3. Characterization.

Vidav has shown in [8] that a unital Banach algebra A with the following conditions:

(V1) $A = H(A) + iH(A)$,

(V2) for each h in $H(A)$ there exists h_1, h_2 in $H(A)$ such that $h_1 + ih_2 = h^2$ and $h_1h_2 = h_2h_1$,

is a B^*-algebra with Vidav-involution and a norm equivalent to the original norm on A.

According to Palmer [7], the condition (V1) implies (V2). Also Berkson [1], Glickfeld [6], and Palmer [7] have shown that if (V1) is satisfied by the algebra A the equivalent norm by Vidav is equal to the original norm on A. So by these results we have the result that if A is a unital Banach algebra satisfying (V1) then A is B^*-algebra under the Vidav-involution. The following lemma extends this result to the non-unital case.

Lemma 3.1 *Let A be a Banach algebra with a bounded approximate identity of norm less than or equal to one. Suppose that every a in A has the form $a = h_1 + ih_2$, for all h_1, h_2 in $H(A)$. Then with the Vidav-involution, A is a B^*-algebra.*

Proof. From Remark 2.1 we have that $J(A^+) = J(A) \times C$. Since $J(A) = A$ (by the hypothesis) we have $J(A^+) = A^+$. Therefore A^+ is a unital B^*-algebra under the Vidav-involution. Furthermore, A is a closed and self adjoint subalgebra of A^+, and is therefore a B^*-algebra under the Vidav-involution.

Finally, combining the results of Theorem 2.1 and Lemma 3.1 we have the following:

Theorem 3.2 *Let A be a Banach algebra with a bounded approximate identity of norm less than or equal to one. Then A is a B^*-algebra under some involution if and only if each element a of A can be written in the form $a = h_1 + ih_2$ where h_1 and h_2 are Hermitian elements of A.*

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