Internat. J. Math. & Math. Sci.
VOL. 21 NO. 1 (1998) 139-144

SYMMETRIC GENERATING SET OF THE GROUPS A_{2n+1} AND S_{2n+1} USING S_n AND AN ELEMENT OF ORDER TWO

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(Received March 18, 1996 and in revised form August 28, 1996)

ABSTRACT. In this paper we will show how to generate in general A_{2n+1} and S_{2n+1} using a copy of S_n and an element of order 2 in A_{2n+1} or S_{2n+1} for all positive integers $n \geq 2$. We will also show how to generate A_{2n+1} and S_{2n+1} symmetrically using n elements each of order 2.

KEY WORD AND PHRASES: Symmetric generators, Involution, Double transitive groups, Group presentation

1991 AMS SUBJECT CLASSIFICATION CODES: 20F05

1. INTRODUCTION

It is shown by Hammas [1] that A_{2n+1} can be presented as

$$G = A_{2n+1} = < X, Y, T \mid < X, Y > = S_n, \; T^2 = [T, S_{n-1}] = 1 >$$

for n=4, 6, where $[T, S_{n-1}]$ means that T commutes with Y and with $X^{-2}YX$, (the generators of S_{n-1}). The relations of the symmetric group $S_n = < X, Y >$ of degree n are found in Coxeter and Moser[2]. Some relations must be added to the presentation of A_{2n+1} in order to complete the coset enumeration. Also, it has been shown by Hammas [1] that for n = 4, 6, the group A_{2n+1} can be symmetrically generated by n elements $T_0, T_1, ..., T_{n-1}$, each of order 2, of the form $T_i = T^{X^i} = X^{-i}TX^i$, where T and X satisfy the relations of the group A_{2n+1}. The set $\{T_0, T_1, ..., T_{n-1}\}$ is called a symmetric generating set of A_{2n+1} (see section 3).

In this paper, we give a generalization of the results obtained by Hammas [1] for all $n \geq 2$. Moreover a proof is given to show that the group

$$G = < X, Y, T \mid < X, Y > = S_n, \; T^2 = [T, S_{n-1}] = 1 >$$

is either A_{2n+1} if n is even or S_{2n+1} if n is odd for all $n \geq 2$. We give permutations that generate A_{2n+1} and S_{2n+1} for all $n \geq 2$ which satisfy the conditions given in the presentation of the group G. Further, we prove that G can be symmetrically generated by n permutations, each of order 2, satisfying the condition given in remark 2.4.

Our research is motivated by the aim of showing groups in their most "natural" role acting on (or permuting) the members of a symmetric generating set. The author has applied the method to obtain the symmetric generating sets and the presentations of the following finite simple groups: Tits group ${}^2F_4(2)'$, Janco groups J_1 and J_2, Mathieu groups M_{12} and M_{24}, and some of the linear groups PSL(2,q). For more details, see Hammas [1].

2. PRELIMINARY RESULTS

In this section, we give some of the preliminary results to be used in later sections. The proofs of these results can be found in many references, see for example [2], [4], and [5].

LEMMA 2.1. Let $1 \leq a \neq b \leq n$ be integers where n is odd. Let G be the group generated by the n-cycle $(1, 2, ..., n)$ and the 3-cycle (n, a, b). If the highest common factor **hcf**$(n, a, b) = 1$, then $G = A_n$.

LEMMA 2.2. Let n be an odd integer and let G be the group generated by the n-cycle $(1, 2, ..., n)$ and the k-cycle $(1, 2, ..., k)$. If $1 < k < n$ and k is an odd integer, then $G = A_n$.

PROOF. Let $\sigma = (1,2,3,...,n)$, and $\tau = (1,2,...,k)$. Since the commutator $[\sigma, \tau] = (1,2,k+1)$, then by Lemma 2.1, $G \cong A_n$.

LEMMA 2.3. Let G be the group generated by n-cycle $(1,2,...,n)$ and the involution $(n,1)(i,j)$ for $1 < i \neq j < n$. If $n \geq 9$ is an odd integer then $G \cong A_n$.

REMARK 2.4. The main condition used in Hammas [1], which we are going to use in this paper, is that T commutes with the generators of the group S_{n-1}.

3. SYMMETRIC GENERATING SETS

Let G be a group and let $\Gamma = \{T_0, T_1, ..., T_{n-1}\}$ be a subset of G, where $T_i = T^{X^i} = X^{-i}TX^i$ for all $i = 0, 1, ..., n-1$. Let S_n be the normalizer of the set Γ in G, which is a copy of the symmetric group of degree n. We define Γ to be a symmetric generating set of G if and only if $G = \langle \Gamma \rangle$ and S_n permutes Γ doubly transitively by conjugation. Equivalently, Γ is realizable as an inner automorphism.

4. PERMUTATIONAL GENERATING SET OF A_{2n+1} and S_{2n+1}

THEOREM 4.1. A_{2n+1} (S_{2n+1}) can be generated using a copy of S_n and an element of order 2 in A_{2n+1} (S_{2n+1}) if n is even (odd) for all $n \geq 2$.

PROOF. Let $X=(1, 2, ..., n)(n+1, n+2, ..., 2n)$, $Y=(n-1, n)(2n-1, 2n)$ and $T=(1, 2n+1)(2, n+2) \dots (n, 2n)$ be three permutations; the first is of order n, the second and the third are of order 2. Let H be the group generated by X and Y. By the Burnside and Moore Theorem (see Coxeter and Moser [2]), the group H is the symmetric group S_n . Let $\bar{G}$ be the group generated by X, Y and T. Consider the commutator $\eta=[X,T]$, which has the form $\eta=(1,n+1,2n+1,n+2,2)$. Then

$$\eta^3\eta^X=(1,2n+1)(2,n+3,3)(n+1,n+2)=\alpha .$$

Therefore $\alpha^2=(2,3,n+3)$. Hence

$$X\eta(\alpha^2)^{X^{-1}}=(1,2,\ldots,n,n+1,\ldots,2n,2n+1).$$

Let $\beta = X\eta(\alpha^2)^{X^{-1}}$. Let $K = <\beta,\alpha^2,T>$ be a subgroup of $\bar{G}$. Since the highest common factor **hcf**$(2,3,n+3) = 1$, then by Lemma 2.1 $<\beta,\alpha^2>=A_{2n+1}$. Now if n is an even integer, then $K=A_{2n+1}$. Since X, Y and T are even permutations then $K=\bar{G}=A_{2n+1}$. Also, if n is an odd integer, then T is an odd permutation and therefore $K=\bar{G}=S_{2n+1}$.

5. SYMMETRIC GENERATING SET OF A_{2n+1} and S_{2n+1}

THEOREM 5.1. Let X,Y and T be the permutations described in Theorem 4.1. Let $\Gamma=\{T_0,T_1,\ldots,T_{n-1}\}$, where $T_i=T^{X^i}$ and $i=0,1,\ldots,n-1$. If n is an even integer, then the set Γ generates the alternating group A_{2n+1} symmetrically, while if n is an odd integer, then the set Γ generates the symmetric group S_{2n+1} symmetrically.

PROOF. Let $T_0=(1,2n+1)(2,n+2)\ldots(n,2n)$, $T_1=(1,n+1)(2,2n+1)\ldots(n,2n)$, ..., $T_{n-1}=T^{X^{n-1}}=(n,2n+1)(1,n+1)\ldots(n-1,2n-1)$. Let $H=<\Gamma>$. We claim that if n is an even integer, then $H\cong A_{2n+1}$ and if n is an odd integer, then $H\cong S_{2n+1}$. To show this, suppose first that n is an even integer. Consider the element

$$\alpha=\prod_{i=0}^{n-1}T^{X^i}.$$

It is not difficult to show that $\alpha=(1,2,n+2,n+3,3,4,n+4,n+5,5,6,\ldots,2n,2n+1,n+1)$ and it is a cycle of length $2n+1$. Let $\beta=T_0T_1$. It is clear that $\beta=(1,2,n+2,2n+2,n+1)$. Let $H_1=<\alpha,\beta>$. We claim that $H_1\cong A_{2n+1}$. To prove this, let θ be the mapping which takes the element in the position i of the cycle α into the element i of the cycle $(1,2,\ldots,2n+1)$. Under this mapping, the group H_1 will be mapped into the group $\theta(H_1)=<(1,2,\ldots,2n+1),(1,2,3,2n,2n+1)>$ which is, by Lemma 2.2, the alternating group A_{2n+1}. Hence $H\cong H_1\cong\theta(H_1)\cong A_{2n+1}$.

Second, suppose that n is an odd integer. Consider the element

$$\delta = \prod_{i=1}^{n} T^{X^i}.$$

It is not difficult to show that $\delta = (1,2n+1,2,n+3,4,n+5,6,n+7,...,2n)$ and it is a cycle of length $n+1$. Let $\mu = \delta^{T_1} T_0$. Since $\delta^{T_1} = (2,2n+1,3,n+4,5,n+6,...,n,n+1)$, then

$$\mu = (1,2n+1,n+3,3,4,n+4,n+5,...,n,n+1,n+2,2)$$

which is a cycle of length $2n+1$. Let $\beta = T_1^{T_2} T_2^{T_3}$, then $\beta = (2,n+2,3)(4,2n+1)(n+3,n+4)$. Therefore $\beta^2 = (2,3,n+2)$. Let $H_2 = <\mu, \beta^2, T_0>$. We claim that $H_2 \cong S_{2n+1}$. To prove this, let θ be the mapping which takes the element in the position i of the cycle μ into the element i of the cycle $(1,2,...,2n+1)$. Under this mapping the group H_2 will be mapped into the group

$$\theta(H_2) = <(1,2,...,2n+1),(2n+1,4,2n),(1,2)(3,4)...(2n-3,2n-2)(2n,2n+1)>.$$

Since the **hcf**$(2n+1,4,2n)=1$, then the group $<(1,2,...,2n+1),(2n+1,4,2n)>$ is the alternating group A_{2n+1}. Since n is an odd integer, then the permutation $(1,2)(3,4)...(2n-3,2n-2)(2n,2n+1)$ is an odd permutation. Therefore the group $\theta(H_2)$ is the symmetric group S_{2n+1}. Hence $H \cong H_2 \cong \theta(H_2) \cong S_{2n+1}$.

The set Γ described above satisfies the conditions of the group G given in section 1. It is important to note that Γ must have **exactly** n elements each of order 2 to generate A_{2n+1} or S_{2n+1}. The following Theorem characterizes all groups obtained by removing m elements of the set Γ for some integer m.

THEOREM 5.2. Let T and X be the permutations described above and let $\Gamma = \{T_1, T_2, ..., T_n\}$. Then, removing m elements of the set Γ for all $1 \le m \le n-3$, the resulting set generates $S_{2(n-m)+1}$, removing $m=(n-2)$ elements of the set Γ, the resulting set generates the dihedral group of order 10 (D_{10}), and removing $m=(n-1)$ elements of the set Γ, the resulting set generates the cyclic group C_2.

PROOF. Using induction on $n-m$, if $n-m=1$, then $\Gamma_1 = \{T_1\}$. Since $T_1 = (1,n+1)(2,2n+1)(3,n+3)...(n,2n)$, then Γ_1 generates C_2. If $n-m=2$, then $\Gamma_2 = \{T_1, T_2\}$. Since T_1 is the permutation described above, $T_2 = (1,n+1)(2,n+2)(3,2n+1)...(n,2n)$, and $T_1T_2 = (2,3,n+3,2n+1,n+2)$, then it is clear that Γ_2 generates D_{10}. Now suppose that $1 \le m \le n-3$. If $n-m=k$, then $\Gamma_k = \{T_1, ..., T_k\}$. Assuming $\alpha = T_1^{(T_2T_3...T_{k-1})} T_k$, then for k *an even* integer we have

$$\alpha = (2,3,n+4,5,n+6,7,n+8,...,k-1,n+k,k+1,n+k+1,2n+1,k,n+k-1,k-2,n+k-3,...,4,n+3,n+2)$$

which is a permutation of length $2k+1$; while if k is an odd integer, then

$$\alpha = (2,n+2,3,n+3,4,n+5,6,n+7,8,n+9,...,k-1,n+k,k+1,n+k+1,2n+1,k,n+k-1,k-2,n+k-3,...,5,n+4,3),$$

it is also a permutation of length $2k+1$. Let $\beta = T_1^{2^{T_2}} T_1T_2T_3$. Since $\beta =(2, n+3)(3, n+2)(4, n+4, 2n+1)$, then $\beta^3=(2,n+3)(3,n+2)$. By Lemma 2.3, α and β^3 generate A_{2k+1}. Hence the group generated by α, β^3 and T_1 is the Symmetric group S_{2k+1}. Therefore the Theorem is true for all m.

REMARK. The above results are summarized in the following table

	n	$G = < X, Y, T >$	$< X, T >$	$< \Gamma >$
1	even	A_{2n+1}	A_{2n+1}	A_{2n+1}
2	odd	S_{2n+1}	S_{2n+1}	S_{2n+1}

where

$$A_{2n+1} = < X,Y,T \mid < X,Y > = S_n, T^2 = [T,Y] = [T, X^{-2}YX] = (XT)^{2n+1} = (YT_{n-2})^{10} >.$$

$$S_{2n+1} = < X,Y,T \mid < X,Y > = S_n, T^2 = [T,Y] = [T, X^{-2}YX] = (XT)^{n(n+1)} = (Y T_{n-2})^{10} >.$$

From the above, we can see that the order of the element XT is $n(n+1)$ when n is an odd integer. As n gets larger, the order of XT becomes very large. For this reason, Hammas [1] had been unable to proceed for large odd values of n.

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