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A NOTE ON SOME APPLICATIONS OF SEMI-OPEN SETS

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ABSTRACT. The object of the present paper is to study the well known notions of semi-closure, semi-interior, semi-frontier and semi-exterior of a set using the concept of semi-open sets A semi-isolated point of a set is also defined and studied.

KEY WORDS AND PHRASES: Semi-closure, semi-interior, semi-isolated point, semi-discrete set, semi-scattered spaces.

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1. INTRODUCTION

A subset A of a topological space (X, τ) is said to be semi-open [1] if there exists an open set U such that $U \subset A \subset Cl(U)$. The complement of a semi-open set is called semi-closed [2] The union of all semi-open sets of X contained in A is called the semi-interior of A [2] and is denoted by $sInt(A)$ The intersection of all semi-closed sets containing A is called the semi-closure of A [2] and is denoted by $sCl(A)$ $sCl(A) - sInt(A)$ is called the semi-frontier of A [3] and is denoted by $sFr(A)$ $sInt(X - A)$ is said to be the semi-exterior of A [3] and is denoted by $sExt(A)$ In this paper, these notions are further investigated We also introduce and study the concepts of semi-isolated points and semi-scattered spaces

THEOREM 1. For a set $A \subset X$, the following are equivalent

(a) A is dense in X

(b) $sCl(A) = X$.

(c) If B is any semi-closed subset of X and $A \subset B$, then $B = X$.

(d) For each $\mathbf{x} \in X$, every semi-open set containing $\mathbf{x}$ has non-empty intersection with A

(e) $sInt(X - A) = \emptyset$.

PROOF. (a) $\Rightarrow$ (b). Let U be an open set with $U \subset X - B \subset Cl(U)$ Since $U \subset X - A$ and A is dense, therefore $U = \emptyset$ and so $Cl(U) = \emptyset$ Hence $B = X$ It follows that the intersection of all semi-closed sets containing A is X, that is $sCl(A) = X$.

(b) $\Rightarrow$ (a) Obvious since $sCl(A) \subset Cl(A)$ for every $A \subset X$

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are obvious.

(d) $\Rightarrow$ (e) If $sInt(X - A) \neq \emptyset$, then $sInt(X - A)$ is a non-empty semi-open set However, $(X - A) \cap A = \emptyset$ and since $sInt(X - A) \subset X - A$, we have $sInt(X - A) \cap A = \emptyset$ This contradicts (d) and means $sInt(X - A) = \emptyset$.

(e) $\Rightarrow$ (b) Since $sInt(X - A) = X - sCl(A)$ [1], therefore $X = sCl(A)$.

THEOREM 2. Let A be a subset of the space X Then

(a) $sFr(sInt(A)) \subset sFr(A)$

(b) $sFr(sCl(A)) \subset sFr(A)$

(c) $sExt(X) = \emptyset$

(d) $sExt(\emptyset) = X$

(e) $sExt(A) = sExt[X - sExt(A)]$

(f) $sInt(A) = A - sFr(A)$

(g) $sInt(A) \subset sExt[sExt(A)]$

(h) $X = sInt(A) \cup sExt(A) \cup sFr(A)$

PROOF. Only the proof of (e) will be given here. We have

$$\begin{aligned} sExt[S - sExt(A)] &= sExt[X - sInt(X - A)] \\ &= sInt[X - (X - sInt(X - A))] \\ &= sInt[sInt(X - A)] \\ &= sInt(X - A) = sExt(A). \end{aligned}$$

THEOREM 3. If $A, B \subset X$ such that $sFr(A) \cap Fr(B) = \emptyset$ and $Fr(A) \cap sFr(B) = \emptyset$, then $sInt(A) \cup sInt(B) = sInt(A \cup B)$.

PROOF. Let $x \in sInt(A \cup B)$. Then there exists a semi-open set U such that $x \in U \subset A \cup B$ If $x \in sFr(A)$ then $x \notin Fr(B)$, so there exists an open set V containing x with $V \subset B$ or $V \subset X - B$ Assume $V \subset B$. Then $x \in U \cap V \subset B$. Since $U \cap V$ is semi-open, $x \in sInt(B)$ On the other hand, if $V \subset X - B$, then $x \in U \cap V \subset A$ and so $x \in sInt(A)$. If $x \notin sFr(A)$ In particular, suppose that $x \notin sCl(A)$, for otherwise, $x \in sInt(A)$. Then $x \in B \subset sCl(B)$ since $x \in A \cup B$. We may assume that $x \notin sFr(A)$ for otherwise, $x \in sInt(B)$ Thus $x \notin Fr(A)$ and the argument now proceeds similarly to the case when $x \notin Fr(B)$.

THEOREM 4. A set $A \subset X$ is nowhere dense iff $Int(sCl(A)) = \emptyset$.

PROOF. The proof is obvious since $Int(ClA) = Int(sCl(A))$ for every $A \subset X$

DEFINITION 1. Let A be a subset of a topological space X. Then

(a) A point $x \in A$ is said to be a semi-isolated point of A if there is a semi-open set U such that $U \cap A = \{x\}$.

(b) A set A is said to be semi-discrete if each point of A is semi-isolated

(c) A space (X, τ) is said to be semi-scattered if every non-empty subset of X has a semi-isolated point.

It is obvious that every isolated point of $A \subset X$ is semi-isolated. But the converse is not true as can be seen from the following example.

EXAMPLE 1. Consider the usual topology on **R**. Let $\mathbf{A} = [0, 1]$ A subset $U = [1, 2)$ of **R** is semi-open and $U \cap A = \{1\}$. $1 \in A$ is a semi-isolated point of A but it is not an isolated point of A

REMARK 1. Let (X, τ) be a topological space and $A \subset X$. Then

(a) A semi-isolated point of X is merely an isolated point. For $\{x\}$ is semi-open iff $\{x\}$ is open. The set of all isolated (semi-isolated) points of a set $A \subset X$ is denoted by $A^s(A^{ss})$.

(b) A space X is a semi-discrete subset of itself iff X is discrete. Every discrete set is semi-discrete. But the converse need not be true as can be seen from the following example.

EXAMPLE 2. The subset $a = [0, 1] \times \{0\} \subset R^2$ is dense-in-itself but it is semi-discrete. For each $x = (r, 0) \in A$, let $U(x)$ be the open unit disk with nonnegative center coordinates which is tangent to A at the point x. Thus $B = U \cap \{x\}$ is semi-open and $\{x\} = B \cap A$. This shows that each point $x \in A$ is a semi-isolated point of A. This implies that A is semi-discrete in R^2. However, A is not discrete since its points are not isolated.

If A'_s denotes the semi-derived set of A, then we have the following theorem

THEOREM 5. If A is a subset of a space X, then

(a) $A'_s \cap A^{ss} = \emptyset$

(b) $sCl(A) = A'_s \cap A^{ss}$

(c) $X = A'_s \cap A^{ss} \cap sExt(A)$

PROOF. (a) $x \in A^{ss} \Leftrightarrow$ there is a semi-open set U containing **x** such that

$$U \cap A = \{x\}$$
$$\Leftrightarrow U \cap (A - \{x\}) = \emptyset \Leftrightarrow x \notin A'_s.$$

(b) $x \in sCl(A) \Leftrightarrow U \cap A \neq \emptyset$ for every semi-open set U containing **x**.

$\Leftrightarrow U \cap (A - \{x\}) \neq \emptyset$ if $x \notin A$ or $U \cap (A - \{x\}) = \emptyset$ if $x \in A$.

$\Leftrightarrow x \in A'_s$ or $x \in A^{ss} \Leftrightarrow x \in A'_s \cup A^{ss}$.

(c) Obvious in view of parts (a) and (b).

THEOREM 6. If $A \subset X$ is dense, then the following hold:

(a) The semi-isolated points of A are precisely the isolated points of A as a subspace

(b) $A \subset A'_s$ iff $A^s = \emptyset$

PROOF. (a) If $\{x\} = B \cap A$, where B is semi-open, then there is an open set U such that $U \subset B \subset Cl(U)$. $U \cap A \neq \emptyset$ since A is dense in X. $B \neq \emptyset$ implies $U \neq \emptyset$. Thus $U \cap A = \{x\}$ and **x** is an isolated point of the subspace A. Converse is obvious.

(b) $A^s = A^{ss}$ because A is dense in X. Since $X = sCl(A) = A'_s \cup A^{ss} = A'_s \cup A^s$ and $A^{ss} \cap A'_s = \emptyset$, therefore $A^s \cap A'_s = \emptyset$. Hence $A = A^s \cup (A \cap A'_s)$. Thus $A \subset A'_s$ iff $A^s = \emptyset$.

THEOREM 7. Every scattered space is semi-scattered.

The following example shows that a semi-scattered space need not be scattered.

EXAMPLE 3. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$ be a topology on X Then the set $A = \{b, c\}$ has no isolated points. But every subset of X has semi-isolated points.

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