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REGULARIZED SUM FOR EIGENFUNCTIONS OF MULTI-POINT PROBLEM IN THE COMMENSURABLE CASE

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ABSTRACT. Consider the eigenvalue problem which is given in the interval $[0,\pi]$ by the differential equation

$$-y''(x)+q(x)y(x)=\lambda y(x); \qquad 0\le x\le \pi \tag{0.1}$$

and multi-point conditions

$$U_1(y)=\alpha_1 y(0)+\alpha_2 y(\pi)+\sum_{k=3}^{n}\alpha_k y(x_k\pi)=0,$$
$$U_2(y)=\beta_1 y(0)+\beta_2 y(\pi)+\sum_{k=3}^{n}\beta_k y(x_k\pi)=0, \tag{0.2}$$

where $q(x)$ is sufficiently smooth function defined in the interval $[0,\pi]$. We assume that the points $x_3,x_4,\dots,x_n$ divide the interval $[0,1]$ to commensurable parts and $\alpha_1\beta_2-\alpha_2\beta_1\neq 0$. Let $\lambda_{k,s}=\rho_{k,s}^2$ be the eigenvalues of the problem (0.1)-(0.2) for which we shall assume that they are simple, where k,s, are positive integers and suppose that $H_{k,s}(x,\xi)$ are the residue of Green's function $G(x,\xi,\rho)$ for the problem (0.1)-(0.2) at the points $\rho_{k,s}$. The aim of this work is to calculate the regularized sum which is given by the form :

$$\sum_{(k)}\sum_{(s)}\left[\rho_{k,s}^{-\sigma}H_{k,s}(x,\xi)-R_{k,s}(\sigma,x,\xi,\rho)\right]=S_\sigma(x,\xi) \tag{0.3}$$

The above summation can be represented by the coefficients of the asymptotic expansion of the function $G(x,\xi,\rho)$ in negative powers of k. In series (0.3) σ is an integer, while $R_{k,s}(\sigma,x,\xi,\rho)$ is a function of variables x,ξ and defined in the square $[0,\pi]x[0,\pi]$ which ensure the convergence of the series (0.3).

KEY WOREDS AND PHRASES : Regularized sum for eigenfunctions, asymptotic formula, Green's function, differential operator.
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1. INTRODUCTION.

It is well-known that the sum of the diagonal elements in a square matrix is equal to the sum of the eigenvalues of its operator in finite dimensional space. In other words the trace of a matrix is equal to the spectral trace in n-dimensional spaces.

It is worth mentioning that this theorem is satisfied also in the case of nuclear operators acting in Hilbert space. Sadovnichii [1] proved this theorem. Thus we might ask the following question. Is the last theorem applicable to the case of unbounded operators?, especially in the case of differential operators since in general case the trace of a matrix and spectral trace do not exist. Consider, for example, the boundary-value problem :

$$-y''(x)+q(x)y(x)=\lambda y(x), \qquad 0\le x\le \pi \tag{1.1}$$

$$y(0)=y(\pi)=0, \tag{1.2}$$

where $q(x)$ is sufficiently smooth function.

The eigenvalues λ_n of problem (1.1), (1.2) has the asymptotic expansion in the form :

$$\lambda_n \sim n^2 + c_0 + \frac{c_2}{n^2} + \frac{c_4}{n^4} + \quad \dots, \tag{1.3}$$

where
$$c_0 = \frac{1}{2\pi}\int_0^{\pi} q(\xi)d\xi \tag{1.4}$$

From Equation (1.3) it is clear that $\sum_{n=1}^{\infty}\lambda_n$ diverges, while $\sum_{n=1}^{\infty}(\lambda_n - n^2 - c_0)$ converges, and is called the regular trace for the problem (1.1), (1.2).

The study of regular trace for differential operators plays an important role in several fields such as mathematical analysis, theoretical physics and quantum mechanics, where the regular traces give the asymptotic expansion for the eigenvalues of operators. We can also use the regular trace in the inverse spectral problems in functional analysis.

A good number of works has been devoted to the deduction of the formulae of regularized traces of differential operators Gelfand, Levitian [2], Charles, Halberg and Kramer [3], Lidsky, Sadovnichii [4, 5, 6], Sadovnichii, Lyubishkin and Belabbasy [7, 8], Saleh [9] and many other authors

The concept of the regularized trace with a weight for the differential operators was introduced by Sadovnichii [10].

The main goal now is to derive asymptotic formulae for the solutions of (0.1) when $|\lambda|\to\infty$ and then use them to obtain the asymptotic formulae for the eigenvalues of the problem (0.1), (0.2). The concluding part of this paper is devoted to the derivation of the regularized sums of eigenfunctions of the second order, and we shall give some examples to illustrate the mentioned concept of regularized sums of eigenfunctions.

2. ASYMPTOTIC FORMLAE FOR THE SOLUTION OF THE STURM-LIOUVILLE EQUATION

The solution of the differential equation (0.1) admits asymptotic expansions in powers of ρ^{-1} which become more precise as the number of derivatives that the function q(x) has increases " Marchenko [11], Naimark [12]. Let $y_1(x,\rho)$ and $y_2(x,\rho)$ be linearly independent solutions of (0.1), then

$$y_1(x,\rho)=e^{i\rho x}\left[1+\sum_{\upsilon=1}^{N}\frac{u_\upsilon(x)}{\rho^\upsilon}+O\left(\frac{1}{\rho^{N+1}}\right)\right],$$

$$y_2(x,\rho)=e^{-i\rho x}\left[1+\sum_{\upsilon=1}^{N}\frac{(-1)^\upsilon u_\upsilon(x)}{\rho^\upsilon}+O\left(\frac{1}{\rho^{N+1}}\right)\right], \tag{2.1}$$

where N positive integer depends on the smoothness of the function $q(x)$ and the functions $u_\upsilon(x)$, $\upsilon=1,2,\dots,N$ admit the representations:

$$u_1(x) = \frac{1}{2i}\int_0^x q(\xi)d\xi,$$

$$u_2(x) = \frac{1}{4}\left[q(x) - q(0) - \frac{1}{2}\left(\int_0^x q(\xi)d\xi\right)^2\right],$$

$$u_{\upsilon+2}(x) = \frac{1}{2i}\int_0^x \{-\frac{d^2}{d\xi^2} + q(\xi)\}u_{\upsilon+1}(\xi)d\xi; \qquad \upsilon = 1,2,3,\ldots,N-2 \qquad (2.2)$$

We note that $u_\upsilon(0) = 0$, for $\upsilon = 1,2,\ldots,N$. By means of the asymptotic formulae (2.1) and Equation (2.2) we can prove that

$$\delta(x,\rho) = \delta(0,\rho) = W[y_1, y_2]\,|_{x=0}$$

$$= -2i\rho + i\left[\frac{q(0)}{2\rho} - \frac{q'''(0) - 4q(0)q'(0)}{4\rho^3} + \ldots\right] \qquad (2.3)$$

3. ASYMPTOTIC FORMULAE FOR EIGENVALUES OF THE PROBLEM (0.1)-(0.2) IN THE COMMENSURABLE CASE

In Saleh [9] proved that the eigenvalues of the problem (0.1)-(0.2) ($\lambda=\rho^2$) are found from the condition :

$$f(\rho) = \frac{\Delta(\rho)}{\delta(\rho)} = 0, \qquad (3.1)$$

where
$$\Delta(\rho) = \det\left\|U_i(y_k)\right\|_{i,k=1}^{2} \qquad (3.2)$$

Upon using Equations (2.1), (2.2) it is easy to see that

$$\Delta(\rho) = \sum_{k=1}^{2n^2-6} A_k(\rho)\left[\gamma_k^{(0)} + \frac{\beta_k^{(1)}}{\rho} + \frac{\beta_k^{(2)}}{\rho^2} + \ldots + \frac{\beta_k^{(N)}}{\rho^N} + \ldots\right], \qquad (3.3)$$

where
$$A_k(\rho) = e^{i\xi_k \pi\rho}, \quad \xi_1 = -1, \quad \xi_2 = -(1 - x_3), \; \xi_{2n^2-6} = -\xi_1 = 1,$$

and $\gamma_k^{(0)}$, $\beta_k^{(j)}$, ($j=1,2,3,\ldots$) are calculated in terms of the constants α_v, β_v ($\upsilon=1,2,\ldots,n$) and the function $q(x)$, for example,

$$\gamma_1^{(0)} = (\alpha_1\beta_2 - \beta_1\alpha_2) = -\gamma_{2n^2-6}^{(0)},$$

$$\beta_1^{(1)} = -\beta_{2n^2-6}^{(1)} = \frac{1}{2i}(\beta_1\alpha_2 - \alpha_1\beta_2)\int_0^\pi q(\xi)d\xi \qquad (3.4)$$

Using the results of Saleh [9], we deduce that in commensurable case the problem (0.1)-(0.2) has 2m series of eigenvalues which have the following asymptotic formula:

$$\rho_{k,s} \sim -2mk - \frac{m}{i\pi}\ln a_0^{(s)} + \frac{a_1^{(s)}}{2a_0^{(s)}k} - \frac{a_1^{(s)}\ln a_0^{(s)}}{4ia_0^{(s)}\pi k^2} + \ldots, \qquad (3.5)$$

where
$$m = \frac{1}{d}; \quad d = \min\{\xi_1, \xi_2, \xi_3, \ldots, \xi_{2n^2-6}\}$$

4. THE ASYMPTOTIC FORMUAE FOR THE GREEN'S FUNCTION OF THE PROBLEM (0.1)-(0.2) IN COMMENSURABLE CASE

It is well-known that the Green's function of differential equation of the second order is given by the formula:

$$G(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\begin{vmatrix} y_1(x,\rho) & y_2(x,\rho) & g(x,\xi,\rho) \\ U_1(y_1) & U_1(y_2) & U_1(g) \\ U_2(y_1) & U_2(y_2) & U_2(g) \end{vmatrix}, \tag{4.1}$$

where
$$g(x,\xi,\rho)=\pm\frac{1}{2\delta(0,\rho)}\left[y_1(x,\rho)y_2(\xi,\rho)-y_2(x,\rho)y_1(\xi,\rho)\right] \tag{4.2}$$

(The positive sign being taken if $x>\xi$, and the negative sign if $x<\xi$).

If we divide the ρ-plane into four regions $\tilde{S}_0, \tilde{S}_1, \tilde{S}_2, \tilde{S}_3$ such that:

$$\tilde{S}_0=\left\{\rho \;:\; |\rho|>R,\; 0<\arg\rho \le \frac{\pi}{2}-\theta\right\},$$
$$\tilde{S}_1=\left\{\rho \;:\; |\rho|>R,\; \frac{\pi}{2}-\theta<\arg\rho\le\pi\right\},$$
$$\tilde{S}_2=\left\{\rho \;:\; |\rho|>R,\; \pi<\arg\rho\le\frac{3\pi}{2}-\theta\right\},$$
$$\tilde{S}_3=\left\{\rho \;:\; |\rho|>R,\; \frac{3\pi}{2}-\theta<\arg\rho\le 2\pi\right\}, \tag{4.3}$$

we see that the Green's function of the problem (0.1)-(0.2) in the commensurable case has the following asymptotic formula:

$$G(x,\xi,\rho)\sim e^{i(x-\xi)\rho}\sum_{\upsilon=0}^{\infty}\frac{\phi^{(1)}_{\upsilon,j}}{\rho^{\upsilon}};\quad \rho\in\tilde{S}^{\delta}_0,\; \xi<x<x_j\pi,$$
$$G(x,\xi,\rho)\sim e^{-i(x,\xi)\rho}\sum_{\upsilon=0}^{\infty}\frac{\phi^{(2)}_{\upsilon,j}}{\rho^{\upsilon}};\quad \rho\in\tilde{S}^{\delta}_0;\; \xi<x_j\pi<x, \tag{4.4}$$

where
$$\tilde{S}^{\delta}_0=\tilde{S}_0\setminus Q^{\delta}_{k,s}$$

and,
$$Q^{\delta}_{k,s}=\left\{\rho \;:\; |\rho-\rho_{k,s}|\le\delta,\; \Delta(\rho_{k,s})=0\right\}$$

Since $H_{k,s}(x,\xi)$ the residue of Green's function $G(x,\xi,\rho)$ for problem (0.1)-(0.2) in the points $\rho_{k,s}$ and from the assumption that the eigenvalues $\rho_{k,s}$ are simple, then

$$H_{k,s}(x,\xi,\rho_{k,s})=\lim_{\rho\to\rho_{k,s}}(\rho-\rho_{k,s})G(x,\xi,\rho) \tag{4.5}$$

Upon using the asymptotic formulae(4.4) for $G(x,\xi,\rho)$ in $\bar{S}^{\delta}_0$ and Equation (4.5) we have for $H_{k,s}(x,\xi)$ the following asymptotic formulae

$$H_{k,s}(x,\xi,\rho_{k,s})\sim e^{i(x-\xi)\rho_{k,s}}\sum_{\upsilon=0}^{\infty}\frac{\phi^{(3)}_{\upsilon,j}}{\rho^{\upsilon}_{k,s}},\quad \rho_{k,s}\in\tilde{S}^{\delta}_0,\; \xi<x<x_j\pi$$
$$H_{k,s}(x,\xi,\rho_{k,s})\sim e^{-i(x-\xi)\rho_{k,s}}\sum_{\upsilon=0}^{\infty}\frac{\phi^{(4)}_{\upsilon,j}}{\rho^{\upsilon}_{k,s}},\quad \rho\in\tilde{S}^{\delta}_0,\; \xi<x_j\pi<x, \tag{4.6}$$

where the functions $\phi^{(1)}_{\upsilon,j},\phi^{(2)}_{\upsilon,j},\phi^{(3)}_{\upsilon,j}$ and $\phi^{(4)}_{\upsilon,j}$ are defined in terms of the constants

α_k, β_k, $(k = 1,2,\ldots,n)$ and the function $q(x)$, $j = 3,4,5,\ldots,n$. We note that for the functions $\Delta(\rho)$, $\Delta'(\rho)$ have asymptotic formulae:

$$\Delta(\rho) \sim e^{-i\rho\pi} \sum_{\upsilon=0}^{\infty} \frac{\psi_{\upsilon}^{(1)}}{\rho^{\upsilon}}, \quad \rho \in \tilde{S}_0^{\delta} \tag{4.7}$$

$$\Delta'(\rho) \sim e^{-i\rho\pi} \sum_{\upsilon=0}^{\infty} \frac{\psi_{\upsilon}^{(2)}}{\rho^{\upsilon}}, \quad \rho \in \tilde{S}_0^{\delta} \tag{4.8}$$

5. REGLARIZED SUM FOR EIGENFUNCTIONS OF THE PROBLEM (0.1)-(0.2) IN THE COMMENSURABLE CASE

Now we wish to evaluate the functions $R_{k,s}(\sigma,x,\xi)$ which ensure the convergence of the series(0.3). We must first estimate the functions $\rho_{k,s}^{-\sigma}H_{k,s}(x,\xi)$ in $\tilde{S}_o^{\sigma}$. From Equation (3.5), we have

$$\rho_{k,s}^{-\sigma} \sim \sum_{\upsilon=0}^{\infty} \frac{Q_{\upsilon}^{(s)}(\sigma)}{k^{\upsilon+\sigma}} \tag{5.1}$$

In the asymptotic formula (5.1)

$$Q_0^{(s)}(\sigma) = (-2m)^{-\sigma}, \quad Q_1^{(s)}(\sigma) = (-1)^{-\sigma+1}(2m)^{-\sigma-1}\frac{m}{i\pi}\ln a_0^{(s)},$$

$$Q_2^{(s)}(\sigma) = (-2m)^{-\sigma}\left[\binom{-\sigma}{1}\frac{a_1^{(s)}}{2a_0^{(s)}(-2m)} + \binom{-\sigma}{2}\frac{(\ln a_0^{(s)})^2}{4\pi^2}\right], \ldots \tag{5.2}$$

From Equation (3.5) we get

$$e^{i\rho_{k,s}(x-\xi)} \sim e^{i(x-\xi)(-2mk-\frac{m}{i\pi}\ln a_0^{(s)})} \sum_{n=0}^{\infty} \frac{\psi_n^{(s)}(x,\xi)}{k^n}, \tag{5.3}$$

where the functions $\psi_n^{(s)}(x,\xi)$ are polynomials of $(x-\xi)$.

Upon using Equations(4.5), (5.1) and (5.2), we have

$$\rho_{k,s}^{-\sigma}H_{k,s}(x,\xi) \sim e^{i(x-\xi)(-2mk-\frac{m}{i\pi}\ln a_0^{(s)})} \sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{\ell=0}^{p} \frac{\psi_{n-p}^{(s)}(x,\xi)Q_l^{(s)}(\sigma+p-l)\phi_{p-l}^{(s)}}{k^{n+\sigma}} \tag{5.4}$$

For large number τ, we consider the function

$$\Phi_{\tau}(\sigma) = \sum_{(k)}\sum_{(s)}\left[\rho_{k,s}^{-\sigma}H_{k,s}(x,\xi) - e^{i(x-\xi)(-2mk-\frac{m}{i\pi}\ln a_0^{(s)})}\sum_{n=0}^{\tau}\sum_{p=0}^{n}\sum_{\ell=0}^{p}\frac{\psi_{n-p}^{(s)}(x,\xi)Q_l^{(s)}(\sigma+p-l)\phi_{p-l}^{(s)}}{k^{n+\sigma}}\right]. \tag{5.5}$$

It is clear that the function $\Phi\tau(\sigma)$ may be extended to analytic function in the half plane $\mathrm{Re}\,\sigma > -\tau$

THEOREM 5.1 If $\mathrm{Re}\,\sigma > -\tau$ then

$$\sum_{(k)}\sum_{(s)}\left[\rho_{k,s}^{-\sigma}H_{K,s}(x,\xi) - e^{i(x-\xi)(-2mk-\frac{m}{i\pi}\ln a_0^{(s)})}\sum_{n=0}^{\tau}\sum_{p=0}^{n}\sum_{l=0}^{p}\frac{\psi_{n-p}^{(s)}(x,\xi)Q_l^{(s)}(\sigma+p-l)\phi_{p-l}^{(s)}}{k^{n+\sigma}}\right].$$

$$= \phi_{-\sigma+1,j}^{(1)} - \sum_{(s)}\sum_{n=0}^{\tau}\sum_{p=0}^{n}\sum_{\ell=0}^{p} e^{i(x-\xi)(\frac{-m}{i\pi}\ln a_0^{(s)})} F(e^{i(x-\xi)(-2m)}, n+\sigma)\, \psi_{n-p}^{(s)}(x,\xi)Q_l^{(s)}(\sigma+p-l)\phi_{p-l}^{(s)}, \tag{5.6}$$

where

$$F(z,\sigma) = \sum_{k=1}^{\infty}\frac{z^k}{k^{\sigma}}.$$

THEOREM 5.2 If $\sigma = -2$, we have

$$\sum_{(k)}\sum_{(s)} \lambda_{k,s} H_{k,s}(x,\xi) - e^{i(x-\xi)(-2mk-\frac{m}{i\pi}\ln a_0^{(s)})} \left[\sum_{n=0}^{3}\sum_{p=0}^{n}\sum_{\ell=0}^{p} \frac{\psi_{n-p}^{(s)}(x,\xi)Q_l^{(S)}(-2+p-l)\varphi_{p-l}^{(s)}}{k^{n-2}}\right]$$

$$= \phi_{3,1}^{(1)} - \sum_{(s)}\sum_{n=0}^{3}\sum_{p=0}^{n} e^{i(x-\xi)(-\frac{m}{i\pi}\ln a_0^{(s)})} F(e^{i(x-\xi)(-2m)}, n-2)\psi_{n-p}^{(s)} Q_\ell^{(s)}\phi_{p-\ell}^{(s)}. \tag{5.7}$$

REMARK. From the definition of $F(z,\sigma)$ it is clear that this function satisfies the following properties

1. $F(z,\sigma) = z\Phi(z,\sigma,1)$, where

$$\Phi(z,\sigma,\upsilon) = \sum_{n=0}^{\infty}(\upsilon+n)^{-\sigma}z^n = \frac{1}{\Gamma(\sigma)}\int_0^{\infty}\frac{t^{\sigma-1}e^{-(\upsilon-1)t}}{e^t - z}dt,$$

Re $\upsilon>0$ and either $|z| \le 1$, $z \ne 1$, Re $\sigma > 0$ or $z = 1$, Re $\sigma>1$.

2. $F(z,-m) = (-1)^{m+1}F(\frac{1}{z},-m)$, $m = 1,2,3,\ldots$

3. $F(z,s) + e^{is\pi}F(\frac{1}{z},s) = \frac{(2\pi)^s}{\Gamma(s)}e^{i\pi s/2}\zeta(1-s,\frac{\log z}{2\pi i})$

4. Equations (0.2), (0.3) furnish the analytical continuation of the series $\sum_{n=1}^{\infty}\frac{z^n}{n^\sigma}$ beyond the circle of convergence $|z| = 1$

If $F_0(z)$ denotes the principal branch of $F(z)$ in the cut z-plane $[0 < \arg(z-1) < 2\pi]$, the cut being imposed from 1 to ∞ along the real axis, the difference of the values of $F_0(z)$. between a point on the upper edge of the cut and a point on the lower edge, according to (0.3),

$$F_0(x,s) - F_0(xe^{2i\pi},s) = 2\pi i(\log x)^{s-1}/\Gamma(s)$$

Hence, if we cross the cut, from the upper half-plane to the lower half-plane, we obtain for the continuation $F_1(z)$ of $F_0(z)$

$$F_1(z) = F_0(z) + 2\pi i(\log z)^{s-1}/\Gamma(s)$$

The analogous formula for the inverse process of continuation is

$$F_2(z) = F_0(z) - 2\pi i(\log z)^{s-1}/\Gamma(s)$$

5. $F(e^{it},-m) = (i\frac{d}{dt})^m \frac{1}{e^{-it}-1}$; $m = 1,2,3\ldots$

The previous properties of $F(z,\sigma)$ are proved in A. Eredelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi [13]

6. D. Klusch [14] considered the generalized zeta function in the from

$$L(x,a,s) = \sum_{n\ge o} e^{(2\pi inx)}(n+a)^{-s}$$

($a \in R^+$; x is not integer, Re $s > 0$; and if x is an integer, Re $s >1$)
and studied some further properties of the function $L(x,a,s)$ resulting from the taylor expansion of the function $W(\zeta) = L(x,a+\zeta,s)$ in the neighbourhood $\xi = 0$
Now, we consider the following examples:

6. EXAMPLES

1. Consider the problem

$$-y''(x) = \lambda y(x) \qquad 0 \le x \le \pi, \tag{6.1}$$

$$y(0) = y(\pi) = 0, \tag{6.2}$$

Its clear that the eigenvalues of problem (6.1), (6.2) are $\lambda_n = n^2$ and the corresponding eigenfunctions are $y_n(x) = \sin nx$, so the regularized trace of problem (6.1), (6.2) is $\sum_{n=1}^{\infty} (\lambda_n - n^2) = 0$, and the regularized sum of eigenfunctions of problem (6.1)-(6.2) is given by the following formula

$$\sum_{(n)} \left[\lambda_n H_n(x,\xi) + \frac{n}{\pi} \sin nx \sin n\xi \right] = 0. \tag{6.3}$$

2. Consider the Sturm Liouville problem

$$-y''(x) + q(x)y(x) = \lambda y, \quad \lambda = p^2, \quad 0 \le x \le \pi \tag{6.4}$$

$$y(0) = y(\pi) = 0, \tag{6.5}$$

where $q(x)$ is a sufficiently smooth function defined in the interval $[0, \pi]$.

Let $y_1(x,\rho)$, $y_2(x,\rho)$ are two independant solutions of Equation (6.4) such that

$$y_j^{(k-1)}(0,\rho) = \begin{cases} 1 & k = j \\ 0 & k \ne j \end{cases}$$

Then from Equations (2.1), (2.2) and (2.3), we have

$$y_1(x,\rho) = \sum_{\upsilon=0}^{N} \frac{A_\nu(x,\rho)}{\rho^\upsilon} + O(\frac{1}{\rho^{N+1}}) \tag{6.6}$$

$$y_2(x,\rho) = \sum_{\upsilon=1}^{N} \frac{B_\upsilon(x,\rho)}{\rho^\upsilon} + O(\frac{1}{\rho^{N+1}}) \tag{6.7}$$

where

$$\begin{aligned}
&A_0(x,\rho) = \cos\rho x \\
&A_1(x,\rho) = iu_1(x)\sin\rho x, \\
&A_2(x,\rho) = u_2(x)\cos\rho x, \\
&A_3(x,\rho) = i\left[u_3(x) + iu_2^1(0)\right]\sin\rho x, \ldots \\
&B_1(x,\rho) = \sin\rho x \\
&B_2(x,\rho) = -iu_1(x)\cos\rho x, \\
&B_3(x,\rho) = -i(iu_2(x) - u_1^1(0))\sin\rho x, \ldots \\
&B_4(x,\rho) = \left[2u_1(x)u_1'(0) - iu_3(x)\right]\cos\rho x, \ldots
\end{aligned} \tag{6.8}$$

and N is a positive integer depending on the smoothness of the function $q(x)$.

Since $\Delta(\rho) = \det\left\| U_j(y_k) \right\|_{j,k}^2$, then

$$\Delta(\rho) = y_2(\pi,\rho) = \sum_{\upsilon=1}^{N} \frac{B_\upsilon(\pi,\rho)}{\rho^\upsilon} + O(\frac{1}{\rho^{N+1}})$$

From the last formula, we can obtain the roots of the function $\Delta(\rho)$ which are eigenvalues of the problem (6.4)-(6.5). Upon using the successive approximation, we get the following asymptotic formula for the zeros of the function $\Delta(\rho)$:

$$\rho_n \sim n + \frac{c_{-1}}{n} + \frac{c_{-3}}{n^3} + \ldots, \tag{6.9}$$

where

$$c_{-1} = \frac{i}{\pi} u_1(\pi), \; c_{-3} = \frac{1}{2\pi i}\left[3\left\{u_1(\pi)u_2(\pi) - iu_1(\pi)u_1'(0) - u_3(\pi)\right\} - u_1^3(\pi)\right], \ldots \tag{6.10}$$

Then

$$\lambda_n \sim n^2 + c_0 + \frac{c_2}{n^2} + \frac{c_4}{n^4} + \ldots, \tag{6.11}$$

where
$$c_0 = \frac{1}{\pi}\int_0^\pi q(t)dt; \quad c_2 = (c_{-1})^2 + 2c_{-3}, \ldots$$

In the paper [2] I. M. Gelfand , B. M. Levitan have proved that

$$\sum_{n=1}^{\infty}(\lambda_n - n^2 - c_0) = \frac{1}{2}c_0 - \frac{1}{4}[q(0) + q(\pi)] \tag{6.12}$$

Upon using the results in H.F. Weinberger[14], we deduce that the Green's function of the problem (6.4)-(6.5) is given by the following formula:

$$G(x,\xi,\rho) = \begin{cases} \dfrac{y_2(\xi,\rho)}{y_2(\pi,\rho)}\left[y_1(x,\rho)y_2(\pi,\rho) - y_2(x,\rho)y_1(\pi,\rho)\right] & x \geq \xi \\ \dfrac{y_2(x,\rho)}{y_2(\pi,\rho)}\left[y_1(\xi,\rho)y_2(\pi,\rho) - y_2(\xi,\rho)y_1(\pi,\rho)\right] & x \leq \xi \end{cases} \tag{6.13}$$

From the definitions of $H_k(x,\xi,\rho_k)$, we deduce that

$$H_k(x,\xi,\rho_k) = \begin{cases} \dfrac{y_2(\xi,\rho_k)\left[y_1(x,\rho_k)y_2(\pi,\rho_k) - y_2(x,\rho_k)y_1(\pi,\rho_k)\right]}{y'_2(\pi,\rho_k)}, & x \geq \xi \\ \dfrac{y_2(x,\rho_k)\left[y_1(\xi,\rho_k)y_2(\pi,\rho_k) - y_2(\xi,\rho_k)y_1(\pi,\rho_k)\right]}{y'_2(\pi,\rho_k)}, & x \leq \xi \end{cases} \tag{6.14}$$

where $y'_2(\pi,\rho_k) = \frac{d}{d\rho}\left[y_2(\pi,\rho)\right] \Big|_{\rho=\rho_k}$

Substituting Equations (6.6), (6.7) and (6.8) in (6.13), (6.14), we get the following asymptotic formulae:

$$G(x,\xi,\rho) \sim \begin{cases} e^{i\rho(x-\xi)} \sum\limits_{\nu=0}^{\infty} \dfrac{\varphi_\nu^{(1)}(x,\xi)}{\rho^\nu}; & x > \xi \\ e^{-i\rho(x-\xi)} \sum\limits_{\nu=0}^{\infty} \dfrac{\varphi_\nu^{(2)}(x,\xi)}{\rho^\nu}; & x \leq \xi \end{cases} \tag{6.15}$$

and

$$H_k(x,\xi,\rho_k) \sim \begin{cases} e^{i\rho(x-\xi)} \sum\limits_{\nu=0}^{\infty} \dfrac{\varphi_\nu^{(3)}(x,\xi)}{\rho_k^\nu}, & x > \xi \\ e^{-i\rho(x-\xi)} \sum\limits_{\nu=0}^{\infty} \dfrac{\varphi_\nu^{(4)}(x,\xi)}{\rho_k^\nu}; & x \leq \xi \end{cases} \tag{6.16}$$

In formulae (6.15) and (6.16), the functions $\varphi_j^{(k)}(x,\xi)$ $(k=1,2,3,4.\ j=0,1,2,3,\ldots)$ can be expressed in terms of the potential $q(x)$ and its derivatives. For example:

$$\varphi_0^{(1)} = \varphi_0^{(2)} = 0, \quad \varphi_1^{(1)} = \varphi_1^{(2)} = \frac{1}{2i}, \quad \varphi_2^{(1)} = \frac{i}{2}(u_1(x) - u_1(\xi)), \quad \varphi_2^{(2)} = \frac{i}{2}(u_1(\xi) - u_1(x)),$$

$$\varphi_3^{(2)} = \varphi_3^{(1)} = \frac{i}{2}\left[u_2(x) + u_1(\xi)u_1(x) + u_2(\xi) + iu'_1(0)\right], \ldots$$

$$\varphi_0^{(3)}=\varphi_0^{(4)}=0,\quad \varphi_1^{(3)}=\varphi_1^{(4)}=\frac{-1}{2\pi},\ \varphi_2^{(3)}=\frac{1}{2\pi}\left[u_1(x)-u_1(\xi)\right],\varphi_2^{(4)}=\frac{1}{2\pi}\left[u_1(\xi)-u_1(x)\right],\ldots$$

$$\phi_3^{(3)}=\frac{1}{2\pi}\left[u_2(x)+u_2(\xi)-u_1(x)u_1(\xi)+iu_1'(0)+\frac{i}{\pi}(u_1(x)+u_1(\pi)-u_1(\xi)-2u_1'(0))-\frac{1}{\pi^2}\right] \tag{6.17}$$

Using the asymptotic formula (6.9) we have

$$\rho_k^2=\sum_{v=0}^{\infty}\frac{Q_v}{k^{v-2}}, \tag{6.18}$$

where $\quad Q_0=1 \quad Q_1=0 \quad Q_2=2C_{-1} \quad Q_3=0, \quad Q_4=2C_{-3}$

When we deal with the problem (6.1)-(6.2), the formula (5.3) takes the form :

$$e^{i\rho_k(x-\xi)}\sim e^{ik(x-\xi)}\sum_{n=0}^{\infty}\frac{\psi_n(x,\xi)}{k^n}., \tag{6.19}$$

where $\quad \psi_0(x,\xi)=1 \qquad \psi_1(x,\xi)=iC_{-1}(x-\xi)$

$$\psi_2(x,\xi)=-\frac{1}{2}C_{-1}^2(x-\xi)^2 \qquad \psi_3=iC_{-3}(x-\xi)+\frac{(iC_{-1}(x-\xi))^3}{6},\ldots \tag{6.20}$$

From the formulae (6.18), (6.19), we have

$$\lambda_k H_k(x,\xi)\sim\begin{cases} e^{ik(x-\xi)}\displaystyle\sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{\ell=0}^{p}\frac{\psi_{n-p}Q_\ell\varphi^{(3)}_{p-\ell}}{k^{n-2}} & x>\xi\\ e^{-ik(x-\xi)}\displaystyle\sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{\ell=0}^{p}\frac{\psi_{n-p}Q_\ell\varphi^{(4)}_{p-\ell}}{k^{n-2}}, & x\le\xi\end{cases} \tag{6.21}$$

Using formulae (6.17), (6.18), (6.20) and (6.21), we have the following theorem.

THEOREM 6.1 For the problem (6.1)-(6.2), the regularized sum for the eigenfunctions is given by the following formulae :

(1) If $x>\xi$, then :

$$\sum_{k=1}^{\infty}\Big[\lambda_k H_k(x,\xi)+\frac{1}{2\pi}e^{ik(x-\xi)}\Big\{k+u_1(\xi)-u_1(x)-2\pi(x-\xi)u_1(\pi)+\frac{1}{k}\Big[u_1(x)u_1(\xi)-u_2(x)-$$
$$-u_2(\xi)-iu_1'(0)+\frac{i}{\pi}(u_1(\xi)-u_1(x)-u_1(\pi)++2u_1'(0))+\frac{1}{\pi^2}+2\pi(x-\xi)u_1(x)-$$
$$-u_1(\xi)u_1(\pi)+2\pi^2(x-\xi)^2u_1^2(\pi)\Big]\Big\}=\frac{i}{2}\left[u_2(x)+u_1(\xi)u_1(x)+u_2(\xi)+u_1'(0)\right]+$$
$$+\frac{1}{2\pi}\Big\{F(e^{i(x-\xi)},-1)+(u_1(\xi)-u_1(x)-2\pi(x-\xi)u_1(\pi))F(e^{i(x-\xi)},0)+$$
$$+\Big[u_1(x)u_1(\xi)-u_2(x)-u_2(\xi)-iu_1'(0)++\frac{i}{\pi}(u_1(\xi)-u_1(x)-u_1(\pi)+2u_1'(0))+$$
$$+\frac{1}{\pi^2}+2\pi(x-\xi)(u_1(x)-u_1(\xi))u_1'(\pi)+2\pi^2(x-\xi)^2u_1^2(\pi)\Big]F(e^{i(x-\xi)}\Big]\Big\} \tag{6.22}$$

(2) If $x<\xi$, then :

$$\sum_{k=1}^{\infty}\Big[\lambda_k H_k(x,\xi)+\frac{1}{2\pi}e^{-ik(x-\xi)}\Big\{k+u_1(x)-u_1(\xi)-2\pi(x-\xi)u_1(\pi)+$$
$$+\frac{1}{k}\Big[u_1(x)u_1(\xi)-u_2(x)-u_2(\xi)-iu_1'(0)+\frac{i}{\pi}(u_1(x)-u_1(\xi)-u_1(\pi)+$$
$$+2u_1'(0))+\frac{1}{\pi^2}+2\pi(x-\xi)(u_1(\xi)-u_1(x))u_1(\pi)+2\pi^2(x-\xi)^2u_1^2(\pi)\Big]\Big\}$$
$$=\frac{i}{2}\left[u_2(x)+u_1(\xi)u_1(x)+u_2(\xi)+u_1'(0)\right]+\frac{1}{2\pi}\Big\{F(e^{-i(x-\xi)},-1)+(u_1(x)-$$
$$-u_1(\xi)-2\pi(x-\xi)u_1(\pi)).F(e^{i(x-\xi)},0)+\Big[u_1(x)u_1(\xi)-u_2(x)-u_2(\xi)-iu_1'(0)+$$
$$+\frac{i}{\pi}(u_1(x)-u_1(\xi)-u_1(\pi)+2u_1'(0))+\frac{1}{\pi^2}+2\pi(x-\xi)(u_1(\xi)-u_1(x))u_1(\pi)+2\pi^2(x-\xi)^2u_1^2(\pi)\Big]\Big\} \tag{6.23}$$

3. Consider the Sturm-Louville problem

$$-y''(x)+q(x)y(x)=\lambda y, \qquad \lambda=\rho^2, \qquad 0\le x\le \pi \tag{6.24}$$

$$y(0)=y(\frac{\pi}{2})+y(\pi)=0, \tag{6.25}$$

where $q(x)$ is a sufficiently smooth function defined on the interval $[0,\pi]$.

Upon using the definition of $\Delta(\rho)$ and the formulae (6.6), (6.7), (6.8), we have:

$$\Delta(\rho)=\frac{\sin\rho\pi+\sin\rho\frac{\pi}{2}}{\rho}+\frac{-i\left[u_1(\pi)\cos\rho\pi+u_1(\frac{\pi}{2})\cos\rho\frac{\pi}{2}\right]}{\rho^2}+$$
$$+\frac{-i\left[(iu_2(\pi)-u_1'(0))\sin\rho\pi+(iu_2(\frac{\pi}{2})-u_1'(0))\sin\rho\frac{\pi}{2}\right]}{\rho^3}+$$
$$+\frac{\left[2u_1(\pi)u_1'(0)-iu_3(\pi)\right]\cos\rho\pi+\left[2u_1(\frac{\pi}{2})u_1'(0)-iu_3(\frac{\pi}{2})\right]\cos\rho\frac{\pi}{2}}{\rho^4}+\ldots \tag{6.26}$$

To find the eigenvalues of problem (6.24)-(6.25), we put

$$V=e^{-i\frac{\pi\rho}{2}}=\sum_{\nu=0}^{\infty}\frac{a_\nu}{\rho^\nu}; \qquad \Delta(\rho)=0 \quad \text{then}$$

$$V^4\left[-\frac{1}{2i\rho}-\frac{iu_1(\pi)}{2\rho^2}+\frac{i[iu_2(\pi)-u_1'(0)]}{3i\rho^3}+\ldots\right]+V^3\left[-\frac{1}{2i\rho}-\frac{iu_1(\frac{\pi}{2})}{2\rho^2}+\frac{i\left[iu_2(\frac{\pi}{2})-u_1'(0)\right]}{3i\rho^3}+\ldots\right]+$$
$$+V\left[\frac{1}{2i\rho}-\frac{iu_1(\frac{\pi}{2})}{2\rho^2}+\frac{i\left[iu_2(\frac{\pi}{2})-u_1'(0)\right]}{3i\rho^3}+\ldots\right]+\left[\frac{1}{2i\rho}-\frac{iu_1(\pi)}{2\rho^2}+\frac{i[iu_2(\pi)-u_1'(0)]}{3i\rho^3}+\ldots\right]=0, \tag{6.27}$$

and

$$\left\{\sum_{\nu=0}^{\infty}\frac{a_\nu}{\rho^\nu}\right\}^4\left[-\frac{1}{2i\rho}-\frac{iu_1(\pi)}{2\rho^2}+\frac{i[iu_2(\pi)-u_1'(0)]}{3i\rho^3}+\ldots\right]+\left\{\sum_{\nu=0}^{\infty}\frac{a_\nu}{\rho^\nu}\right\}^3\left[-\frac{1}{2i\rho}-\frac{iu_1(\frac{\pi}{2})}{2\rho^2}+\frac{i\left[iu_2(\frac{\pi}{2})-u_1'(0)\right]}{3i\rho^3}+\ldots\right]+$$
$$+\left\{\sum_{\nu=0}^{\infty}\frac{a_\nu}{\rho^\nu}\right\}\left[\frac{1}{2i\rho}-\frac{iu_1(\frac{\pi}{2})}{2\rho^2}+\frac{i\left[iu_2(\frac{\pi}{2})-u_1'(0)\right]}{3i\rho^3}+\ldots\right]+\left[\frac{1}{2i\rho}-\frac{iu_1(\pi)}{2\rho^2}-\frac{i[iu_2(\pi)-u_1'(0)]}{3i\rho^3}+\ldots\right]=0 \tag{6.28}$$

Equating the coefficients of ρ^{-k} ($k=1,2,3,\ldots$) to zero, we have

$$-a_0^4-a_0^3+a_0+1=0 \tag{6.29}$$

Solving the Equation (6.29), we have

$$a_0^{(1)}=1, \quad a_0^{(2)}=-1, \quad a_0^{(3)}=\frac{-1+i\sqrt{3}}{2}, \qquad a_0^{(4)}=\frac{-1-i\sqrt{3}}{2} \tag{6.30}$$

Upon using the result in [9], we have for the eigenvalues of the problem (6.24)-(6.25), the following asymptotic formula

$$\rho_{k,s}-4k-\frac{2}{i\pi}\ln a_0^{(s)}+\frac{a_1^{(s)}}{2a_0^{(s)}k\pi}-\frac{a_1^{(s)}\ln a_0^{(s)}}{4ia_0^{(s)}\pi k^2}+\ldots \tag{6.31}$$

where $s=1,2,3,4$, and

$$a_1^{(1)}=\frac{1}{6}u_1(\pi),\ a_1^{(2)}=-\frac{1}{2}u_1(\pi),\ a_1^{(3)}=\frac{u_1(\pi)}{4(a_0^{(3)})^3+3(a_0^{(3)})^2-1},\ a_1^{(4)}=\frac{u_1(\pi)}{4(a_0^{(3)})^3+3(a_0^{(3)})^2-1}, \tag{6.32}$$

Using the formula (6.31), we have

$$\rho_{k,s}^2=\sum_{v=0}^{\infty}\frac{Q_v^{(s)}}{k^{v-2}},\tag{6.33}$$

where

$$Q_0^{(s)}=16,\ Q_1^{(s)}=-16i\ln a_0^{(s)},\ Q_2^{(s)}=-\frac{4}{\pi^2}(\ln a_0^{(s)})^2-\frac{4a_1^{(s)}}{a_0^{(s)}\pi},\ Q_3^{(s)}=\frac{-2i}{\pi^2}(\pi-1)a_1^{(s)}\ln a_0^{(s)},\ldots\tag{6.34}$$

According to (4.1), we see that the Green's function $G(x,\xi,\rho)$ of the problem (6.24)-(6.25) has the following asymptotic formulae:

$$G(x,\xi,\rho)\sim\begin{cases} e^{i\rho(x-\xi)}\sum\limits_{v=0}^{\infty}\dfrac{\varphi_{v,1}^{(1)}}{\rho^v}; & \xi<x<\frac{\pi}{2};\ \rho\in\tilde{S}_0^\sigma \\ e^{-i\rho(x-\xi)}\sum\limits_{v=0}^{\infty}\dfrac{\varphi_{v,1}^{(2)}}{\rho^v}; & x<\xi<\frac{\pi}{2};\ \rho\in\tilde{S}_0^\sigma \\ e^{i\rho(x-\xi)}\sum\limits_{v=0}^{\infty}\dfrac{\varphi_{v,2}^{(1)}}{\rho^v}; & \frac{\pi}{2}<\xi<x;\ \rho\in\tilde{S}_0^\sigma \\ e^{-i\rho(x-\xi)}\sum\limits_{v=0}^{\infty}\dfrac{\varphi_{v,2}^{(2)}}{\rho^v}; & \frac{\pi}{2}<x<\xi;\ \rho\in\tilde{S}_0^\sigma, \end{cases}\tag{6.35}$$

where

$$\varphi_{0,1}^{(1)}=\varphi_{0,1}^{(2)}=\varphi_{0,2}^{(1)}=\varphi_{0,2}^{(2)}=0\qquad \varphi_{1,1}^{(1)}=\varphi_{1,1}^{(2)}=\varphi_{1,2}^{(1)}=\varphi_{1,2}^{(2)}=\frac{1}{2i}$$

$$\varphi_{2,1}^{(1)}=\varphi_{2,2}^{(1)}=-\varphi_{2,1}^{(2)}=-\varphi_{2,2}^{(2)}=\frac{i}{2}(u_1(x)-u_1(\xi))$$

$$\varphi_{3,1}^{(1)}=\varphi_{3,1}^{(2)}=\varphi_{3,2}^{(1)}=\varphi_{3,2}^{(2)}=\frac{i}{2}\left[u_2(x)+u_1(\xi)u_1(x)+u_2(\xi)+iu_1'(0)\right]\tag{6.36}$$

Upon using the asymptotic formulae (6.35) for $G(x,\xi,\rho)$ in $\tilde{S}_0^\sigma$ and the definition of the function $H_{k,s}(x,\xi)$, we have

$$H_{k,s}(x,\xi)=\begin{cases} e^{i\rho_{k,s}(x-\xi)}\sum\limits_{v=0}^{\infty}\dfrac{\phi_{v,1}^{(3)}}{\rho_{k,s}^v}; & \xi<x<\frac{\pi}{2};\ \rho_{k,s}\in\tilde{S}_0^\sigma, \\ e^{-i\rho_{k,s}(x-\xi)}\sum\limits_{v=0}^{\infty}\dfrac{\phi_{v,1}^{(4)}}{\rho_{k,s}^v}; & \xi<x<\frac{\pi}{2};\ \rho_{k,s}\in\tilde{S}_0^\sigma, \\ e^{i\rho_{k,s}(x-\xi)}\sum\limits_{v=0}^{\infty}\dfrac{\phi_{v,2}^{(3)}}{\rho_{k,s}^v}; & \frac{\pi}{2}<\xi<x;\ \rho_{k,s}\in\tilde{S}_0^\sigma, \\ e^{-i\rho_{k,s}(x-\xi)}\sum\limits_{v=0}^{\infty}\dfrac{\phi_{v,2}^{(4)}}{\rho_{k,s}^v}; & \frac{\pi}{2}<x<\xi;\ \rho_{k,s}\in\tilde{S}_0^\sigma, \end{cases}\tag{6.37}$$

where

$$\phi_{0,1}^{(3)}=\phi_{0,1}^{(4)}=\phi_{0,2}^{(3)}=\phi_{0,2}^{(4)}=0;\quad \phi_{1,1}^{(3)}=\phi_{1,1}^{(4)}=\phi_{1,2}^{(3)}=\phi_{1,2}^{(4)}=-\frac{1}{2\pi};$$

$$\phi^{(3)}_{2,1}=\phi^{(3)}_{2,2}=-\phi^{(4)}_{2,1}=-\phi^{(4)}_{2,2}=\frac{1}{2\pi}(u_1(\xi)-u_1(x))$$

$$\phi^{(3)}_{3,1}=\phi^{(3)}_{3,2}=-\phi^{(4)}_{3,1}=-\phi^{(4)}_{3,2}=\frac{1}{2\pi}\Big[u_2(x)+u_2(\xi)-u_1(x)u_1(\xi)+$$

$$+iu_1'(0)+\frac{i}{\pi}(u_1(x)+u_1(\pi)-u_1(\xi)-2u'(0)-\frac{1}{\pi^2}\Big],\ldots \tag{6.38}$$

From formulae (6.33) and (6.37), we get

$$\rho^2_{k,s}H_{k,s}(x,\xi)\begin{cases} e^{-2i(x-\xi)(2k-\frac{1}{\pi}\ln a^{(s)}_0}\displaystyle\sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{\ell=0}^{p}\frac{\psi^{(s)}_{n-p}Q^{(s)}_{\ell}\phi^{(3)}_{p-\ell,1}}{k^{n-2}} & \xi<x<\frac{\pi}{2} \\ e^{2i(x-\xi)(2k-\frac{1}{\pi}\ln a^{(s)}_0}\displaystyle\sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{\ell=0}^{p}\frac{\psi^{(s)}_{n-p}Q^{(s)}_{\ell}\phi^{(4)}_{p-\ell,1}}{k^{n-2}} & x<\xi<\frac{\pi}{2} \\ e^{-2i(x-\xi)(2k-\frac{1}{\pi}\ln a^{(s)}_0}\displaystyle\sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{\ell=0}^{p}\frac{\psi^{(s)}_{n-p}Q^{(s)}_{\ell}\phi^{(3)}_{p-\ell,2}}{k^{n-2}} & \frac{\pi}{2}<\xi<x \\ e^{2i(x-\xi)(2k-\frac{1}{\pi}\ln a^{(s)}_0)}\displaystyle\sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{\ell=0}^{p}\frac{\psi^{(s)}_{n-p}Q^{(s)}_{\ell}\phi^{(4)}_{p-\ell,2}}{k^{n-2}}; & \frac{\pi}{2}<x<\xi \end{cases} \tag{6.39}$$

where,...

$$\psi^{(s)}_0(x,\xi)=1,\quad \psi^{(s)}_1(x,\xi)=\frac{ia^{(s)}_1(x-\xi)}{2a^{(s)}_0\pi};\quad \psi^{(s)}_2=-\frac{1}{8}(x-\xi)^2\Big(\frac{a^{(s)}_1}{a^{(s)}_0\pi}\Big)^2,$$

$$\psi^{(s)}_3(x,\xi)=\frac{-(a^{(s)}_1\ln a^{(s)}_0)(x-\xi)}{\pi a^{(s)}_0}+\frac{i\Big(\frac{a^{(s)}_1(x-\xi)}{2a^{(s)}_0\pi}\Big)^3}{6},\ldots \tag{6.40}$$

Upon using formulae (6.36), (6.38), (6.39) and (6.40), we have the following theorem :

THEOREM 6.2

1) If $\xi<x<\frac{\pi}{2}$, then the regularized sum of the first order for eigenfunctions of the problem (6.24)-(6.25) are given by the followig formulae

$$\sum_{k=1}^{\infty}\sum_{s=1}^{4}\left[\lambda_{k,s}H_{k,s}(x,\xi)-e^{i(x-\xi)(-4k-\frac{2}{i\pi}\ln a^{(s)}_0)}\sum_{n=0}^{3}\sum_{p=0}^{n}\sum_{\ell=0}^{p}\frac{\psi^{(s)}_{n-p}Q^{(s)}_{\ell}\phi^{(3)}_{p-\ell,1}}{k^{n-2}}\right].$$

$$=\frac{i}{2}\Big[u_2(x)+u_1(\xi)u_1(x)+u_2(\xi)+iu_1'(0)\Big]-\sum_{s=1}^{4}\sum_{n=0}^{3}\sum_{p=0}^{n}\sum_{\ell=0}^{p}e^{i(x-\xi)(-\frac{2}{i\pi}\ln a^{(s)}_0)}$$

$$.F(e^{-4i(x-\xi)},n-2)\psi^{(s)}_{n-p}Q^{(s)}_{\ell}\phi^{(3)}_{p-\ell,1} \tag{6.41}$$

(2) If $x<\xi<\frac{\pi}{2}$, then the regularized sum of the first order for eigenfunctions of the problem (6.24)-(6.25) are given by the followig formulae

$$\sum_{k=1}^{\infty}\sum_{s=1}^{4}\left[\lambda_{k,s}H_{k,s}(x,\xi)-e^{-i(x-\xi)(-4k-\frac{2}{i\pi}\ln a^{(s)}_0)}\sum_{n=0}^{3}\sum_{p=0}^{n}\sum_{\ell=0}^{p}\frac{\psi^{(s)}_{n-p}Q^{(s)}_{\ell}\phi^{(4)}_{p-\ell,1}}{k^{n-2}}\right]$$

$$=\frac{i}{2}\Big[u_2(x)+u_1(\xi)u_1(x)+u_2(\xi)+iu_1'(0)\Big]-\sum_{s=1}^{4}\sum_{n=0}^{3}\sum_{p=0}^{n}\sum_{\ell=0}^{p}e^{-i(x-\xi)(-\frac{2}{i\pi}\ln a^{(s)}_0)}$$

$$. F(e^{4i(x-\xi)}, n-2)\psi_{n-p}^{(s)} Q_\ell^{(s)} \phi_{p-\ell,1}^{(4)} \tag{6.42}$$

(3) If $\frac{\pi}{2} < \xi < x$, then the regularized sum of the first order for eigenfunctions of the problem (6.24)-(6.25) are given by the followig formulae.

$$\sum_{k=1}^{\infty}\sum_{s=1}^{4}\left[\lambda_{k,s}H_{k,s}(x,\xi) - e^{i(x-\xi)(-4k-\frac{2}{i\pi}\ln a_0^{(s)})}\sum_{n=0}^{3}\sum_{p=0}^{n}\sum_{\ell=0}^{p}\frac{\psi_{n-p}^{(s)} Q_\ell^{(s)} \phi_{p-\ell,2}^{(3)}}{k^{n-2}}\right]$$

$$= \frac{i}{2}\left[u_2(x) + u_2(\xi) + u_1(x)u_1(\xi) + iu_1'(0)\right] - \sum_{s=1}^{4}\sum_{n=0}^{3}\sum_{p=0}^{n}\sum_{\ell=0}^{p} e^{i(x-\xi)(-\frac{2}{i\pi}\ln a_0^{(s)})} .$$

$$. F(e^{4i(x-\xi)}, n-2)\psi_{n-p}^{(s)} Q_\ell^{(s)} \phi_{p-\ell,2}^{(3)} \tag{6.43}$$

(4) If $\frac{\pi}{2} < x < \xi$, then the regularized sum of the first order for eigenfunctions of the problem (6.24)-(6.25) are given by the followig formulae

$$\sum_{k=1}^{\infty}\sum_{s=1}^{4}\left[\lambda_{k,s}H_{k,s}(x,\xi) - e^{-i(x-\xi)(-4k-\frac{2}{i\pi}\ln a_0^{(s)})}\sum_{n=0}^{3}\sum_{p=0}^{n}\sum_{\ell=0}^{p}\frac{\psi_{n-p}^{(s)} Q_\ell^{(s)} \phi_{p-\ell,2}^{(4)}}{k^{n-2}}\right]$$

$$= \frac{i}{2}\left[u_2(x) + u_2(\xi) + u_1(x)u_1(\xi) + iu_1'(0)\right] - \sum_{s=1}^{4}\sum_{n=0}^{3}\sum_{p=0}^{n}\sum_{\ell=0}^{p} e^{i(x-\xi)(-\frac{2}{i\pi}\ln a_0^{(s)})}$$

$$F(e^{4i(x-\xi)}, n-2)\psi_{n-p}^{(s)} Q_\ell^{(s)} \phi_{p-\ell,2}^{(4)} \tag{6.44}$$

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