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ON EQUIVALENCE OF GRADED RINGS

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ABSTRACT. Let $R = \underset{g\in G}{\oplus} R_g$ be a G-graded ring. In this paper we define the "homogeneous-equivalence" concept between graded rings We discuss some properties of the G-graded rings and investigate which of these are preserved under homogeneous-equivalence maps. Furthermore, we give some results in graded ring theory and also some applications of this concept to Z-graded rings

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1. INTRODUCTION

Let G be a group with identity e. Then a ring R is said to be *G-graded ring* if there exist additive subgroups R_g of R such that $R = \underset{g\in G}{\oplus} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. We consider $\text{supp}(R) = \{g \in G : R_g \neq 0\}$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$ where x_g is the component of x in R_g. Also, we write $h(R) = \bigcup_{g\in G} R_g$.

In this paper we define an equivalence relation on the set of all graded rings and give some applications of this relation. In Section 1, we define the homogeneous-equivalence concept between graded rings and give the necessary and sufficient conditions for two graded rings to be homogeneously-equivalent. In Section 2, we discuss the relation between this new concept and some related concepts given in [1] In Section 3, we give some properties of graded rings and see which of these are preserved under homogeneous-equivalence maps. In Section 4, we give some useful results and applications of this new concept to Z-graded rings. We give the necessary and sufficient conditions for two first strongly Z-graded rings to be homogeneously-equivalent.

1. HOMOGENEOUS-EQUIVALENCE OF GRADUATIONS

In this section we define the homogeneous-equivalence concept between graded rings and give some of its properties. Also, we give the necessary and sufficient conditions for two graded rings to be homogeneously-equivalent.

DEFINITION 1.1. Let G, H be groups, R be a G-graded ring and S be an H-graded ring. We say that R is *homogeneously equivalent* (shortly "h.e.") to S if there exists a ring isomorphism $f : R \to S$ sending $h(R)$ onto $h(S)$. We call such an f a *homogeneous-equivalence* of R with S.

From now on G, H are groups, R is a G-graded ring and S is an H-graded ring unless otherwise indicated.

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REMARK 1.2. The relation h.e. is an equivalence relation.

PROPOSITION 1.3. An isomorphism $f : R \to S$ of rings is a homogeneous equivalence if and only if there is a bijection b of supp (R) onto supp (S) such that $f(R_g) = S_{b(g)}$ for all $g \in \text{supp}(R)$ In that case the bijection b is uniquely determined by the isomorphism f If $R \neq 0$, then the bijection b sends the identity element $e_G \in \text{supp}(R)$ of G onto the identity element $e_H \in \text{supp}(S)$ of H.

PROOF. Suppose $f : R \to S$ is a homogeneous equivalence map. Define $b : \text{supp}(R) \to \text{supp}(S)$ by $b(g) = h$ where $f(R_g) = S_h$. Then clearly b is a well-defined bijective map. The converse is obvious

Suppose $R \neq 0$. To show that $b(e_G) = e_H$ it is enough to show $f(R_{e_G}) = S_{e_H}$. Since $e_H \in H$, there exists $g \in G$ such that $f(R_g) = S_{e_H}$ and then $f(R_g)f(R_g) = S_{e_H}$, i.e., $S_{e_H} \subseteq f(R_{g^2})$. Therefore, $f(R_{g^2}) = S_{e_H}$. But f is 1-1 implies $R_g = R_{g^2} \neq 0$. Hence $g = g^2$ and then $g = e_G$, i.e., $f(R_{e_G}) = S_{e_H}$

Let I be an ideal of a graded ring R. Then I is a *graded ideal* of R if $I = \bigoplus_{g \in G} (R_g \cap I)$.

PROPOSITION 1.4. Suppose R is h.e. to S by f. Then I is a graded ideal of R if and only if $f(I)$ is a graded ideal of S.

PROOF. An ideal I in a G-graded ring R is graded if and only if it is generated as an ideal by its subset $I \cap h(R)$. Obviously this property is preserved under homogeneous equivalence maps.

REMARK 1.5. It follows from Proposition 1.4 that any property definable solely in terms of graded ideals of R must be preserved under homogeneous equivalence maps.

PROPOSITION 1.6. Let R be h.e. to S by f and I be a graded ideal of R. Then R/I is h.e. to $S/f(I)$.

PROOF. By Proposition 1.4, $f(I)$ is a graded ideal of S and then $S/f(I)$ is an H-graded ring Define $\varphi : R/I \to S/f(I)$ by $\varphi(r + I) = f(r) + f(I)$. Then clearly φ is a ring isomorphism. Let $h \in \text{supp}(S/f(I))$. Then $(S/f(I))_h \neq 0$ and hence there exists $s_h \in S_h - f(I)$, i.e., $h \in \text{supp}(S)$ Since R is h.e. to S by f, there exists $g \in \text{supp}(R)$ such that $f(R_g) = S_h$ by Proposition 1.3

CLAIM. $\varphi\big((R/I)_g\big) = (S/f(I))_h$.

Let $r_g + I \in (R/I)_g$. Then $\varphi(r_g + I) = f(r_g) + f(I) \in (S/f(I))_h$. So, $\varphi((R/I)_g) \subseteq (S/f(I))_h$. Conversely, let $t_h + f(I) \in (S/f(I))_h$. Then $f^{-1}(t_h) \in R_g$ and hence $f^{-1}(t_h) + I \in (R/I)_g$. So, $\varphi(f^{-1}(t_h) + I) = t_h + f(I) \in \varphi((R/I)_g)$. Therefore, $\varphi((R/I)_g) = (S/f(I))_h$ and hence R/I is h.e to $S/f(I)$

2. VARIOUS EQUIVALENCE OF GRADUATIONS

In this section we discuss the relation between the homogeneous-equivalence concept and the equivalence and almost-equivalence concepts given in [1].

DEFINITION 2.1 ([1]). We say that R is *almost equivalent* (shortly "a.e.") to S if there exists a ring isomorphism $f : R \to S$ such that for each $h \in H$, there exists $g \in G$ with $f(R_g) = S_h$. If R is a.e. to S and S is a.e. to R then we say R is *equivalent* to S.

PROPOSITION 2.2. If R is a.e. to S then R is h.e. to S.

PROOF. Follows from Proposition 1.3

However, the converse of this proposition need not be true in general as we see in the following example.

EXAMPLE 2.3. Let K be a field and $R = S = K[x]$ is the polynomial ring over K in one variable x. Let $G = Z_3$. Then R is a G-graded ring with $R_j = \{kx^{3r+j} : k \in K, r = 0, 1, \ldots\}$ for $j \in Z_3$. Let $H = Z_6$. Then S is an H-graded ring with

$$S_0 = \{kx^{3r} : k \in K, r = 0, 1, \ldots\}$$
$$S_2 = \{kx^{3r+1} : k \in K, r = 0, 1, \ldots\}$$
$$S_4 = \{kx^{3r+2} : k \in K, r = 0, 1, \ldots\} \quad \text{and} \quad S_j = 0 \quad \text{otherwise.}$$

Clearly R is h.e. to S. If R is a.e. to S by f then there exists $g \in G$ with $f(R_g) = S_3 = 0$, i e , $R_g = 0$ a contradiction.

PROPOSITION 2.4. R is a.e to S if and only if the following two conditions are satisfied:

(i) R is h.e. to S.

(ii) If H-supp $(S) \neq \emptyset$ then G-supp $(R) \neq \emptyset$

PROOF. Suppose R is a.e. to S by f. Then (i) follows from Proposition 2.2 Assume H-supp $(S) \neq \emptyset$. Then there exists $h \in H$ such that $S_h = 0$. Since R is a.e to S there exists $g \in G$ such that $f(R_g) = 0$. Thus $R_g = 0$, i.e., G-supp $(R) \neq \emptyset$.

Conversely, suppose R is h.e. to S by f. Let $h \in H$. If $h \in$ supp(S) then by Proposition 1.3, there exists $g \in G$ such that $f(R_g) = S_h$. If $h \notin$ supp(S) then by (ii) there exists $g \notin$ supp(R) So, $0 = f(R_g) = S_h$, i.e., R is a.e to S by f.

COROLLARY 2.5. R is equivalent to S if and only if the following two conditions are satisfied:

(i) R is h.e. to S.

(ii) G-supp$(R) \neq \emptyset$ if and only if H-supp$(S) \neq \emptyset$.

PROPOSITION 2.6. Suppose R is h.e. to S and $|G| = |H| = n < \infty$. Then R is equivalent to S.

PROOF. Similar to the proof of Proposition 2.3 in [1].

3. PROPERTIES PRESERVED UNDER HOMOGENEOUS-EQUIVALENCE MAPS

In this section we give some properties of graded rings and see which of these are preserved under homogeneous-equivalence maps. For more details about the properties one can look in [1,2,3].

DEFINITION 3.1. For a G-graded ring R we say

1. R is *semiprime* if R has no non-zero nilpotent graded ideals.

2. R is *prime* if the product of any two non-zero graded ideals (of the same type right or left) of R is non-zero.

3. R is *Noetherian* (*Artinian*) if R satisfies the ascending (descending) chain conditions on graded ideals of R.

4. R is *simple* if O, R are the only graded ideals of R.

PROPOSITION 3.2. Suppose R is h.e. to S. Then

1. R is semiprime if and only if S is semiprime.
2. R is prime if and only if S is prime
3. R is Noetherian (Artinian) if and only if S is Noetherian (Artinian).
4. R is simple if and only if S is simple.

PROOF. Follows from Remark 1.5.

DEFINITION 3.3. For a G-graded ring R we say

1. R is *strong* if $R_gR_h = R_{gh}$ for all $g, h \in G$. Also, R is strong if $1 \in R_gR_{g^{-1}}$ for all $g \in G$ (Proposition 1.6 of [4]).

2. R is *first strong* if $1 \in R_gR_{g^{-1}}$ for all $g \in$ supp(R).

3 R is *second strong* if $R_gR_h = R_{gh}$ for all g, $h \in$ supp(R) and supp(R) is a monoid in G.

It is easy to see that a G-graded ring R is first strong if and only if supp(R) is a subgroup of G and R is strong as a supp(R)-graded ring. It follows that the bijection b : supp$(R) \to$ supp(S) given in Proposition 1.3 which is induced by a homogeneous-equivalence f of R onto an H-graded ring S is an isomorphism of groups whenever R is first strong. Similarly any such f induces an isomorphism b of supp(R) onto supp(S) as monoids whenever R is second strong. So, we have the following results.

PROPOSITION 3.4. Suppose R is h.e. to S. Then

1. R is first strong if and only if S is first strong.
2. R is second strong if and only if S is second strong.
3. If R is strong then supp(S) is a subgroup of H and G is isomorphic to supp(S) as groups.

DEFINITION 3.5. A G-graded ring R is *nondegenerate* if for any $a_g \in R_g - 0$, $a_g R_{g^{-1}} \neq 0$ and $R_{g^{-1}} a_g \neq 0$.

PROPOSITION 3.6. Suppose R is h.e. to S Then R is nondegenerate if and only if S is nondegenerate.

PROOF. Similar to the proof of Proposition 2.7 in [1].

DEFINITION 3.7. A G-graded ring R is *regular* if $a_g \in a_g R_{g^{-1}} a_g$ for all $g \in G$ and $a_g \in R_g$

PROPOSITION 3.8. Suppose R is h.e. to S. Then R is regular if and only if S is regular.

PROOF. Suppose R is h.e. to S by f, and R is regular. Let $a_h \in S_h$ If $a_h = 0$ then $a_h \in a_h S_{h^{-1}} a_h$. If $a_h \neq 0$, then there exists $g \in \text{supp}(R)$ and a non-zero element $b_g \in R_g$ such that $f(b_g) = a_h$. Thus, $a_h = f(b_g) \in f\left(b_g R_{g^{-1}} b_g\right) = a_h S_{h^{-1}} a_h$, i.e., S is regular. The other part is obvious since h.e. is symmetric.

DEFINITION 3.9. A G-graded ring R is *faithful* if for any $a_g \in R_g - 0$, $a_g R_h \neq 0$ and $R_h a_g \neq 0$ for all $g, h \in G$.

In the following example we show that faithfulness and strongness need not be preserved under homogeneous-equivalence maps. .

EXAMPLE 3.10. Let $R = S = Z[i] = \{a + ib : a, b \in Z\}$ (The Gaussian integers). Let $G = Z_2$ and $H = Z_4$. Then R is a G-graded ring with $R_0 = Z$, $R_1 = iZ$. Also, S is an H-graded ring with $S_0 = Z$, $S_2 = iZ$ and $S_1 = S_3 = 0$.

Clearly R is h.e. to S. But R is faithful and strong while S is not strong because $S_1 S_1 \neq S_2$. Moreover, S is not faithful because $i \in S_2 - 0$ and $iS_3 = 0$.

4. APPLICATIONS TO Z-GRADED RINGS

In this section we give necessary and sufficient conditions for two first strongly Z-graded rings to be homogeneously-equivalent. Also, we define homogeneous-equivalence order-preserving maps between Z-graded rings.

PROPOSITION 4.1. Let R and S be first strong graduations and assume that supp(R), supp(S) are cyclic groups. Then R is h.e. to S if and only if there exists a ring isomorphism $f : R \rightarrow S$ and there exists generator g of supp(R) and generator h of supp(S) such that $f(R_g) = S_h$ and $f\left(R_{g^{-1}}\right) = S_{h^{-1}}$.

PROOF. The first part is obvious because homogeneous-equivalence of first strong graded rings induce isomorphisms between their support subgroups. Suppose $f : R \rightarrow S$ is a ring isomorphism. Let g, h be generators of supp(R), supp(S) respectively with $f(R_g) = S_h$ and $f\left(R_{g^{-1}}\right) = S_{h^{-1}}$ Let $h' \in \text{supp}(S)$. Then there exists $n \in Z$ such that $h' = h^n$. If $n = 0$, then $h' = e_H$ and hence $f(R_{e_G}) = f\left(R_g R_{g^{-1}}\right) = S_h S_{h^{-1}} = S_{e_H} = S_{h'}$. If $n > 0$ then $S_{h'} = S_h S_h ... S_h$ (n-times), and hence $S_{h'} = f(R_{g^n})$. If $n < 0$ then $S_{h'} = S_{h^{-1}} S_{h^{-1}} ... S_{h^{-1}}$ (($-n$)-times), and hence $S_{h'} = f(R_{g^n})$. Therefore, R is h.e. to S by f.

DEFINITION 4.2. Let R be a Z-graded ring. Then R is said to be

(1) *Right limited* if there exists $j \in Z$ such that $R_i = 0$ for all $i \geq j$.

(2) *Left limited* if there exists $j \in Z$ such that $R_i = 0$ for all $i \leq j$

The property of right and left limited are not preserved between h.e. Z-graded rings as we see in the following example.

EXAMPLE 4.3. Let K be a field and $R = K[x]$ is the polynomial ring over K in one variable x Then R is a Z-graded ring with $R_0 = K$, $R_i = Kx^i$ for $i > 0$ and $R_i = 0$ for $i < 0$.

Let $S = K[x]$. Then S is a Z-graded ring with $S_0 = K$, $S_j = Kx^{-j}$ for $j < 0$ and $S_j = 0$ for $j > 0$. Clearly R is h.e. to S; R is left limited while S is not left limited.

Now, we will add extra conditions to ensure that these properties are preserved between h.e Z-graded rings.

DEFINITION 4.4. Let R and S be Z-graded rings such that R is h e. to S by f. Then f is *order-preserving* map if whenever $r, s \in h(R) - 0$ with $\deg(r) < \deg(s)$ we have $\deg(f(r)) < \deg(f(s))$

REMARKS 4.5.

1) f is an order-preserving map if and only if f^{-1} is order-preserving

2) The map f which shows R is h.e. to S in Example 4.3, is not an order-preserving map.

DEFINITION 4.6 ([5]) Let R be a Z-graded ring and I be an ideal of R

1) We denote by $I^{\sim}$ the ideal of R generated by

$\{r \in h(R) :$ there exists $x \in I$ with $x = x_1 + ... + x_n + r;$

$x_i \in R_{s_i} - 0, s_i \in Z$ and $\deg(x_1) < \deg(x_2) < ... < \deg(x_n) < \deg(r)\}$.

2) We denote by $I_{\sim}$ the ideal of R generated by

$\{r \in h(R) :$ there exists $x \in I$ with $x = r + x_1 + ... + x_n;$

$x_i \in R_{s_i} - 0, s_i \in Z$ and $\deg(r) < \deg(x_1) < \deg(x_2) < ... < \deg(x_n)\}$.

One can show that $I^{\sim}$ and $I_{\sim}$ are graded ideals. Moreover, if I, J are ideals of R with $I \subseteq J$ then $I^{\sim} \subseteq J^{\sim}$.

PROPOSITION 4.7. Suppose R and S are Z-graded rings such that R is h.e. to S by an order-preserving map f. Let I be an ideal of R. Then

1) $f(I^{\sim}) = (f(I))^{\sim}$

2) $f(I_{\sim}) = (f(I))_{\sim}$

3) $(R/I)^{\sim}$ is h.e. to $(S/f(I))^{\sim}$ and
$(R/I)_{\sim}$ is h.e. to $(S/f(I))_{\sim}$.

PROOF. 1) Let $I^{\sim}$ be the ideal of R generated by $A = \{a \in h(R) :$ there exists $x \in I$ with $x = x_1 + x_2 + ... + x_n + a;\ x_i \in R_{s_i} - 0,\ s_i \in Z$ and $\deg(x_1) < \deg(x_2) < ... < \deg(x_n) < \deg(a)\}$. Let $a \in A$ and assume $x = x_1 + x_2 + ... + x_n + a$. Then $f(x) = f(x_1) + f(x_2) + ... + f(x_n) + f(a)$. Since f is an order-preserving map, $\deg(f(x_1)) < \deg(f(x_2)) < ... < \deg(f(x_n)) < \deg(f(a))$. But $f(x) \in f(I)$ implies $f(a) \in (f(I))^{\sim}$. Therefore, $f(A) \subseteq (f(I))^{\sim}$ and hence $f(I^{\sim}) \subseteq (f(I))^{\sim}$ Also, $(f(I))^{\sim} \subseteq f(I^{\sim})$ since f^{-1} is an order-preserving map.

2) Similar to 1).

3) Follows from 1), 2) and Proposition 1.6.

PROPOSITION 4.8. Suppose R and S are Z-graded rings such that R is h.e. to S by an order-preserving map f. Then R is right (left) limited if and only if S is right (left) limited.

PROOF. Suppose R is right limited. Then there exists $k \in Z$ such that $R_i = 0$ for all $i \geq k$ and $R_{k-1} \neq 0$. Hence there exists $j \in \mathrm{supp}(S)$ such that $f(R_{k-1}) = S_j$.

CLAIM. $S_i = 0$ for all $i \geq j + 1$.

Suppose to the contrary that there exists $t \geq j + 1$, such that $S_t \neq 0$. Then there exists $m \in \mathrm{supp}(R)$ such that $f(R_m) = S_t$. Clearly, $m < k - 1$ and since f is order-preserving we have $t < j$. But $t \geq j + 1$ implies $j + 1 < j$ a contradiction. Therefore, $S_i = 0$ for all $i \geq j + 1$, i.e., S is right limited. The converse is obvious. The proposition in the case of R is left limited, can be similarly proved.

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