Internat. J. Math. & Math. Sci.
VOL. 20 NO. 3 (1997) 503-510

SOME PROPERTIES AND CHARACTERIZATIONS OF *A*-NORMAL FUNCTIONS

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(Received January 11, 1994 and in revised form October 23, 1995)

ABSTRACT. Let M be the set of all functions meromorphic on $D = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in (0, 1]$, a function $f \in M$ is called a-normal function of bounded (vanishing) type or $f \in N^a$ (N_0^a), if $\sup_{z\in D}(1 - |z|)^a f^{\#}(z) < \infty$ $(\lim_{|z|\to 1}(1 - |z|)^a f^{\#}(z) = 0)$. In this paper we not only show the discontinuity of N^a and N_0^a relative to containment as a varies, which shows $\bigcup_{0<a<1} N^a \subset UBC_0$, but also give several characterizations of N^a and N_0^a which are real extensions for characterizations of N and N_0.

KEY WORDS AND PHRASES: A-normal function, UBC_0.
1991 AMS SUBJECT CLASSIFICATION CODES: 30C45, 31A20

1. INTRODUCTION.

Throughout this paper, let $D = \{z : |z| < 1\}$ be the unit disk in $\mathbb{C}$ and $dm(z)$ the two-dimensional Lebesgue measure on D. Also let $\phi_w(z) = (w - z)/(1 - \overline{w}z)$ to be a canonical Möbius map of D onto D determined by $w \in D$, and let $D(w, r) = \{z \in D : |\phi_w(z)| < r\}$ a pseudohyperbolic disk with center $w \in D$ and radius $r \in [0, 1]$. Suppose that $g(z, w) = -log|\phi_w(z)|$ is the Green function of D with logarithmic singularity at $w \in D$. Also assume that $a \in (0, 1]$ and M is the class of functions meromorphic on D. For $f \in M$, let $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$, which is the spherical derivative of f. Further we say f is an a-normal function of bounded type if

$$\|f\|_{N^a} = \sup_{z\in D}(1 - |z|)^a f^{\#}(z) < \infty, \tag{1.1}$$

and f is an a-normal function of vanishing type if

$$\lim_{|z|\to 1}(1 - |z|)^a f^{\#}(z) = 0. \tag{1.2}$$

The families of all a-normal functions of bounded and vanishing type are denoted by N^a and N_0^a, respectively. It is easy to observe that $N_0^a \subset N^a$ and that for $a \in (0, 1)$, N_0^a and N^a are proper subsets of N and N_0, which are the classical sets of normal and little normal functions, namely, $N = N^1$ and $N_0 = N_0^1$, respectively.

There has been much interesting research on N and N_0 (see [1-3]), and hence we look for N^a and N_0^a to have some analogous properties. In this paper, we first consider the continuity of the families N^a and N_0^a as a varies, and we find that both are discontinuous , morever that $\bigcup_{0<a<1/2} N^a$ and $\bigcup_{1/2\le a<1} N^a$ are proper subsets of $D_\#$ and UBC_0, respectively, where $D_\#$ is the family of functions $f \in M$ satisfying

$$\|f\|_{D_\#}^2 = \int_D [f^\#(z)]^2\, dm(z) < \infty, \tag{1.3}$$

and UBC_0 is the family of functions $f \in M$ satisfying

$$\lim_{|w|\to 1} \int_D [f^\#(z)]^2 g(z,w)\, dm(z) = 0. \tag{1.4}$$

Here, it is worth while mentioning that $D_\# \subset UBC_0$ and that UBC_0 is an important meromorphic counterpart of $VMOA$—the space of analytic functions with vanishing mean oscillation on D (see [4,8]). We then characterize functions in N^a and N_0^a and obtain three criterions which are extensions of criteria for N and N_0.

2. CONTINUITY OF N^a AND N_0^a.

In this section, we pay attention to the continuity of N^a and N_0^a. Firstly, we see the monotonicity of N^a and N_0^a. More precisely we have

THEOREM 2.1 Let $a_1, a_2 \in (0,1]$. If $a_1 < a_2$ then

(i). $N^{a_1} \subset N^{a_2}$.

(ii). $N_0^{a_1} \subset N_0^{a_2}$.

PROOF. It sufficies to prove $N^{a_1} \subset N_0^{a_2}$ for $a_1 < a_2$. Let $f \in N^{a_1}$, then $\|f\|_{N^{a_1}} < \infty$ and

$$(1-|z|)^{a_2} f^\#(z) \le (1-|z|)^{a_2-a_1} \|f\|_{N^{a_1}}.$$

This gives $f \in N_0^{a_2}$, i.e., $N^{a_1} \subseteq N_0^{a_2}$. As to the strict inculsion, we take a function $f_1(z) = (1-z)^{1-a_3}$, $a_3 \in (a_1, a_2)$. A simple computation just gives $f_1 \in N_0^{a_2} \backslash N^{a_1}$. In fact,

$$\|f_1\|_{N^{a_1}} = (1-a_3) \sup_{z\in D} \frac{(1-|z|)^{a_1}|1-z|^{-a_3}}{1+|1-z|^{2(1-a_3)}} = \infty.$$

At the same time,

$$\lim_{|z|\to 1} (1-|z|)^{a_2} f_1^\#(z) = (1-a_3) \lim_{|z|\to 1} \frac{(1-|z|)^{a_2}|1-z|^{-a_3}}{1+|1-z|^{2(1-a_3)}} = 0.$$

Thus, $f_1 \in N_0^{a_2} \setminus N^{a_1}$. So, $N^{a_1} \subset N_0^{a_2}$.

Denote by $D_\#^\infty$ the class of functions $f \in M$ with

$$\|f\|_{D_\#^\infty} = \sup_{z\in D} f^\#(z) < \infty. \tag{2.1}$$

For $a \in (0,1)$, it is easy to see that $D_\#^\infty \subset N_0^a$. Furthermore, Theorem 2.1, together with $N^a \subset N$, $N_0^a \subset N_0$ and [3,4] suggest that we consider the continuity of N^a and N_0^a. For this purpose, we need a corollary which can be viewed as an application of Theorem 2.1.

COROLLARY 2.2 Let $a, b \in (0,1]$. Then

(i). $\bigcup_{a<b} N_0^a = \bigcup_{a<b} N^a$.

(ii). $\bigcap_{a<b} N_0^b = \bigcap_{a<b} N^b$.

PROOF. (i). On the one hand, the relation: $\bigcup_{a<b} N_0^a \subseteq \bigcup_{a<b} N^a$ is clear. On the other hand, if $f \in \bigcup_{a<b} N^a$, then f must be in some N^a, saying, N^a, where $a \in (0,b)$. However, for $a' \in (a,b)$ we have $f \in N_0^{a'}$ by the proof of Theorem 2.1. So $f \in \bigcup_{a<b} N_0^a$, and hence $\bigcup_{a<b} N_0^a = \bigcup_{a<b} N^a$.

(ii). This part can be proved similarly.

Now, we can state the discontinuity of N^a and of N_0^a .

THEOREM 2.3 Let $a, b \in (0,1]$. Then

(i). $\bigcup_{a<b} N^a \subset N_0^b$.

(ii). $\bigcap_{a<b} N_0^b \supset N^a$.

PROOF. Owing to Theorem 2.1 and Corollary 2.2, we only require proving (1). $\bigcup_{a<b} N_0^a \neq N_0^b$ and (2). $\bigcap_{a<b} N^b \neq N^a$.

First, let us consider (1). If $f_2(z) = \sum_{k=1}^{\infty} \frac{z^{2^k}}{k2^{k(1-b)}}$, then we get that f_2 is bounded on D. Since $\lim_{k\to\infty} \frac{2^{k(b-a)}}{k} = \infty$ for $b > a$, $f_2 \notin \bigcup_{a<b} N_0^a$ from [7, Theorem 1]. But it follows that $f_2 \in N_0^b$ again from [7, Theorem 1]. Note that we have here used a fact: $f^\#$ is equivalent to $|f'|$ once f is bounded and analytic on D. The above facts tell us that $\bigcup_{a<b} N_0^a \neq N_0^b$ is true.

Second, let us consider (2). For this, we pick $f_3(z) = \sum_{k=1}^{\infty} \frac{kz^{2^k}}{2^{k(1-a)}}$. It is clear that f_3 is bounded on D. Moreover $f_3 \notin N^a$ by using [7,Theorem 1]. However, $\lim_{k\to\infty} \frac{k}{2^{k(b-a)}} = 0$ as $b > a$, and then $f_3 \in \bigcap_{a<b} N^b$. That is to say, $\bigcap_{a<b} N^b \neq N^a$.

This completes the proof.

Finally, we discuss a special case of Theorem 2.3. Theorem 2.3 implies that $\bigcup_{0<a<1} N^a \subset N_0$. Noting the inclusion: $UBC_0 \subset N_0$ [8], we will naturally ask what is connection between $\bigcup_{0<a<1} N^a$ and UBC_0. It is a little bit surprising to us that $\bigcup_{0<a<1} N^a$ is a proper subset of UBC_0. This result shows that there is a big gap from $\bigcup_{0<a<1} N^a$ or $\bigcup_{0<a<1} N_0^a$ to N^a or N_0^a. Exactly speaking, we obtain

THEOREM 2.4

(i). $\bigcup_{0<a<1/2} N^a \subset D_\#$.

(ii). $\bigcup_{1/2\leq a<1} N^a \not\subseteq D_\# \not\subseteq \bigcup_{1/2\leq a<1} N^a \subset UBC_0$.

(iii). $\bigcap_{0<a<1} N_0^a \supset D_\#^\infty$.

PROOF. First of all, setting $f_4(z) = log(1-z)$, we check that $f_4 \in D_\# \setminus \bigcup_{0<a<1} N^a$. Indeed, we have

$$\begin{aligned} \|f_4\|_{D_\#}^2 &= \int_D \frac{1}{|1-z|^2[1+|log(1-z)|^2]^2}\, dm(z) \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \frac{1}{t[1+\theta^2+log^2 t]^2}\, dt d\theta \\ &\leq 2\pi \arctan\frac{\pi}{2} \end{aligned} \tag{2.2}$$

and

$$\|f_4\|_{N^a} \geq \lim_{t\to 1} \frac{(1-t)^{a-1}}{1+log^2(1-t)} = \infty \tag{2.3}$$

for any $a \in (0,1)$.

Now we turn to the proofs of (i), (ii) and (iii).

(i). If $f \in \bigcup_{0<a<1/2} N^a$, then there is an $a \in (0,1/2)$ such that $f \in N^a$ and thus

$$\|f\|_{D_\#}^2 \leq \|f\|_{N^a}^2 \int_D \frac{1}{(1-|z|)^{2a}}\, dm(z). \tag{2.4}$$

(2.3) and (2.4) imply that (i) is true.

(ii). From Theorem 2.1, (2.1), (2.2) and (2.3) it is seen that we only have need to demonstrate (1). $\bigcup_{1/2\leq a<1} N^a \not\subseteq D_\#$ and (2). $\bigcup_{1/2\leq a<1} N^a \subset UBC_0$. For (1), we take a function $f_5(z) = \sum_{k=0}^{\infty} \frac{z^{2^k}}{2^{k/2}}$. It is clear that f_5 is bounded and also in $N^a \setminus D_\#$, $a \in [1/2,1)$, from [7, Theorem 1]. Thus (1) holds. For (2), we may fix $a \in [1/2,1)$. For $w \in D$ and $f \in N^a$, we have

$$\int_D [f^\#(z)]^2 g(z,w)\,dm(z) \leq \|f\|_{N^a}^2 \int_D \frac{g(z,w)}{(1-|z|)^{2a}}\,dm(z)$$
$$\leq 2^{2a}\|f\|_{N^a}^2 \int_D \frac{-(1-|w|^2)^{2(1-a)} log|z|}{(1-|z|^2)^{2a}|1-\overline{w}z|^{4(1-a)}}\,dm(z)$$

By noting that $-log|z| \leq 8(1-|z|)$ for $|z| \geq \frac{1}{4}$, we also get

$$\int_{D\setminus D(0,\frac{1}{4})} \frac{-log|z|}{(1-|z|^2)^{2a}|1-\overline{w}z|^{4(1-a)}}\,dm(z) \leq 8\int_D \frac{(1-|z|^2)^{1-2a}}{|1-\overline{w}z|^{4(1-a)}}\,dm(z)$$

and

$$\int_{D(0,\frac{1}{4})} \frac{-log|z|}{(1-|z|^2)^{2a}|1-\overline{w}z|^{4(1-a)}}\,dm(z) \leq (\frac{4}{3})^{4-2a} 2\pi \int_0^{\frac{1}{4}} t log\frac{1}{t}\,dt$$

Furthermore, we can obtain a constant $C > 0$ so that

$$\int_D [f^\#(z)]^2 g(z,w)\,dm(z) \leq C\|f\|_{N^a}^2 (1-|w|^2)^{2(1-a)}. \tag{2.5}$$

Here we have used Lemma 4.2.2 in [10]. The above (2.5) gives $f \in UBC_0$, in other words, (2) holds.

(iii). We pick $f_5(z) = \sum_{k=1}^{\infty} \frac{z^{2^k}}{2^k}$. By [7,Theorem 1] it follows that $f_5 \in \bigcap_{0<a<1} N_0^a$. However, it is very easy to observe that $f_5 \notin D_\#^\infty$. So, $\bigcap_{0<a<1} N_0^a \neq D_\#^\infty$.

3. CHARACTERIZATIONS OF N^a AND N_0^a.

In this section, we characterize functions in N^a and N_0^a for $a \in (0,1]$ in terms of the weighted average, the pseudo-hyperbolic disk and the Green function, respectively. We use $|E|$ to denote the measure of the set $E \subseteq D$ relative to $dm(z)$, i.e., $|E| = \int_E dm(z)$.

THEOREM 3.1 Let $f \in M, a \in (0,1]$ and $p \in (1,\infty)$. Then the following statements are equivalent:

(i). $f \in N^a$.

(ii). There is an $r_0 \in (0,1)$ such that for any $r \in (0,r_0]$,

$$\sup_{w\in D} \frac{1}{|D(w,r)|^{1-a}} \int_{D(w,r)} [f^\#(z)]^2\,dm(z) < \pi.$$

(iii). There is an $r_0 \in (0,1)$ such that for any $r \in (0,r_0]$,

$$\sup_{w\in D} \int_{D(w,r)} [f^\#(z)]^2 (1-|z|^2)^{2(a-1)}\,dm(z) < \pi.$$

(iv).

$$\sup_{w\in D} \int_D [f^\#(z)]^2 (1-|z|^2)^{2(a-1)} g^p(z,w)\,dm(z) < \infty.$$

PROOF. We prove this theorem in accordance with the order $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i) \Rightarrow (iv) \Rightarrow (iii)$.

Step 1. $(i) \Rightarrow (ii)$. Let $f \in N^a$, then $\|f\|_{N^a} < \infty$. For $w \in D$ and $r \in (0,1)$ we have

$$|D(w,r)| = \frac{\pi r^2(1-|w|^2)^2}{(1-r^2|w|^2)^2} \tag{3.1}$$

and

$$\int_{D(0,r)} \frac{1}{(1-|z|^2)^{2a}|1-\overline{w}z|^{2(2-a)}}\,dm(z) < \frac{\pi r}{(1-r)^4}. \tag{3.2}$$

Further, we readily get

$$I_1(w,r) = \frac{1}{|D(w,r)|^{1-a}} \int_{D(w,r)} [f^{\#}(z)]^2\, dm(z) \le \frac{\pi(4r)^a}{(1-r)^4} \|f\|^2_{N^a}. \tag{3.3}$$

From (3.3) it follows that there exists an $r_0 \in (0,1)$ for which $\sup_{w\in D} I_1(w,r) < \pi$ for any $r \in (0, r_0]$, i.e., (ii) holds.

Step 2. *(ii)* $\Rightarrow$ *(iii)*. For $w \in D$ and $r \in (0,1)$, by (3.1) we have

$$\begin{aligned} I_2(w,r) &= \int_{D(w,r)} [f^{\#}(z)]^2 (1-|z|^2)^{2(a-1)}\, dm(z) \\ &\le \sup_{z\in D(w,r)} (1-|z|^2)^{2(a-1)} \int_{D(w,r)} [f^{\#}(z)]^2\, dm(z) \\ &\le \frac{\pi(4r)^{(1-a)}}{(1-r)^{4(1-a)}} \frac{1}{|D(w,r)|^{(1-a)}} \int_{D(w,r)} [f^{\#}(z)]^2\, dm(z). \end{aligned} \tag{3.4}$$

Once assuming (ii), we can choose $r_0 \in (0,1)$ such that

$$\sup_{w\in D} \frac{1}{|D(w,r)|^{(1-a)}} \int_{D(w,r)} [f^{\#}(z)]^2\, dm(z) < \pi$$

for any $r \in (0, r_0]$. Further, when $r \in (0, r_0]$, we have

$$\sup_{w\in D} I_2(w,r) < \frac{\pi^2(4r)^{1-a}}{(1-r)^{4(1-a)}}. \tag{3.5}$$

Thus, there exists an $r_1 \in (0, r_0]$ such that $\sup_{w\in D} I_2(w,r) < \pi$ for any $r \in (0, r_1]$, and hence (iii) holds.

Step 3. *(iii)* $\Rightarrow$ *(i)*. If (iii) holds, then there exists an $r_0 \in (0,1)$ satisfying

$$C_0 = \sup_{w\in D} \frac{1}{\pi} \int_{D(w,r_0)} [f^{\#}(z)]^2 (1-|z|^2)^{2(a-1)}\, dm(z) < 1. \tag{3.6}$$

Consequently, for all $w \in D$,

$$S(r_0, f, w) = \frac{1}{\pi} \int_{D(w,r_0)} [f^{\#}(z)]^2\, dm(z) \le C_0 \dot{<} 1. \tag{3.7}$$

Dufresnoy's lemma [5,Lemma II, p.216] then yields that

$$(1-|w|^2)^{2a} [f^{\#}(w)]^2 \le \frac{S(r_0, f, w,)(1-|w|^2)^{2(a-1)}}{r_0^2(1 - S(r_0, f, w,))}. \tag{3.8}$$

Also,

$$\begin{aligned} &\int_{D(w,r_0)} [f^{\#}(z)]^2 (1-|z|^2)^{2(a-1)}\, dm(z) \\ &\ge \inf_{z\in D(w,r_0)} (1-|z|^2)^{2(a-1)} \int_{D(w,r_0)} [f^{\#}(z)]^2\, dm(z) \\ &\ge (1-r_0)^{4(1-a)} (1-|w|^2)^{2(a-1)} \int_{D(w,r_0)} [f^{\#}(z)]^2\, dm(z) \\ &= \pi(1-r_0)^{4(1-a)} (1-|w|^2)^{2(a-1)} S(r_0, f, w). \end{aligned} \tag{3.9}$$

From (3.7), (3.8) and (3.9) it derives that for all $w \in D$,

$$(1-|w|^2)^a f^{\#}(w) \le \left[\frac{C_0}{\pi r_0^2(1-C_0)(1-r_0)^{4(1-a)}}\right]^{\frac{1}{2}}, \tag{3.10}$$

namely, $f \in N^a$.

Step 4. $(i) \Rightarrow (iv)$. Under $f \in N^a$, we have

$$\begin{aligned} I_3(w) &= \int_D [f^{\#}(z)]^2(1-|z|^2)^{2(a-1)} g^p(z,w)\, dm(z) \\ &\le C_1 \|f\|^2_{N^a}, \end{aligned} \tag{3.11}$$

where $C_1 = 2\pi \int_0^1 t(1-t^2)^{-2} log^p \frac{1}{t}\, dt < \infty$ for $p \in (1,\infty)$. So, (iv) follows.

Step 5. $(iv) \Rightarrow (iii)$. If (iv) holds, then, for $w \in D$ and $r \in (0,1)$, we have

$$\int_{D(w,r)} [f^{\#}(z)]^2(1-|z|^2)^{2(a-1)}\, dm(z) \le -\frac{I_3(w)}{logr}. \tag{3.12}$$

That is to say, we can choose an $r_0 \in (0,1)$ so that $\sup_{w\in D} I_2(w,r) < \pi$ for any $r \in (0, r_0]$.

This completes the proof.

For N_0^a we have a similar result.

THEOREM 3.2 Let $f \in M$, $a \in (0,1]$ and $p \in (1,\infty)$. Then the following statements are equivalent:

(i). $f \in N_0^a$.

(ii). There is an $r_0 \in (0,1)$ such that for any $r \in (0, r_0]$,

$$\lim_{|w|\to 1} \frac{1}{|D(w,r)|^{1-a}} \int_{D(w,r)} [f^{\#}(z)]^2\, dm(z) = 0.$$

(iii). There is an $r_0 \in (0,1)$ such that for any $r \in (0, r_0]$,

$$\lim_{|w|\to 1} \int_{D(w,r)} [f^{\#}(z)]^2(1-|z|^2)^{2(a-1)}\, dm(z) = 0.$$

(iv).

$$\lim_{|w|\to 1} \int_D [f^{\#}(z)]^2(1-|z|^2)^{2(a-1)} g^p(z,w)\, dm(z) = 0.$$

PROOF. We show this theorem according to the order $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i) \Rightarrow (iv) \Rightarrow (iii)$.

Step 1. $(i) \Rightarrow (ii)$. Suppose that $f \in N_0^a$. Then for any compact subset $E \subset D$ and all $z \in E$, such a function f satisfies

$$\lim_{|w|\to 1} (1-|\phi_w(z)|^2)^a f^{\#}(\phi_w(z)) = 0. \tag{3.13}$$

Thus, for any $\epsilon > 0$, there exists a $\rho \in (0,1)$ such that for $|w| > \rho$,

$$\int_{D(w,r)} [f^{\#}(z)]^2\, dm(z) \le \epsilon \int_{D(0,r)} \frac{|1-\overline{w}z|^{4(a-1)}}{(1-|w|^2)^{2(a-1)}(1-|z|^2)^{2a}}\, dm(z).$$

As its consequence,

$$I_1(w,r) \le \frac{\epsilon(\pi r)^a}{(1-r)^{1-4a}}, \tag{3.14}$$

and hence it turns out that there is an $r_0 \in (0,1)$ to make $I_1(w,r) < \epsilon$ for all $w \in D \setminus D(0,\rho)$ and any $r \in (0, r_0]$. i.e., (ii) holds.

Step 2. $(ii) \Rightarrow (iii)$. This follows readily from (3.4).

Step 3. $(iii) \Rightarrow (i)$. Assuming that $f \in M$ is of (iii), for any $\epsilon \in (0,1)$, we can find $\rho \in (0,1)$ to make $I_2(w, r_0) < \pi\epsilon$, and consequently $S(r_0, f, w,) \leq \epsilon < 1$ for all $w \in D \backslash D(0,\rho)$. Combining (3.8) and (3.9) we get

$$(1-|w|^2)^a f^{\#}(w) \leq [\frac{\epsilon}{(1-r_0)^{(5-4a)}}]^{\frac{1}{2}} \tag{3.15}$$

for all $w \in D \setminus D(0,\rho)$. Therefore $f \in N_0^a$.

Step 4. $(i) \Rightarrow (iv)$. Provided that (i) is true. Since $C_1 < \infty$, for any $\epsilon > 0$ there is an $r_2 \in (0,1)$ such that

$$\int_{D\backslash D(0,r_2)} \frac{log^p \frac{1}{|z|}}{(1-|z|^2)^2}\, dm(z) < \epsilon. \tag{3.16}$$

Also, for this r_2 and all $w \in D$,

$$\begin{aligned} &I_3(w) \\ &= (\int_{D(0,r_2)} + \int_{D\backslash D(0,r_2)})[f^{\#}(\phi_w(z))]^2 \frac{(1-|w|^2)^{2a}(1-|z|^2)^{2(a-1)}}{|1-\overline{w}z|^{4a}} log^p \frac{1}{|z|}\, dm(z) \\ &= (\int_{D(0,r_2)} + \int_{D\backslash D(0,r_2)})(...)\, dm(z). \end{aligned}$$

From the condition: $f \in N_0^a$ it follows that there exists a $\rho_1 \in (0,1)$ such that for $|w| > \rho_1$,

$$\int_{D(0,r_2)} (...)\, dm(z) \leq (\frac{16}{15})^2 \epsilon^2 \int_0^{r_2} log^p \frac{1}{t}\, dt \tag{3.17}$$

and

$$\int_{D\backslash D(0,r_2)} (...)\, dm(z) \leq \|f\|_{N^a}^2 \int_{D\backslash D(0,r_2)} \frac{log^p \frac{1}{|z|}}{(1-|z|^2)^2}\, dm(z). \tag{3.18}$$

Combining (3.16), (3.17) and (3.18) deduces (iii) right away.

Step 5. $(iv) \Rightarrow (iii)$. This is a simple consequence of (3.12).

This completes the proof.

REMARK. (i). A special case of $(i) \Leftrightarrow (iii)$ in Theorem 3.1 was stated by Wulan and Yan [6]. (ii). The case: $a = 1$ of $(i) \Leftrightarrow (iv)$ in Theorem 3.1 and in Theorem 3.2 was given by Aulaskari and Lappan [3]. (iii). The ideas and examples of this paper are suitable for the a-Bloch and little a-Bloch fuctions (see [7,9]). (iv). It is an open question as to which of the results from this paper are valid for $a \in (1,\infty)$. Similar questions may also be asked about corresponding classes of harmonic functions (Cf [3]).

ACKNOWLEDGMENT. The author is grateful to Dr. L.Zhong, Mr. R. Zhao and Mr. H.Wulan for their helpful comments. The author is also indebted to the referees for their valuable suggestions. This work was supported by the National Natural Science Foundation of China.

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