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ON NORMALLY FLAT EINSTEIN SUBMANIFOLDS

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ABSTRACT. The purpose of this paper is to study the second fundamental form of some submanifolds M^n in Euclidean spaces $\mathbb{E}^m$ which have *flat normal connection*. As such, Theorem 1 gives precise expressions for the (essentially 2) Weingarten maps of all 4-dimensional *Einstein* submanifolds in $\mathbb{E}^6$, which are specialized in Corollary 2 to the *Ricci flat* submanifolds. The main part of this paper deals with *flat* submanifolds. In 1919, E. Cartan proved that every flat submanifold of dimension ≤ 3 in a Euclidean space is totally cylindrical. Moreover, he asserted without proof the existence of flat non-totally cylindrical submanifolds of dimension > 3 in Euclidean spaces We will comment on this assertion, and in this respect will prove, in Theorem 3, that every flat submanifold M^n with flat normal connection in $\mathbb{E}^m$ is totally cylindrical (for all possible dimensions n and m).

KEY WORDS AND PHRASES. Submanifolds, normal connection, Ricci flat submanifolds.
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1. INTRODUCTION.

This paper deals first of all with the second fundamental form of an Einstein submanifold of codimension 2.

A Riemannian manifold is Einstein if its Ricci tensor field is proportional (with a constant coefficient of proportionality) to the Riemannian metric. We recall that every space of constant sectional curvature is Einstein

The converse statement is true also in 2 and 3 dimensions, as shown by J.A Schouten and D.J Struik in 1921

FACT A (see [8] or [5] or [1]). If a Riemannian manifold M of dimension n ($n \leq 3$) is Einstein, then it is a space of constant curvature.

T.Y. Thomas in 1936 and A. Fialkow in 1938 classified the Einstein hypersurfaces of the real space forms In particular, we have

FACT B (see [9] or [6] or [10] and [1]). Let M^n be a hypersurface immersed in $\mathbb{E}^{n+1}$, where $n \geq 3$. If M^n is Einstein, then:

(B.1) the Riemannian scalar curvature, say s, of M is constant and non-negative,

(B.2) if $s = 0$, then M is locally Euclidean;

(B.3) if $s > 0$, then every point of M is umbilical and M is locally a hypersphere S^m.

Theorem 1 of this paper determines all possible expressions of the second fundamental form of all Einstein 2-codimensional submanifolds with flat normal connection in $\mathbb{E}^6$, and in Corollary 2 we specify these expressions for all Ricci flat 2-codimensional submanifolds with flat normal connection in $\mathbb{E}^6$. The proofs of these two results use the flatness of the normal connection and are based on the following well-known characterization of 4-dimensional Einstein spaces by I.M. Singer and T.A. Thorpe.

FACT C (see [9] or [1]). Let M be a Riemannian 4-manifold. Then M is Einstein if and only if, for every $m \in M$, for any 2-plane P at m, the sectional curvature of P is equal to the sectional curvature of the 2-plane $P^{\perp}$ perpendicular to P at m.

The method of the proof of Theorem 1 inspires us to establish in Theorem 3 a relation between flatness and cylindricity. The importance of this relation will be justified in Fact D.

2. STATEMENTS OF THE MAIN RESULTS.

THEOREM 1. Let M be a 4-manifold isometrically immersed with flat normal connection in $\mathbb{E}^6$. Then M is Einstein if and only if for each point $m \in M$:

(1.1) either M is cylindrical at m;

(1.2) or M is umbilical (non-geodesic) with respect to a normal direction N_1 at m and cylindrical in another normal direction N_2 perpendicular to N_1 at m;

(1.3) or with respect to a suitable orthonormal tangent frame of M at m and an orthonormal normal frame $\{N_1, N_2\}$ at m, the Weingarten operators A_{N_1}, A_{N_2} admit respectively one among the following matricial representations:

$$A_{N_1} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{N_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \tag{1.3.1}$$

where $ab = cd$;

$$A_{N_1} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}, \tag{1.3.2}$$

where a is a non-zero real number, and N_2 is cylindrical;

$$A_{N_1} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \frac{b}{a} & 0 & 0 \\ 0 & 0 & \epsilon\frac{b}{a} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{N_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.3}$$

where $\epsilon = \pm 1, a, b, p, q$ are real numbers such that $ab \neq 0$ and $pq = \epsilon(a^2 - \frac{b^2}{a^2})$;

$$A_{N_1} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \frac{b}{a} & 0 & 0 \\ 0 & 0 & \frac{c}{a} & 0 \\ 0 & 0 & 0 & \frac{d}{a} \end{pmatrix}, \quad A_{N_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \tag{1.3.4}$$

where a, b, c, d, p, q, u are real numbers such that $a \neq 0$, and

$$pq = d - \tfrac{bc}{a^2}, \qquad pu = c - \tfrac{bd}{a^2},$$

$$qu = b - \tfrac{cd}{a^2}, \text{ and } \left(b - \tfrac{cd}{a^2}\right) \cdot \left(c - \tfrac{bd}{a^2}\right) \cdot \left(d - \tfrac{bc}{a^2}\right) > 0.$$

With respect to case 1.3.1 of Theorem 1, we give in particular the following

EXAMPLE AND REMARK 1. Let $M_1(c)$ and $M_2(c)$ be two surfaces of constant Gauss curvature c in the Euclidean 3-space $\mathbb{E}^3$. Then

(1) the Riemannian product $M^4 = M_1(c) \times M_2(c)$ canonically isometrically immersed in $\mathbb{E}^6$ is an Einstein 2-dimensional submanifold with flat normal connection. It is not a space of constant curvature and moreover it is not Ricci flat, unless $c = 0$.

(2) In particular, for $c < 0$, for instance $M_1(c)$ and $M_2(c)$ both being a pseudo-sphere in $\mathbb{E}^3$ of the same pseudo-radius c, the Riemannian product manifold M^4 is an Einstein submanifold with flat normal connection in $\mathbb{E}^6$ which has strictly negative scalar curvature. Thus, in contrast to the fact that for 1-codimensional Einstein submanifolds in Euclidean spaces the scalar curvature $s \in \mathbb{R}^+$, *there exists 2-codimensional Einstein submanifolds with any given real number as scalar curvature.*

COROLLARY 2. Let M be a 4-dimensional manifold isometrically immersed with flat normal connection in $\mathbb{E}^6$.

Then M is Ricci flat if and only if for each $m \in M$;

(2.1) either M is flat (hence cylindrical) at m;

(2.2) or with respect to a suitable orthonormal tangent frame at m and an orthonormal normal frame $\{N_1, N_2\}$ at m, the Weingarten operators A_{N_1}, A_{N_2} admit respectively one of the following matricial representations:

$$A_{N_1} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -\frac{a}{2} & 0 & 0 \\ 0 & 0 & -\frac{a}{2} & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad A_{N_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.2 1}$$

where $pq = \frac{3}{4} a^2 > 0$.

$$A_{N_1} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \frac{b}{a} & 0 & 0 \\ 0 & 0 & \frac{c}{a} & 0 \\ 0 & 0 & 0 & \frac{d}{a} \end{pmatrix}, \quad A_{N_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \tag{2.2.2}$$

where $a \neq 0, pq = d - \frac{bc}{a^2}, pu = c - \frac{bd}{a^2}, qu = b - \frac{cd}{a^2}$
and $b + c + d = 0, (d - \frac{bc}{a^2}) \cdot (c - \frac{bd}{a^2}) \cdot (b - \frac{cd}{a^2}) > 0$.

THEOREM 3. Let M^n be a n-dimensional manifold isometrically immersed with flat normal connection in $\mathbb{E}^{n+N}$.

Then M^n is flat if and only if it is cylindrical.

3. DEFINITIONS [3].

We consider a manifold M isometrically immersed with codimension N in the Euclidean space $\mathbb{E}^{n+N}$.

3.1. Let ξ be a normal vector field on M.

We shall say that M is *quasi-umbilical* in the direction ξ if the Weingarten tensor A_ξ of ξ admits an eigenvalue λ_ξ with multiplicity $n-1$ or n.

In particular:

(i) if $\lambda_\xi = 0$, we say that M is *cylindrical* in the direction ξ;

(ii) if λ_ξ has multiplicity n, we say that M is *umbilical* in the normal direction ξ.

3.2. M is *(totally) cylindrical* [resp. *quasi-umbilical*] if, locally around each point, there exists an orthonormal normal frame field composed with cylindrical [resp. quasi-umbilical] directions.

Now we prove our results.

4.1 PROOF OF THE THEOREM 1.

Let M be a 4-manifold isometrically immersed in the Euclidean 6-space $\mathbb{E}^6$.

Suppose that M is Einstein. Then by Fact C, for any $m \in M$ and any 2-plane P in T_mM, its sectional curvature is the same as the sectional curvature of its orthogonal 2-plane $P^\perp$ in T_mM.

To exploit this statement, we suppose moreover that the normal connection of M in $\mathbb{E}^6$ is flat. Then, at each point $m \in M$, there exists an orthonormal tangent frame $\{e_1(m), \cdots, e_4(m)\}$ which diagonalizes simultaneously all Weingarten tensors of M (at m). We denote by $c_{ij}(m)$ the sectional curvature of the 2-plane $\{e_i(m), e_j(m)\}$ for $1 \leq i < j \leq 4$. Then M is Einsteinian if and only if for each $m \in M$.

$$\begin{cases} c_{12}(m) = c_{34}(m) \\ c_{13}(m) = c_{24}(m) \\ c_{14}(m) = c_{23}(m). \end{cases} \tag{*}$$

Now we fix the point m in M. Either M is geodesic at m: then the problem is solved. Or M is non-geodesic at m; we can assume that $\sigma_m(e_1(m), e_1(m)) \neq 0$ where σ_m is the second fundamental form at m. We can put $N_1 = \frac{\sigma_m(e_1(m), e_1(m))}{\|\sigma_m(e_1(m), e_1(m))\|}$ and denote N_2 the unit normal vector perpendicular to N_1. By our choice of the tangent frame $\{e_1(m), \cdots, e_4(m)\}$, the Weingarten tensors A_{N_1}, A_{N_2} relative to N_1, N_2 respectively can be represented by the matrices:

$$A_{N_1} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & \cdot 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \quad A_{N_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & \mu_4 \end{pmatrix}$$

Hence the previous system (*) is equivalent to the following one:

$$\begin{cases} \lambda_1\lambda_2 = b \quad (1) \\ \lambda_1\lambda_3 = c \quad (2) \\ \lambda_1\lambda_4 = d \quad (3) \\ \lambda_3\lambda_4 + \mu_3\mu_4 = b \quad (4) \\ \lambda_2\lambda_4 + \mu_2\mu_4 = c \quad (5) \\ \lambda_2\lambda_3 + \mu_2\mu_3 = d \quad (6), \end{cases} \qquad (**)$$

where $b = c_{12} = c_{34}, c = c_{13} = c_{24}, d = c_{14} = c_{23}$.

To resolve this system of 6 equations with 7 unknowns, let us first compute λ_1 Using the equations (1), (2), (3) and the equality $b + c + d = \frac{1}{4}s$ (where s is the constant scalar curvature of M) we find that λ_1 is a solution of the equation:

$$x^2 - 4 < H, N_1 > x + \tfrac{1}{4}s = 0 \qquad (***)$$

where x is unknown and H is the mean curvature vector at m. Such an equation admits a solution $\lambda_1 = a$ since:

$$4 < H, N_1 >^2 - \tfrac{1}{4}s = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^2 - \lambda_1(\lambda_2 + \lambda_3 + \lambda_4) \geq 0.$$

To determine the unknowns $\lambda_2, \lambda_3, \lambda_4, \mu_2, \mu_3, \mu_4$, we discuss on the index of nonnullity $\pi(m)$ of M at m, i.e., the rank of the Riemannian curvature operator $\mathcal{R}$ at m. Because of the system (*), $\pi(m) \in \{0, 2, 4, 6\}$.

CASE 1: $\pi(m) = 0$. Then M is flat (hence Ricci flat) at m. By the system (**), M is cylindrical at m.

CASE 2: $\pi(m) = 2$. Then we obtain the situation (1.3.1).

CASE 3: $\pi(m) = 4$. It is easy to check that this is impossible.

CASE 4: $\pi(m) = 6$. From a simple discussion on the rank of A_{N_2}, we deduce either (1.2) or (1.3.2) or (1.3.3), or (1.3.4). This proves the Theorem 1

4.2. REMARK.

In accordance with each of the possibilities from Theorem 1 and Corollary 2, we can construct local parametrization of submanifolds of codimension 2 in $\mathbb{E}^6$ with flat normal connection which are, at a particular point, Einstein or in particular Ricci flat.

5. ON FLAT SUBMANIFOLDS.

5.1. A flat manifold is in particular Einstein. In 1919 [2], Elie Cartan studied the second fundamental form of flat submanifolds of a Euclidean space.

FACT D. ([2]). (D.1) Every n-dimensional flat submanifold;d of $\mathbb{E}^{n+N}$ with $n \leq 3$ is cylindrical. Moreover: (D.2) E. Cartan stated without proof, that the assertion (D.1) fails if $n \geq 4$.

With respect to (D.2), we consider the case of dimension $n = 4$.

Assume $h: \mathbb{E}^4 \times \mathbb{E}^4 \to \mathbb{E}^N$ is a flat bilinear symmetric map and consider the dimension of the vector space $[Imh]$ generated by the image of h. We may suppose without loss of generality that $N = dim[Imh]$. Since the dimension of the space of all symmetric bilinear forms on $\mathbb{E}^4$ is equal to 10, we can restrict ourselves to $0 \leq N \leq 10$. Using techniques as for the proof of Fact (D.1), it is easy to demonstrate that, if $N \in \{7, 8, 9, 10\}$, we can reduce N so that $N \in \{0, 1, 2, 3, 4, 5, 6\}$. In the same paper where E. Cartan proved Fact (D.1), he showed also that for the case $N \in \{0, 1, 2, 3, 4\}$ the flatness implies the cylindricity. Consequently, the only unknown cases are: "$N = 5$" and "$N = 6$". In 1986 [7], an example of a 4-submanifold in $\mathbb{E}^{10}$ which is, at a particular point, flat without being cylindrical is constructed. However, a full justification of Fact (D.2) is still lacking for the moment; in other words the method of resolution of the so-called Gauss equation of a flat submanifold in a Euclidean space is still unknown in dimension n and in codimension N with $N \geq n + 1$, even for the case of dimension $n = 4$. One first resolution for such a problem is given in Theorem 3 for the particular case of flat normal connection.

5.2. PROOF OF THEOREM 3. For this purpose, we apply the following Fact E and Lemma (*) which we state and prove below:

FACT E (see [7] and [2]). Let ν be a vector space. let ω be another vector space, endowed with a scalar product $< \cdot, \cdot >$.

Suppose $\phi: \nu \times \nu \to \omega$ is a bilinear symmetric map, flat with respect to $< \cdot, \cdot >$ (i.e., $< \phi(x, y), \phi(z, w) > = < \phi(x, w), \phi(y, z) >$ for any x, y, z, w in ν. Assume moreover that the orthogonal projection of ϕ on a subspace W of ω is cylindrical.

Then the orthogonal projection of ϕ on the orthogonal supplementary subspace $W^\perp$ of W in ω is flat too.

LEMMA (*). Let $\sigma: \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{E}^N$ be a flat bilinear symmetric map satisfying the following property (F): "There exists an orthonormal frame $B = \{e_1, \cdots, e_n\}$ in $\mathbb{E}^n$ which diagonalizes simultaneously all projections $< \sigma, \xi >$ of σ in any direction $\xi \in \mathbb{E}^N$".

Then σ is cylindrical.

PROOF. We shall prove this lemma by induction on N, and suppose σ is not geodesic. The lemma is true for $N = 1$.

Consider the case $N = 2$. Let $\{\xi^1, \xi^2\}$ be an orthonormal frame in $\mathbb{E}^2$. The property (F) implies that each component $< \sigma, \xi^\alpha >$ can be represented in the frame B by the matrix

$$< \sigma, \xi^\alpha > = \begin{pmatrix} \lambda_1^\alpha & & & \\ & & & 0 \\ & \lambda_2^\alpha & & \\ 0 & & \ddots & \\ & & & \lambda_n^\alpha \end{pmatrix}.$$

The sectional curvature c_{ij} of each 2-plane generated by $\{e_i, e_j\}$ is given by

$$c_{ij} = \sum_{\alpha=1}^{n} \lambda_i^\alpha \lambda_j^\alpha.$$

We may suppose that $\sigma(e_1, e_1) \neq 0$ and ξ^2 is collinear to $\sigma(e_1, e_1)$. By this manner: $\lambda_1^1 = 0$ and $\lambda_1^2 \neq 0$. Since σ is flat, the c_{ij} are both null. This implies:

$$\lambda_i^2 = 0 \text{ for } 2 \leq i \leq n.$$

Hence σ is flat and $< \sigma, \xi^2 >$ is cylindrical. By Fact E, $< \sigma, \xi^1 >$ if flat too. Since $< \sigma, \xi^1 >$ is a (flat) bilinear symmetric form, it is well-known that it is cylindrical. Hence the lemma is proved for $N = 2$.

Now suppose Lemma (*) is true for a certain integer k and any dimension N with $N \leq k$. Let us prove that it then is also true for $N = k + 1$. By our hypothesis, $\sigma: \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{E}^{k+1}$ if flat and enjoys the property (F). When we reason as for the case $N = 2$, we easily find that $< \sigma, \xi^{k+1} >$ is cylindrical. We apply Fact E again and deduce that the projection σ on the hypersurface $\mathbb{E}^k$ of $\mathbb{E}^{k+1}$ perpendicular to ξ^{k+1} is flat too, $\sigma: \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{E}^k$.

Our hypothesis of induction obviously asserts that σ is cylindrical too. Hence $\sigma: \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{E}^{k+1}$ is cylindrical. This completes the proof of Lemma (*). □

6. OPEN PROBLEMS.

For the moment, the following questions related to this paper remain still without answer.

PROBLEM 1. How to classify all Einstein 4-manifolds, and in particular all Ricci flat 4-manifolds? (see [1]).

PROBLEM 2. Resolve the Gauss equation of a flat submanifold M^4 of codimension 5 or 6 in the Euclidean space; i.e., find all bilinear symmetric map $\sigma: \mathbb{E}^4 \times \mathbb{E}^4 \to \mathbb{E}^N$ (for $N = 5$ or 6) satisfying the equality: $< \sigma(x, y), \sigma(z, w) > - < \sigma(x, z), \sigma(y, w) > = 0$ for any x, y, z, w in $\mathbb{E}^4$ (consider only the case when the kernel Ker σ of σ is trivial!).

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