Internat. J. Math. & Math. Sci.
VOL. 21 NO. 3 (1998) 475-478

ON A MODIFIED HYERS-ULAM STABILITY OF HOMOGENEOUS EQUATION

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(Received February 21, 1996)

ABSTRACT. In this paper, a generalized Hyers-Ulam stability of the homogeneous equation shall be proved, i.e., if a mapping f satisfies the functional inequality $\|f(yx) - y^k f(x)\| \leq \varphi(x,y)$ under suitable conditions, there exists a unique mapping T satisfying $T(yx) = y^k T(x)$ and $\|T(x) - f(x)\| \leq \Phi(x)$

KEY WORDS AND PHRASES: Functional equation, homogeneous equation, stability.
1991 AMS SUBJECT CLASSIFICATION CODES: 39B72, 39B52.

1. INTRODUCTION

It is well-known that if a real-valued mapping f defined on non-negative real numbers is a solution of the homogeneous equation, i.e. if f satisfies

$$f(yx) = y^k f(x), \tag{1.1}$$

where k is a given real number, then $f(x) = cx^k$ for some $c \in \mathbb{R}$.

In this note, we shall investigate a generalized Hyers-Ulam stability of the homogeneous equation (1.1) with extended domain and range by using ideas from the paper of Găvruta [1].

Let $(X, +, \cdot)$ be a field and $(X, +, \|\|)$ a real Banach space. In addition, we assume $\|xy\| = \|x\|\,\|y\|$ for all $x, y \in X$. For convenience, we write $x^2, x^3, \cdots$ instead of $x \cdot x, (x \cdot x) \cdot x, \cdots$. If there is no confusion we use 0 and 1 to denote the 'zero-element' and the unity (the neutral element with respect to '$\cdot$') in X, respectively. By x^{-1} we denote the multiplicatively inverse element of x. Suppose k is a natural number. Let $\varphi : X \times X \to [0, \infty)$ be a mapping such that

$$\Phi_z(x) = \sum_{j=0}^{\infty} \|z\|^{-(j+1)k} \varphi\left(z^j x, z\right) < \infty \tag{1.2}$$

or

$$\tilde{\Phi}_z(x) = \sum_{j=0}^{\infty} \|z\|^{jk} \varphi\left(z^{-(j+1)} x, z\right) < \infty \tag{1 3}$$

for some $z \in X$ with $\|z\| > 1$ and all $x \in X$. Moreover, we assume that

$$\begin{cases} \Phi_z(w^n x) = o\big(\|w\|^{nk}\big) & (\text{if } \Phi_z(x) < \infty) \\ \tilde{\Phi}_z(w^n x) = o\big(\|w\|^{nk}\big) & \big(\text{if } \tilde{\Phi}_z(x) < \infty\big) \end{cases}, \tag{1.4}$$

as $n \to \infty$, for some $w \in X$ and all $x \in X$. Let a mapping $f : X \to X$ satisfy

$$\big\|f(yx) - y^k f(x)\big\| \le \varphi(x, y) \tag{1.5}$$

and

$$\begin{cases} \varphi(z^n x, y) = o(\|f(z^n x)\|) \text{ as } n \to \infty & (\text{if } \Phi_z(x) < \infty) \\ \varphi(z^{-n} x, y) = o(\|f(z^{-n} x)\|) \text{ as } n \to \infty & \big(\text{if } \tilde{\Phi}_z(x) < \infty\big) \end{cases}, \tag{1.6}$$

for all x and $y \ne 0$ in X. If (1.3) holds true then we further assume $f(0) = 0$. Our main result is the following theorem.

THEOREM. There exists a unique mapping $T : X \to X$ satisfying (1.1) and

$$\|T(x) - f(x)\| \le \begin{cases} \Phi_z(x) & (\text{if } \Phi_z(x) < \infty) \\ \tilde{\Phi}_z(x) & \big(\text{if } \tilde{\Phi}_z(x) < \infty\big) \end{cases}, \tag{1.7}$$

for all $x \in X$

2. PROOF OF THEOREM

We use induction on n to prove

$$\big\|y^{-nk} f(y^n x) - f(x)\big\| \le \sum_{j=0}^{n-1} \|y\|^{-(j+1)k} \varphi\big(y^j x, y\big) \tag{2.1}$$

for any $n \in \mathbb{N}$. By (1.5), it is clear for $n = 1$. If we assume that (2.1) is true for n, we get for $n + 1$

$$\begin{aligned} \big\|y^{-(n+1)k} f\big(y^{n+1} x\big) - f(x)\big\| &\le \|y\|^{-(n+1)k} \big\|f(yy^n x) - y^k f(y^n x)\big\| + \big\|y^{-nk} f(y^n x) - f(x)\big\| \\ &\le \|y\|^{-(n+1)k} \varphi(y^n x, y) + \sum_{j=0}^{n-1} \|y\|^{-(j+1)k} \varphi\big(y^j x, y\big) \\ &= \sum_{j=0}^{n} \|y\|^{-(j+1)k} \varphi\big(y^j x, y\big) \end{aligned}$$

by using (1.5) and (2.1).

(a) First, we assume that $\Phi_z(x) < \infty$ for some $z \in X$ with $\|z\| > 1$ and all $x \in X$ Let $n > m > 0$. It then follows from (2.1) and (1.2) that

$$\begin{aligned} \big\|z^{-nk} f(z^n x) - z^{-mk} f(z^m x)\big\| &= \|z\|^{-mk} \big\|z^{-(n-m)k} f(z^{n-m} z^m x) - f(z^m x)\big\| \\ &\le \|z\|^{-mk} \sum_{j=0}^{n-m-1} \|z\|^{-(j+1)k} \varphi\big(z^j z^m x, z\big) \\ &= \sum_{j=m}^{n-1} \|z\|^{-(j+1)k} \varphi\big(z^j x, z\big) \to 0 \quad \text{as} \quad m \to \infty. \end{aligned}$$

Therefore, $(z^{-nk} f(z^n x))$ is a Cauchy sequence. Since X is a Banach space, we may define

$$T(x) = \lim_{n \to \infty} z^{-nk} f(z^n x)$$

for all $x \in X$. From the definition of T, (1.2) and (2.1) we can easily verify the truth of the first relation in (1.7).

Suppose x and $y \ne 0$ to be arbitrary elements of X. By (2.1) we have

$$\big\|y^{-k} f(yz^n x) - f(z^n x)\big\| \le \|y\|^{-k} \varphi(z^n x, y).$$

It follows from the inequality just above and (1.6) that

$$\left\|f(z^n x)^{-1} y^{-k} f(yz^n x) - 1\right\| \leq \|y\|^{-k} \|f(z^n x)\|^{-1} \varphi(z^n x, y) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, it holds

$$\lim_{n\to\infty} f(z^n x)^{-1} y^{-k} f(yz^n x) = 1. \tag{2.2}$$

By (2.2) we can show that for all x and $y \neq 0$ in X

$$\begin{aligned} T(yx) &= \lim_{n\to\infty} z^{-nk} f(z^n yx) \\ &= y^k \lim_{n\to\infty} z^{-nk} f(z^n x) \lim_{n\to\infty} f(z^n x)^{-1} y^{-k} f(z^n yx) \\ &= y^k T(x). \end{aligned}$$

Besides, it is not difficult to show that $T(0) = 0$. Hence, $T(yx) = y^k T(x)$ holds true for all $x, y \in X$

Let $U : X \to X$ be another mapping which fulfills (1.1) and (1.7). By using (1.1), (1 7) and (1.4) we get

$$\|T(x) - U(x)\| = \|w\|^{-nk} \|T(w^n x) - U(w^n x)\| \leq 2\|w\|^{-nk} \Phi_z(w^n x) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, it is clear that $T(x) = U(x)$ for all $x \in X$.

(b) Now, we consider the case $\tilde{\Phi}_z(x) < \infty$ for some $z \in X$ with $\|z\| > 1$ and all $x \in X$ By replacing x in (2.1) with $y^{-n}x$ we get

$$\left\|f(x) - y^{nk} f(y^{-n} x)\right\| \leq \sum_{j=0}^{n-1} \|y\|^{jk} \varphi\big(y^{-(j+1)} x, y\big) \tag{2 3}$$

for any $n \in \mathbb{N}$. As in part (a), if $n > m > 0$ then we obtain

$$\left\|z^{nk} f(z^{-n} x) - z^{mk} f(z^{-m} x)\right\| \leq \sum_{j=m}^{n-1} \|z\|^{jk} \varphi\big(z^{-(j+1)} x, z\big) \to 0 \quad \text{as} \quad m \to \infty,$$

by using (2.3) and (1.3). We may define

$$T(x) = \lim_{n\to\infty} z^{nk} f(z^{-n} x)$$

for all $x \in X$. Hence, the second inequality in (1.7) is obvious in view of (2.3).

For arbitrary x and $y \neq 0$ in X, it follows from (2.1) and (1.6) that

$$\lim_{n\to\infty} f(z^{-n} x)^{-1} y^{-k} f(yz^{-n} x) = 1 \tag{2 4}$$

as in part (a) above. By using (2.4), we get for x and $y \neq 0$ in X

$$\begin{aligned} T(yx) &= \lim_{n\to\infty} z^{nk} f(z^{-n} yx) \\ &= y^k \lim_{n\to\infty} z^{nk} f(z^{-n} x) \lim_{n\to\infty} f(z^{-n} x)^{-1} y^{-k} f(yz^{-n} x) \\ &= y^k T(x). \end{aligned}$$

Since $f(0) = 0$ is assumed in the case of $\tilde{\Phi}_z(x) < \infty$, it also holds $T(yx) = y^k T(x)$ for $y = 0$

The uniqueness of T can be proved as in (a).

3. EXAMPLES

EXAMPLE 1. Let $\varphi(x, y) = \delta + \theta\|x\|^a \|y\|^b$ $(\delta \geq 0, \theta \geq 0, 0 \leq a < k, b \geq 0)$ be given in the functional inequality (1.5). If a mapping $f : X \to X$ satisfies the first condition in (1.6) then there exists a unique mapping $T : X \to X$ fulfilling (1.1) and

$$\|T(x) - f(x)\| \leq \delta\left(\|z\|^k - 1\right)^{-1} + \theta\|z\|^b\left(\|z\|^k - \|z\|^a\right)^{-1}\|x\|^a$$

for any $x,\ z \in X$ with $\|z\| > 1$. In particular, if $\delta > 0$ and $\theta = 0$ then f itself satisfies (1.1).

EXAMPLE 2. Assume that $\varphi(x,y) = \theta\|x\|^a\|y\|^b$ $(\theta \geq 0, a > k, b \geq 0)$ is given in the functional inequality (1.5) If a mapping $f : X \to X$ satisfies the second condition in (1.6) then there exists a unique mapping $T : X \to X$ which satisfies (1.1) and

$$\|T(x) - f(x)\| \leq \theta\|z\|^b\left(\|z\|^a - \|z\|^k\right)^{-1}\|x\|^a$$

for all $x, z \in X$ with $\|z\| > 1$.

If $\varphi(x,y) = \theta\|x\|^k g(\|y\|)$ for some mapping $g : [0,\infty) \to [0,\infty)$ then our method to get stability for the homogeneous equations (1 1) cannot be applied. By modifying an example in the paper of Rassias and Šemrl [2] we shall introduce a mapping $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.5) and (1.6) with some φ and such that $|f(x)|\,|x|^{-k}$ (for $x \neq 0$) is unbounded.

EXAMPLE 3. Let us define $f(x) = x^k \log|x|$ for $x \neq 0$ and $f(0) = 0$. Then f satisfies (1.5) and both conditions of (1..6) with $\varphi(x,y) = |x|^k\,|y|^k\,|\log|y||$ $(y \neq 0)$ and $\varphi(x,0) = 0$, even though φ satisfies neither (1.2) nor (1.3). In this case we can expect no analogy to the results of Example 1 and 2 Really, it holds

$$\lim_{n\to\infty} |T(x) - f(x)|\,|x|^{-k} = \infty$$

for each mapping $T : \mathbb{R} \to \mathbb{R}$ fulfilling (1.1).

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