Internat. J. Math. & Math. Sci.
VOL. 20 NO. 3 (1997) 599-604

WAVE GROWTH PATTERNS IN A NON-LINEAR DISPERSIVE SYSTEM WITH INSTABILITY AND DISSIPATION

DINESH GERA, MRIDUL GAUTAM and HOTA V. S. GANGARAO[1]

Department of Mechanical & Aerospace Engineering
[1] CFC, Department of Civil Engineering
West Virginia University
Morgantown WV 26506 U.S.A

(Received November 30, 1995)

ABSTRACT. A simple one-dimensional non linear equation including effects of instability, dissipation, and dispersion is examined numerically. Periodic solution of a non linear dispersive equation is presented for different values of α, β, and γ characterizing the constants for instability, dissipation, and dispersion respectively. In this paper, the growth pattern for the wave at different time intervals is discussed. Various equilibrium states with different initial configuration have been observed depending on initial conditions.

KEY WORDS AND PHRASES: Nonlinear dispersive waves, nonlinear instability and dissipation.
1991 AMS SUBJECT CLASSIFICATION CODES: 76E30.

1.0 INTRODUCTION

One of the intrinsic properties attributed to dispersion in a non linear system with instability and dissipation has been pointed out by a numerical initial value problem concerning a simple one-dimensional model equation [1], given by:

$$U_t + UU_x + \alpha U_{xx} + \beta U_{xxx} + \gamma U_{xxxx} = 0 \qquad (1)$$

Dispersion works as an effective impedance in non linear mode coupling processes and results in saturation at higher amplitudes for strong dispersion leading to a non linear equilibrium, i.e., a row of saturated soliton like pulses. However, the wave evolutions are chaotic in the absence of sufficient dispersion [2,3].

It is interesting to notice from the linear dispersion relation, $\Omega = \alpha k^2 - \gamma k^4 + i\beta k^3$, which is obtained by substitution of $U \propto \exp(ikx + \Omega t)$ into the linear version of Equation-1, small amplitude sinusoidal waves with long wavelengths are linearly unstable. Thus small amplitude sinusoidal waves grow or damp according to whether *Re* $\Omega>0$ or *Re* $\Omega<0$. The maximum growth rate is always given at the wave number $k_{max}=(\alpha/2\gamma)^{1/2}$ [4]. It is anticipated that these unstable waves may destroy a steady row of pulses when the distance between adjacent soliton like pulses become too long [5].

One of the simplest non-linear effects which is capable of saturating the growth of a linearly unstable wave or spectrum of waves is resonant mode coupling [6]. The existence of both instability and dispersion indicates the possibility of a steady state, because the energy influx due to the self excitation is transferred through mode coupling to short wave length and is expected to be balanced by damping due to the fourth order dissipation term. When the rate of energy influx from the linear instability is balanced by the rate of outflow from mode coupling, the steady state is achieved.

Cohen *et al.* [7] applied the similar one dimensional partial differential equation as equation (1) in analyzing the non-linear saturation of the dissipative trapped ion mode The dissipative trapped ion mode is a low frequency, electrostatic drift wave propagating in the electron diamagnetic direction. The wave is destabilized by ion collisional damping and Landau damping due to both circulating and trapped ions [8].

Equation (1) also represents the modified K-dV equation which includes the energy dissipation terms Ott and Sudan [9] observed a solitary wave pattern for a small change in the amount of such dissipation Ott and Sudan [9] modified equation (1) and investigated three cases, viz, (a) magnetic waves damped by electron-ion collisions, (b) ion-sound waves with electron Landau damping [8], and (c) shallow water waves damped by viscosity.

A non linear evolution equation for the free interface displacement from planar shape is found to possess classical form as Eq. (1), with interfacial viscosities supporting the existence of a dispersive term [10-12]. In such a system, u will represent the perturbed interface. Therefore, it is important to understand the role of dispersion in non linear systems, including both growth and damping mechanisms in relation to instability waves in fluid systems.

2.0 NUMERICAL ANALYSES

Equation-1 with periodic boundary conditions on the interval [0,L] is integrated numerically by a finite difference method in space and time. Five point, central difference approximations were used for spatial derivatives so that possible leading errors were of an order less than the fourth power of the spatial mesh size Spatial mesh points were taken to be 150 in the periodicity length L=2, with the periodic boundary condition at x=0 and x=L. Initial conditions assigned were (a) -cos(πx), (b) -sin(πx)-cos(πx), (c) step wave with a unit amplitude, and (d) uniformly distributed random numbers.

For the case when $\beta u_{xxx} \sim u u_x >> \alpha u_{xx} \sim \gamma u_{xxxx}$, then the Eq.(1) describing slow changes of the amplitude of the Koteweg de Vries solution may be derived by means of two-time asymptote equation expansion with slow time scale defined by T=ϵt (where ϵ is a small perturbation parameter) The steady solution of equation (1) is given asymptotically to be [5].

$$u = A\,\mathrm{sech}^2 Bx \left\{ 1 - \epsilon \frac{6\alpha}{5\beta B} \tanh Bx \ \ln(\cosh Bx) \right\} \tag{2}$$

where $A=(21\alpha\beta/5\gamma)$ and $B=(7\alpha/20\gamma)^{1/2}$. Temporal evolution of the solution of eq. (1) is also derived by Toh and Kawahara [5]. The asymptotic solution (Eq. (2)) is used later for the qualitative comparison of our numerical algorithm.

3.0 RESULTS AND DISCUSSION

The results of numerical integration of Equation-1 with the initial condition (a) is presented in Figure-1 Values of α, β, and γ for the different cases I,II, and III are given in Table-1.

Table-1 Instability, Dissipation and Dispersion coefficients for three cases

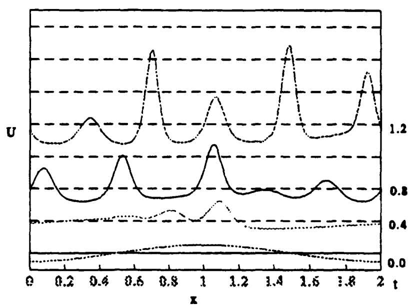
	I	II	III
α	0.01	2.0e-03	5.1e-06
β	0.1	5.0e-04	5 0e-04
γ	0.001	5.0e-04	5.0e-04

It is evident from Figure-1 that case-I grows at the fastest rate as compared to the other two cases, this is attributed to the fact that it had the largest value of α. Figure 1(b) depicts the moderate growth of the wave, and Figure 1(c) shows a stable wave form. The stable wave form is predicted because of very small value of α. It is seen from Figure-1(a) that the temporal evolution develops to a solitary wave form after 1 2 seconds for case-I, while the case-II is under damped and case-III has a stable amplitude. As α decreases (refer to Figure-1(a) through 1(c)) stability of the wave increases. This is also qualitatively observed by Toh and Kawahara [5], who used the steady solution as given by equation 2. The other case with the periodic boundary condition of $(-\sin(\pi x) - \cos(\pi x))$ is presented in Figure-2 (the α, β, and γ values for this case and the rest of the other boundary condition cases are the same as I). It grows to the waves of constant amplitude after 1.2 seconds. Similar trends were also observed by Oron and Edwards [10] and Cohen *et al.* [7]. It is interesting to notice from Figure-3, if the step wave is taken to as the initial disturbance, then after t=0 4 seconds, it also converges to the cyclic wave Last case was tried by taking random variables as an initial condition. It is inferred from Figure-4 that it also grows to a stable wave form after an elapse of initial 2 0 seconds

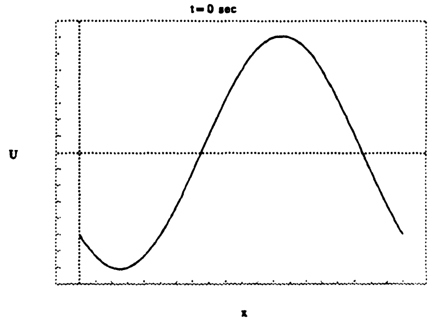
A wave with sufficiently small amplitude and large width grows because the growth term αU_{xx} is more important than the damping term γU_{xxxx} for small wave numbers. Meanwhile, the dispersion can inhibit mode coupling and result in saturation at higher amplitudes for sufficient dispersion; for which the growth just balances the damping and also the dispersion balances the non linearity.

Computer solutions [5,7,10,12] indicate that the growth of an initial perturbation is followed by formation of a row of solitons for the strongly dispersive case Also, it is interesting to notice that the existence of a dispersive effect can bring about a kind of organization in the system that exhibits a turbulent like behavior if the effect of dispersion is completely neglected. Numerical results revealed, that the number of pulses increases as the amplitude of the initial disturbance increases Thus, the equilibrium state is initial condition dependent. The intervals between pulses are not regular at the stage when the saturated pulses are first developed but become regular at later time.

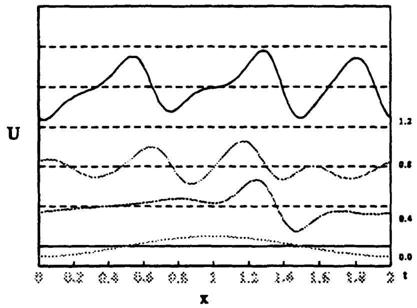
Equation (1) with β=0 reduces to that equation describing the chemical reactions which exhibit a turbulent like behavior [13] A time evolution of eq.(1) with β=0 for arbitrary initial disturbances was pursued



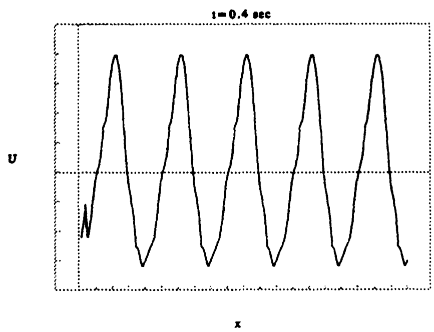
1 (a)



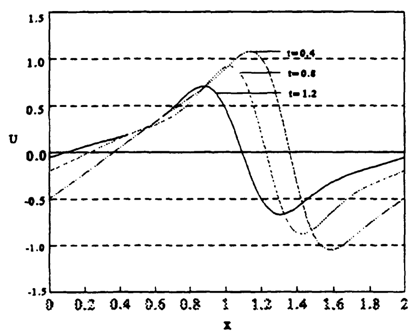
2 (a)



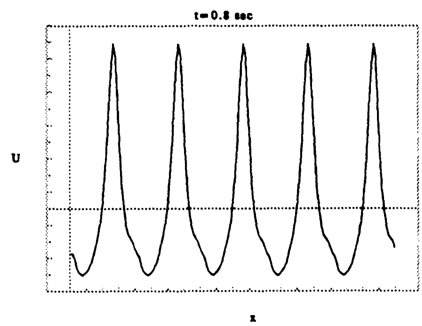
1(b)



2(b)



1(c)

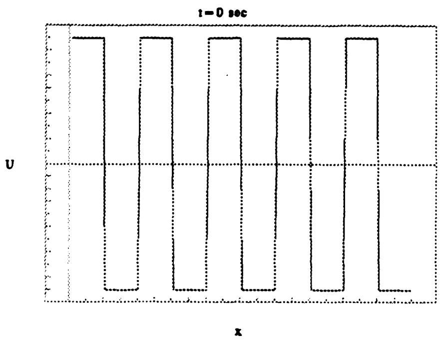


2 (c)

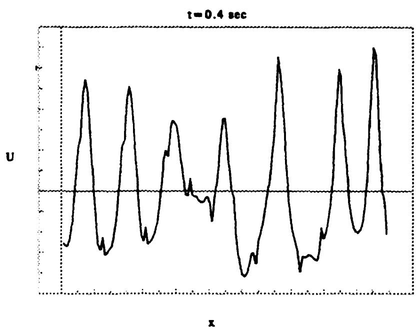
Figure 1 Temporal evolution of U for initial condition -cos(πx) for (a) Case-I*, (b) Case-II*, and (c) Case-III*.

Figure 2 Temporal evolution of U for initial condition -cos(πx)-sin(πx) at (a) t=0.0 sec, (b) t=0.4 sec © t=0.8 sec

* Refer to Table 1

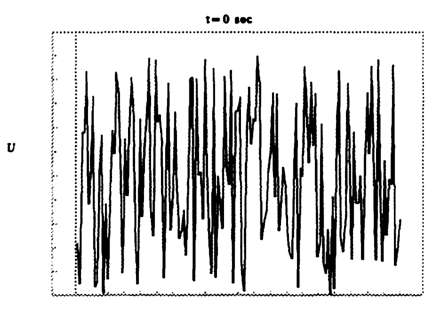


3(a)

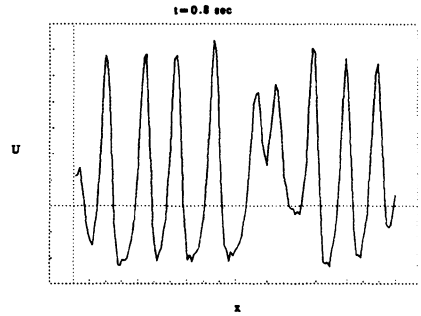


3(b)

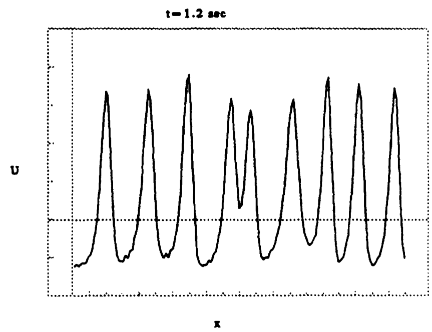
Figure 3 Temporal evolution of U for initial condition of step wave at (a) 0.0 sec, and (b) 0.4 sec



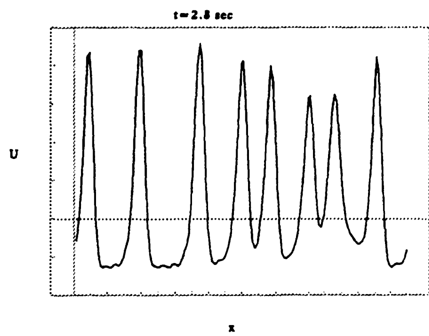
4 (a)



4 (b)



4 (c)



4 (d)

Figure 4 Temporal evolution of U for initial condition of random numbers at t= (a) 0.0 sec, (b) 0.4 sec, (c) 1.2 sec, and (d) 2.8 sec numerically by Yamada & Kuramoto [13]. It was concluded, at an early stage of evolution, that a spatially periodic structure determined by the maximum growth rate develops but it finally breaks into a turbulent state

4.0 CONCLUSIONS

Short wavelength components due to initial uniformly distributed random variables assigned at individual mesh points quickly damp out and soon generate a wave form. It should be emphasized that not all of the young humps generated at the initial stage grow up to saturated soliton-like pulses. It should be noted here that the number of solitons which emerge from a given initial condition in the purely dispersive case has no direct relation to the final number of soliton like pulses in the case with non zero α and γ.

REFERENCES

[1] Kawahara, T., and Toh, S., "Nonlinear Dispersive Periodic Waves in the Presence of Instability and Damping", *Phys. Fluids*, **28(6)**, 1636 (1985).

[2] Rabinovich, M.I., and Fabrikant, A.L., "Stochastic Self-Modulation of Waves in Nonequilibrium Media", *Sov. Phys. JETP*, **50(2)**, 311 (1979).

[3] Moon, H. T., Huerre, P., and Redekopp, L.G., "Three Frequency Motion and Chaos in the Ginzburg-Landau Equation", *Phys. Rev. Lett.* **49**, 458 (1982).

[4] Whitham, G.B., *Linear and Nonlinear Waves*, Wiley-Interscience, New York (1974)

[5] Toh, S., and Kawahara, T., "On the Stability of Soliton-like Pulses in a Nonlinear Dispersive System with Instability and Dissipation", *J. Phys. Soc. Jpn.* **54**, 1257 (1985).

[6] Ott, E., Manheimer, W.M., Book, D.L., and Boris J P., "Model Equations for Mode Coupling Saturation in Unstable Plasmas", *Phys. Fluids*, **16(6)**, 855 (1973)

[7] Cohen, B.I., Krommes, J.A., Tang, W.M., and Rosenbluth, M.N., "Non-Linear Saturation of the Dissipative Trapped-Ion Mode By Mode Coupling", *Nuclear Fusion*, **16(6)**, 971 (1976)

[8] Ott, E., and Sudan, R N , "Nonlinear Theory of Ion Acoustic Waves with Landau Damping", *Phys. Fluids*, **12(11)**, 2388 (1969).

[9] Ott, E , and Sudan, R.N., "Damping of Solitary Waves", *Phys. Fluids*, **13(6)**, 1432 (1970)

[10] Oron, A., and Edwards, D.A., "Stability of a Falling Liquid Film in the presence of Interfacial Viscous Stress", *Phys. Fluids A*, **5(2)**, 506 (1993).

[11] Topper, J., and Kawahara, T., "Approximate Equations for Long Nonlinear Waves on a Viscous Fluid", *J. Phys. Soc. Jpn.* **44**, 663 (1978)

[12] Jurman, L.A., Bruno, K., and McCready, M.J., "Periodic and Solitary Waves on Thin, Horizontal, Gas-Sheared Liquid Films" *Int. J. Multiphase Flow*, **15(3)**, 371 (1989).

[13] Yamada, T , and Kuramoto, Y., "A Reduced Model Showing Chemical turbulence", *Prog. Theor. Phys.* **56**, 681 (1976)