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FUNCTIONAL EVOLUTION EQUATIONS WITH NONCONVEX LOWER SEMICONTINUOUS MULTIVALUED PERTURBATIONS

A.G. IBRAHIM

Department of Mathematics
Faculty of Science
Cairo University
Giza, EGYPT

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ABSTRACT. In this paper we prove some existence theorems concerning the solutions and integral solution for functional (delay) evolution equations with nonconvex lower semicontinuous multivalued perturbations

KEY WORDS AND PHRASES: Functional evolution equations, m-accretive operators, integral solutions

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1. INTRODUCTION

Let E be a Banach space, r, $T \in \mathbb{R}^+$ and $I = [a, b]$ Let us denote

$C_E([-r, T])$ the vector space of all continuous functions from $[-r, T]$ to E endowed with the uniform topology

For all $t \geq 0$, $s_t : C_E([-r, t]) \rightarrow C_E([-r, 0])$,

$$(s_t f)(\theta) = f(t + \theta), \quad \forall \theta \in [-r, 0].$$

$A : I \times E \rightarrow 2^E$ such that $A(t, .)$ is an m-accretive multivalued operator

$P_{wc}(E)$ the family of nonempty weakly compact subsets of E

In this paper we are concerned with the following problems

(1) Existence of solutions of the perturbated evolution equation with delay

$$(P)\begin{cases} u'(t) \in -A(t, u(t)) + F(t, s_t u) & \text{a e on } I, \\ u \equiv \psi & \text{on } [-r, 0] \end{cases}$$

where $F : I \times C_E([-r, 0]) \rightarrow P_{wc}(E)$ is a multivalued function such that $F(t, .)$ is lower semicontinuous and $\psi \in C_E([-r, 0])$ is arbitrary but fixed.

(2) Existence of solutions of the perturbated evolution equation with delay

$$(Q)\begin{cases} u'(t) \in -N_{\Gamma(t)}(u(t)) + F(t, s_t u) & \text{a e on } I, \\ u \equiv \psi & \text{on } [-r, 0] \end{cases}$$

where $N_{\Gamma(t)}(x)$ is the normal cone of the convex set $\Gamma(t)$ at the point $x \in E; t \in I$ It should be noticed that the problem (Q) is not a special case of the problem (P)

(3) Existence of integral solutions of (P), when the operator A is independent of t, under conditions that are weaker than those imposed in (P)

The results obtained in the present paper generalized the following interesting known cases

Problem (P) for which the dual of E is uniformly convex, $A(t, .)$ is an m-accretive single-valued operator and F is a Lipschitz single-valued function cf Kartsatos and Parrott [1]

Problem (P) for which E is reflexive, $A(t,.)$ is an m-accretive multivalued operator and F is a Lipschitz single-valued function cf Tanaka [2]

Problems (P) and (Q) without delay cf Cichon [3], [4], Ibrahim [5] and the references therein

2. NOTATIONS AND DEFINITIONS

Let E^* be the dual of E, E_σ the Banach space E endowed with the weak topology $\sigma(E,E^*)$ If B is a multivalued operator from E to 2^E then B is said to be accretive if for each $\lambda > 0, x_1, x_2 \in D(B)$ (the domain of B), $y_1 \in B(x_1)$ and $y_2 \in B(x_2)$ we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|.$$

We say that B is m-accretive if B is accretive and if there exists $\lambda > 0$ such that $R(I + \lambda B) = E$, where I is the identity map It is known that if B is m-accretive, then for every $\lambda > 0$ the resolvant $J_\lambda B = (I + \lambda B)^{-1}$ and the Yosida approximation of B; $B_\lambda = (I - J_\lambda B)/\lambda$, are defined everywhere The generalized domain of B is defined by

$$D^*(B) = \left\{x \in E : |B(x)| = \lim_{\lambda\to\infty} \|B_\lambda x\| < \infty\right\}.$$

For the properties of m-accretive multivalued operators refer to [6] and [7]

If C is a convex subset of E and $x \in C$, then the normal cone of C at x is defined by

$$N_C(x) = \{y \in E^* : \langle y, z - x\rangle \leq 0, \forall\, z \in C\}.$$

Now we recall some concepts concerning multivalued functions Let Y be a locally convex space and let $G : E \to 2^Y - \{\phi\}$ We say that G is lower semicontinuous (resp. upper semicontinuous) if for every open V in Y the set $\{x \in E : G(x) \cap V \neq \phi\}$ (resp $\{x \in E : G(x) \subset V\}$) is open in E. We say that G is lower semicontinuous (resp. upper semicontinuous) in the Kuratowski sense iff for all $v_n \to v$ in E, $G(v) \subseteq \lim_{n\to\infty}\inf G(v_n)$ (resp $\lim_{n\to\infty}\sup G(v_n) \subseteq G(v)$), where

$$\lim_{n\to\infty} \inf G(v_n) = \left\{z \in Y : z = \lim_{n\to\infty} z_n, z_n \in G(v_n), \forall\, n \geq 1\right\},$$
$$\lim_{n\to\infty} \sup G(v_n) = \left\{z \in Y : z = \lim_{n\to\infty} z_{n_k}, z_{n_k} \in G(v_{n_k}), \forall\, k \geq 1\right\}.$$

If E is metrizable then lower semicontinuity and lower semicontinuity in the Kuratowski sense are equivalent (cf [8], [9])

The following known result will be used in the sequel

LEMMA 2.1 [6]. For every $t \in I$, let $A(t,.)$ be an m-accretive multivalued operator from E to $2^E - \{\phi\}$ satisfying the following condition:

(C_1) There exist $\lambda_0 > 0$, a continuous function $h : I \to E$ and a nondecreasing continuous function $L : [0,\infty) \to [0,\infty)$ such that for all $\lambda \in (0,\lambda_0)$ and for almost $t, s \in I$,

$$\|A_\lambda(t,x) - A_\lambda(s,x)\| \leq \|h(s) - h(t)\| L(\|x\|), \quad \forall\, x \in E.$$

Then $D^*(A(t,.))$ and $\overline{D}(A(t,.))$ are independent of t

So if A is as in Lemma 2.1 we may write $D^*(A) := D^*(A(t,.))$ and $\overline{D(A)} := \overline{D(A(t,.))}$; $t \in I$ respectively

LEMMA 2.2 [10]. Let E be a Banach space and M a compact metric space If T is a lower semicontinuous multivalued function on M and with nonempty closed decomposable values in $L^1_E(I)$, then T has a continuous selection.

3. EXISTENCE OF SOLUTIONS FOR THE PROBLEMS (P) AND (Q)

To prove our results we need the following lemmas

LEMMA 3.1. Let ψ be an element of $C_E([-r,0])$ and β be a positive real number The set

$$\chi = \left\{ u \in C_E([-r,0]) : u \equiv \psi \text{ on } [-r,0] \text{ and } u(t) = \psi(0) + \int_0^t f(s)ds;\ f \in K_\beta \right\},$$

is nonempty and convex, where $K_\beta = \{f \in L_E^1(I) : |f(t)| \leq \beta \text{ a e on } I\}$ If E is reflexive then χ is compact subset of $C_{E_\sigma}([-r,T])$ If, in addition, E is separable then χ is metrizable

PROOF. It is obvious that χ is nonempty, convex and equicontinuous and that the set $\{u(t) : u \in \chi\};\ t \in I$, is bounded So, if E is reflexive then, χ is relatively compact in $C_{E_\sigma}([-r,T])$ by Ascoli's theorem Let us verify that χ is closed in $C_{E_\sigma}([-r,T])$ Let (u_n) be a sequence in χ converging to $u \in C_{E_\sigma}([-r,T])$ Then $u \equiv \psi$ on $[-r,0]$ and for each $n \geq 1$ there exists $f_n \in K_\beta$ such that $u_n(t) = \psi(0) + \int_0^t f_n(s)ds;\ t \in I$ Since E is reflexive, K_β is weakly compact in $L_E^1(I)$ Hence, the sequence (f_n) has a subsequence, denoted again by (f_n), converging weakly to $f \in K_\beta$ Then $u(t) = \psi(0) + \int_0^t f(s)ds;\ t \in I$ This proves that χ is closed in $C_{E_\sigma}([-r,T])$ Now if E is separable then so is $L_E^1(I)$ Consequently, K_β is metrizable Since χ is isomorphic to $\{\psi(0)\} \times K_\beta$, then χ is metrizable

LEMMA 3.2. Let G be a multivalued function from E_σ to the nonempty closed subsets of E such that G is lower semicontinuous in the Kuratowski sense. If (x_n) is a sequence converging to x in E_σ, then for every $z \in E$,

$$\lim_{n\to\infty} \sup d(z, G(x_n)) \leq d\Big(z, \lim_{n\to\infty} \inf G(x_n)\Big) \leq d(z, G(x)).$$

PROOF. Let $y \in \lim_{n\to\infty} \inf G(x_n)$ Then there exists a sequence (y_n) such that $y_n \in G(x_n);\ n \geq 1$ and $y_n \to y$ as $n \to \infty$ For any $z \in E$ we have

$$\lim_{n\to\infty} \sup d(z, G(x_n)) \leq \lim_{n\to\infty} \sup \|z - y_n\| = \|z - y\|,$$

which proves the first inequality The second inequality follows from the lower semicontinuity of G

THEOREM 3.1. Let E be a reflexive separable Banach space Let $A(t,.);\ t \in I$ be an m-accretive multivalued operator from E to $2^E - \{\phi\}$ satisfying condition (C_1) together with the following conditions

(C_2) There exist $\mu > 0$ such that for all $x \in E$, the function $w_x : t \to (I + \mu A(t,.))^{-1}$ belongs to $L_E^2(I)$

(C_3) For all $r > 0$ there exists $\delta(r) > 0$ such that for all $\lambda > 0$ and all $x \in \overline{D}(A)$ with $\|x\| < r$,

$$\|J_\lambda A(0,x) - x\| \leq \lambda\delta(r).$$

Let F be a measurable multivalued function from $I \times C_E([-r,0])$ to $P_{wc}(E)$ satisfying the following conditions

(F_1) There exists $\alpha > 0$ such that

$$\sup\{\|y\| : y \in F(t,u)\} \leq \alpha, \quad \forall (t,u) \in I \times C_E([-r,0]).$$

(F_2) For all $t \in I, F(t,.)$ is lower semicontinuous in the sense of Kuratowski from $C_{E_\sigma}([-r,0])$ to E

(F_3) For all $u \in C_E([-r,0])$ the multivalued function $t \to F(t, s_t u)$ admits a measurable selection

Then for every $\psi \in C_E([-r,0])$ with $\psi(0) \in D^*(A)$, the problem (P) has a solution.

PROOF. We split the proof into the following three steps

(1) Let $f \in K_\alpha = \{g \in L_E^1(I) : \|g(t)\| \leq \alpha \text{ a.e on } I\}$. Since A satisfies conditions (C_1), (C_2) and (C_3), then by Theorem 4 of [5], there exists a unique absolutely continuous function $u_f : I \to E$ such that·

(i) $u_f'(t) \in -A(t, u(t)) + f(t)$ a e. on $I, u_f(0) = \psi(0)$,

(ii) $\|u_f(t)\| \leq \beta_1 = (\alpha+1)T + L(r)\sup_{t\in I}\|h(t)\| + \delta(r), \forall t \in I$, where
$r = \alpha(1 + L(\|\psi(0)\|)) + |A(0,x_0)|$,

(iii) the function $f \to u_f$ is continuous from K_α to $C_{E_\sigma}(I)$

(2) Set $\chi_1 = \left\{u \in C_E([-r,T]), u \equiv \psi \text{ on } [-r,0] \text{ and } u(t) = \psi(0) + \int_0^t f(s)ds, f \in K_\beta\right\}$ By Lemma 3 1, χ_1 is a compact subset of $C_\sigma([-r,T])$ and is metrizable. Define a multivalued function T_1 on χ_1 by $T_1(u) = \{f \in K_\alpha : f(t) \in F(t, s_t u) \text{ a e on } I\}$ In this step we prove that T_1 has a continuous selection $V_1 : \chi_1 \to K_\alpha$ For this purpose, we show that T_1 satisfies the conditions of Lemma 2 2 Condition (F_3) assures that the values of T_1 are nonempty Moreover, if D is a measurable subset of I and $g_1, g_2 \in T_1(u)$ for some $u \in \chi_1$, then the function $g = N_D g_1 + N_{I-D} g_2$ belongs to $T_1(u)$, where N is the characteristic function. Then the values of T_1 are decomposable It remains to prove that T_1 is lower semicontinuous Since χ_1 is compact metrizable in $C_{E\sigma}([-r,T])$, it suffices to show that T_1 is lower semicontinuous in the Kuratowski sense So, let (u_n) be a sequence in χ_1 converging to $u \in \chi_1$, with respect to the topology on $C_{E\sigma}([-r,T])$ and let $g \in T_1(u)$ Since F is measurable, then for all $n \geq 1$ the multivalued function

$$t \to B_n(t) = \{z \in F(t, s_t u_n) : \|g(t) - z\| = d(g(t), F(t, s_t u_n))\}$$

has a measurable selection $g_n : I \to E$. Thus, by Lemma 3 2, for all $t \in I$,

$$\begin{aligned}\lim_{n\to\infty} \|g(t) - g(t_n)\| &\leq \lim_{n\to\infty} \sup d(g(t), F(t, s_t u_n)) \\ &\leq d\Big(g(t), \lim_{n\to\infty} inf\, F(t, s_t u_n)\Big) \\ &= d(g(t), F(t, s_t u)) = 0.\end{aligned}$$

This means that T_1 is lower semicontinuous and hence there exists a continuous function $V_1 : \chi_1 \to K_\alpha$ such that $V_1(x) \in T(x), \forall x \in \chi_1$

(3) Define a function $\theta : \chi_1 \to \chi_1$ by $\theta(x) = u_f, f = V_1(x)$ By (iii) of the first step, θ is continuous Hence, by Tichonoff's fixed point theorem, there exists $u \in \chi_1$ such that $u = u_f, f = V_1(u) \in T_1(u)$ This means that $u'(t) \in -A(t, u(t)) + f(t)$ and $f(t) \in F(t, s_t u)$ a e on I The theorem is thus proved.

THEOREM 3.2. Let H be a Hilbert space and F be a measurable multivalued function from $I \times C_H([-r,0])$ to $P_{wc}(H)$ satisfying conditions (F_1), (F_2) and (F_3) Let Γ be a multivalued function from I to the family of nonempty closed convex subsets of H, with compact graph G and satisfies the following conditions.

(Γ_1) There exists $\gamma > 0$ such that $\|x - proj_{\Gamma(t)} x\| \leq \gamma(\tau - t)$ for all $(t,x) \in G$ and all $\tau \in I, (t < \tau)$

(Γ_2) The function $(t,x) \to \delta^x(x, \Gamma(t)) = \sup\{(x,y) : y \in \Gamma(t)\}$ is lower semicontinuous on $I \times B_\sigma$, where B_σ is the relative weak topology

Then for all $\psi \in C_E([-r,0])$ with $\psi(0) \in \Gamma(0)$, the problem (Q) has a solution

PROOF. We split the proof into the following three steps

(1) Let $f \in K_\alpha$ Since Γ has a compact graph and satisfies conditions (Γ_1) and (Γ_2) then by Theorem 3 1 [11], there exists a unique absolutely continuous function $u_f : I \to H$ such that
   - (i) $u'_f(t) \in -N_{\Gamma(t)}(u(t)) + f(t)$ a.e. on I,
   - (ii) $u_f(0) = \psi(0), u_f(t) \in \Gamma(t), \forall t \in I$,
   - (iii) $\|u_f(t)\| \leq \beta_2 = T(\gamma + \alpha), \forall t \in I$ and the function $f \to u_f$ is continuous from K_α to C_{H_σ}

(2) Set $\chi_2 = \left\{u \in C_H([-r,T]) : u = \psi \text{ on } [-r,0] \text{ and } u(t) = \psi(0) + \int_0^t f(s)ds, f \in K_{\beta_2}\right\}$ and define a multivalued function T_2 on χ_2 by $T_2(u) = \{f \in K_\alpha : f(t) \in F(t, s_t u) \text{ a.e on } I\}$ As in the second step of the proof of Theorem 3.1 we can show that T_2 has a continuous selection $V_2 : \chi_2 \to K_\alpha$

(3) Define the function $\theta : \chi_2 \to \chi_2$ by $\theta(x) = u_f, f = V_2(x)$ As in the third step of the proof of Theorem 3.1, we can show that there exists a unique $u \in \chi_2$ such that $u = u_f, f \in T_2(u)$ Clearly u is a solution of (Q)

4. EXISTENCE OF INTEGRAL SOLUTIONS FOR THE PROBLEM (P) WHEN THE OPERATOR A IS INDEPENDENT OF TIME

In this section A denotes a multivalued operator from E to $2^E - \{\phi\}$ Consider the evolution equation

$$(P^*)\begin{cases} u'(t) \in -A(u(t)) + f(t) & \text{a e on } I \\ u(0) = x_0 \in \overline{D(A)}, \end{cases}$$

where $f \in L^1_E(I)$ By an integral solution of (P^*) we mean a continuous function $u : I \to \overline{D(A)}$ with $u(0) = x_0$ such that

$$\|u(t) - z\| \leq \|u(s) - z\| + \int_s^t [u(r) - z, f(r) - y]_+ dr,$$

for each $z \in D(A), y \in A(z)$ and $0 \leq s \leq t < T$, where

$$[x_1, x_2]_+ = \lim_{h \downarrow 0} (\|x_1 + hx_2\| - \|x_1\|)/h, \ \forall x_1, x_2 \in E.$$

It is known that [7] if A is an m-accretive operator then for each $(x_0, f) \in \overline{D(A)} \times L^1_E(I)$, the problem (P^*) has a unique integral solution u_f, such that the function $f \to u_f$ is continuous In this section we are concerned with the existence of integral solutions of the functional evolution equation

$$(P^{**})\begin{cases} u'(t) \in -A(u(t)) + F(t, s_t u) & \text{a e on } I \\ u \equiv \psi & \text{on } [-r, 0], \end{cases}$$

where F is a multivalued function from $I \times C_E([-r, 0])$ to $2^E - \{\phi\}, S_t; t > 0$ is the operator of translation defined in section 1 and ψ is a given function, belongs to $C_E([-r, 0])$ with $\psi(0) \in \overline{D(A)}$ By an integral solution of (P^{**}) we mean a continuous function $u : [-r, T] \to E$ with $u \equiv \psi$ on $[-r,]0$, such that u is an integral solution of the evolution equation $u'(t) \in -A(u(t)) + f(t), u(0) = \psi(0)$, where $f \in L^1_E(I)$ and $f(t) \in F(t, s_t u)$, a e on I

We say that the operator $A : E \to 2^E - \{\phi\}$ has the (M)-property ([7], [12]) if for each $x_0 \in D(A)$ and each uniformly integrable subset Q of $L^1_E(I)$, the set $\{u_g : g \in Q\}$ is a relatively compact subset of $C_E(I)$ where u_g is the unique integral solution of the evolution equation $u'(t) \in -A(u(t)) + g(t)$ a e on I; $u(0) = x_0$. It is well known that ([7], [12]) if the proper operator $-A$ generates a compact semigroup (via Crandall-Liggett's exponential formula [3], [13]), then A has the property (M)

THEOREM 4.1. Let E be a Banach space and A an m-accretive multivalued operator from E to $2^E - \{\phi\}$ having the (M)-property. Let F be a measurable multivalued function from $I \times C_E([-r, 0])$ to the non-empty closed subsets of E satisfying the condition (F_3) together with the following conditions

(F_4) There exists a function $h \in L^1_{\mathbb{R}}(I)$ such that

$$\sup\{\|z\| : z \in F(t, u)\} \leq h(t), \quad \forall (t, u) \in I \times C_E([-r, 0]).$$

(F_5) For all $t \in I, F(t, .) : C_E([-r, 0]) \to E$ is lower semicontinuous in the Kuratowski sense

Then for all $\psi \in C_E([-r, 0])$ with $\psi(0) \in \overline{D(A)}$, the problem (P^{**}) has an integral solution

PROOF. Consider the set $Q = \{f \in L^1_E(I) : \|f(t)\| \leq h(t)$ a e. on $I\}$ One can easily show that Q is nonempty and uniformly integrable subset of $L^1_E(I)$ As mentioned above, for each $f \in Q$ there exists a unique continuous function $u_f : I \to \overline{D(A)}$ such that u_f is the unique integral solution of the evolution equation $u'(t) \in A(u(t)) + f(t), u(0) = \psi(0)$ and the function $f \mapsto u_f$ is continuous from Q to $C_E(I)$. Let $\chi^* = \overline{\{u^*_f \in C_E([-r, T]) : f \in Q\}}$, where $u^*_f \equiv \psi$ on $[-r, 0]$ and $u^*_f \equiv u_f$ on I Since a has the property (M), χ^* is compact in the metric space $C_E([-r, T])$ Now, define a multivalued function T on χ^* by $T(x) = \{f \in L^1_E(I) : f(t) \in F(t, s_t x)$ a e on $I\}$ As in the second step of the proof of Theorem 3 1, we can show that T has a continuous selection $V : \chi^* \to L^1_E(I)$

Also, define a function $\Phi : \chi^* \to \chi^*, \Phi(x) = u_f^*, f = V(x)$ The function Φ is clearly continuous and hence has a fixed point $x \in \chi^*$ It is obvious that x is the desired solution

5. EXAMPLES

In this section we give some examples illustrating the scope of the results developed in sections 3 and 4

EXAMPLE 1. Let for all $t \in I$, $A(t) = B - h(t)$ where $h : I \to E$ is integrable and B is an m-accretive operator on E Clearly $A(t)$ is m-accretive for all $t \in I$ Let $\lambda > 0, s, t \in I$ and $x \in E$ Then

$$\|A_\lambda(t,x) - A_\lambda(s,x)\| \leq \frac{1}{\lambda} \|J_\lambda A(t,x) - J_\lambda A(s,x)\| \leq \|h(t) - h(s)\|.$$

Hence condition (C_1) of Lemma 2.1 holds

EXAMPLE 2. In [6] there are several examples for operators A such that for every $t \in I, A(t)$ is m-accretive and satisfies condition (C_1)

EXAMPLE 3. Let H be a real Hilbert space with inner product $(.,.)$ and let $\Phi : H \to H$ be a proper lower semicontinuous convex function. The set $\partial\Phi(x) = \{z \in H : \Phi(x) \leq \Phi(y) + \langle x - y, z\rangle$ for each $y \in H\}$ is called the subdifferential of Φ at the point x We recall that $D(\partial\Phi) = \{x \in H : \partial\Phi(x)$ is nonempty$\}$. Now if we define an operator $A : D(A) = D\partial(\Phi) \to 2^H$ by $A(x) = \partial\Phi(x)$, then A is m-accretive and the following conditions are equivalent [7]

(i) For each $\lambda > 0$, the resolvent $J_\lambda A$ is a compact operator

(ii) The function Φ is of compact type

(iii) The semigroup generated by the operator $-A$ is compact

EXAMPLE 4. Take $E = L^2_{\mathbb{R}}([0,\pi])$ and let us define $A : D(A) \subseteq E \to E$ by $Au = -u^{(2)}(t)$ for each $u \in D(A)$ where $D(a) = \{u \in E : u^{(2)} \in E, u(0) = u(\pi) = 0\}$ The operator A is m-accretive and the semigroup $\{S(t) : t > 0\}$ generated by $-A\big(S(t) = \lim_{n\to\infty}\big(I+\frac{t}{n} A\big)^{-n}\big)$ is compact [7]

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