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ON COUNTABLE CONNECTED HAUSDORFF SPACES IN WHICH THE INTERSECTION OF EVERY PAIR OF CONNECTED SUBSETS IS CONNECTED

V. TZANNES

Department of Mathematics
University of Patras
Patras 26110 Greece

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ABSTRACT. We prove that a countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected, cannot be locally connected, and also that every continuous function from a countable connected, locally connected Hausdorff space, to a countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected, is constant.

KEY WORDS AND PHRASES. Countable connected, locally connected.

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1. INTRODUCTION.

The problem of existence of countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected was posed by Čvid in [1], and was answered in [2]. Recently, Gruenhage [3] assuming the continuum hypothesis constructed a perfectly normal space in which the only non-degenerate connected subsets of it, are the cofinite sets. Also assuming Martin' s Axiom he constructed a completely regular and a countable Hausdorff space with this property. Obviously, in these spaces the intersection of every pair of connected subsets is connected. None of the spaces in [2] and [3] is locally connected, or has a dispersion point.

We prove that a countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected, cannot be locally connected, and also that every continuous function from a countable connected, locally connected Hausdorff space, to a countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected, is constant. Both these results hold in a Hausdorff connected space with a dispersion point: The first is obvious and the second, for not necessarily countable spaces, was proved by Coppin in [4]. Improvements of Coppin's result, as well as results concerning the constancy of functions between two spaces, can be found in the papers by Chew and Doyle [5], and by Sanderson [6].

Let X be a connected topological space. A point t is called a cut point of X if the space $X \setminus \{t\}$ is not connected. Thus, it t is a cut point of X, then the subspace $X \setminus \{t\}$ is the union of two mutually separated sets $A(t), B(t)$. (Two sets A, B are called separated if $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$.) Obviously, if $A(t), B(t)$ are connected, the separation is unique. Let $x, y \in X$. A cut point t of X is said to separate the points x, y if the above sets $A(t), B(t)$ can be chosen so that $x \in A(t)$ and $y \in B(t)$. The set of cut points of X separating the points x, y will be denoted by $E(x, y)$. The empty set and the singletons are considered to be connected. All spaces are assumed to have more than one point.

2. RESULTS.

PROPOSITION 1. Let X be a Hausdorff connected space such that $E(a, b) \neq \emptyset$, for every $a, b \in X$. Then there exists a continuous non-constant real valued function on X, separating the points a and b.

PROOF. The proof is reduced to the Urysohn' s Lemma in the following manner: For every point $t \in E(a, b)$ there exist two sets $M_a(t)$, $M_b(t)$ such that $a \in M_a(t)$, $b \in M_b(t)$, $\overline{M_a(t)} = M_a(t) \cup \{t\}$, $\overline{M_b(t)} = M_b(t) \cup \{t\}$ and $X \setminus \{t\} = M_a(t) \cup M_b(t)$. Hence the sets

$$F_1 = \bigcap_{t \in E(a,b)} (M_a(t) \cup \{t\}) \quad \text{and} \quad F_2 = \bigcap_{t \in E(a,b)} (M_b(t) \cup \{t\})$$

are both closed disjoint cointaining the points a, b respectively, and not containing any cut point of X separating the points a, b. Consequently, for every point d of the set of positive dyadic rational numbers we can define an open set $(M_a(t))(d)$ such that if $d < r$, then $\overline{M_a(t)(d))} \subseteq M_a(t)(r)$. But then the function $f(x) = inf\{d : x \in (M_a(t))(d)\}$, if $x \notin F_2$, and $f(x) = 1$, if $x \in F_2$ is continuous separating the points a, b.

PROPOSITION 2. Does not exist a countable connected, locally connected Hausdorff space in which the intersection of every pair of connected subsets is connected.

PROOF. As it is proved in [7, Theorem 9.1] a connected locally connected space X is a Hausdorff space in which the intersection of every pair (indeed every collection) of connected sets is connected, if and only if no two point of X are conjugate. That is, $E(x, y) \neq \emptyset$, for every $x, y \in X$. But then, Proposition 1 implies that there exists a non-constant continuous real valued function on X, which is impossible for countable connected spaces.

PROPOSITION 3. Let X be a countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected. Then

(1) The subset D of X at every point of which X is not locally connected, is dense.

(2) The subset L at every point of which X is locally connected is totally disconnected or empty.

PROOF (1). By Proposition 2, $D \neq \emptyset$. Hence at every point $x \in X \setminus \overline{D}$, the space X is locally connected and therefore if U_x is an open connected neighbourhood of x for which $U_x \cap \overline{D} \neq \emptyset$, then U_x is also a locally connected space in which the intersection of every pair of connected subsets is connected, which is impossible, by Proposition 2.

(2). Obvious.

THEOREM. Every continuous function from a countable connected, locally connected Hausdorff space, to a connected Hausdorff space in which the intersection of every pair of connected subsets is connected, is constant.

PROOF. Let f be a continuous non-constant function from X to Y. Obviously the space $Z = f(X)$ is countable connected Hausdorff in which the intersection of every pair of connected subsets is connected. Let x, y be distinct points of X such that $f(x) \neq f(y)$ and let $U_{f(x)}$, $U_{f(y)}$ be disjoint open neighbourhoods of $f(x), f(y)$, respectively. Since X is locally connected there exists an open connected neighbourhood U_x of x such that $f(U_x) \subseteq U_{f(x)}$. If $f(U_x) = \{f(x)\}$ then we consider the set $A = \{a \in X : f(a) = f(x)\}$. Since the set $\overline{A} \setminus \overset{\circ}{A}$ is not empty, it follows that there exist a point $a \in \overline{A} \setminus \overset{\circ}{A}$ and a connected open neighbourhood U_a of a, such that $f(U_a) \subseteq U_{f(x)}$ and $f(U_a) \neq \{f(x)\}$. Therefore the component $C_{f(x)}$ of $f(x)$ in $\overline{U}_{f(x)}$ is not a singleton.

Consider the component K of $f(y)$ in $Z \setminus C_{f(x)}$. If $K = \{f(y)\}$ then for the component M of Y in $X \setminus f^{-1}(C_{f(x)})$ it holds that $f(M) = \{f(y)\}$ and $f(\overline{M}) = \{f(y)\}$. Since the subspace $X \setminus f^{-1}(C_{f(x)})$ is locally connected it follows that M is open-and-closed (in $X \setminus f^{-1}(C_{f(x)})$), and hence $\overline{M} \cap f^{-1}(C_{f(x)}) \neq \emptyset$ which is impossible. Therefore the component K of y in $Z \setminus C_{f(x)}$ is not a singleton.

Thus, by [8, Vol. II, Ch. V, Theorem 5, III], for the connected subsets $C_{f(x)}$ and K it follows that the set $Z \setminus K$ is connected and hence either (1) $\overline{(Z \setminus K)} \cap K \neq \emptyset$, or (2) $(Z \setminus K) \cap \overline{K} \neq \emptyset$ or (3) $\overline{(Z \setminus K)} \cap \overline{K} \neq \emptyset$.

In case (1), let $p, q \in \overline{(Z \setminus K)} \cap K$, and $p \neq q$. Then for the connected subsets $(Z \setminus K) \cup \{p, q\}$ and K it holds that $((Z \setminus K) \cup \{p, q\}) \cap K = \{p, q\}$ which is impossible because by assumption the intersection of every pair of connected subsets of Z must be connected. Therefore $\overline{(Z \setminus K)} \cap K$ is a singleton. We set $\overline{(Z \setminus K)} \cap K = \{p\}$. The set K is closed because if a is a limit point of K and $a \notin K$ then for the connected subsets $K \cup \{a\}$ and $\overline{(Z \setminus K)}$ the subset $\overline{(Z \setminus K)} \cap (K \cup \{a\}) = \{a, p\}$ must be connected, which is impossible. Hence if we consider the component M of y in $X \setminus f^{-1}(C_{f(x)})$ then $f(\overline{M}) \subseteq K$ which is also impossible because $\overline{M} \cap f^{-1}(C_{f(x)}) \neq \emptyset$.

In case (2) it can be proved in the same manner as in case (1) that $(Z \setminus K) \cap \overline{K}$ is a singleton and that $Z \setminus K$ is closed. We set $(Z \setminus K) \cap \overline{K} = \{q\}$. Since $\overline{K} = K \cup \{q\}$ it follows that $(Z \setminus K) \setminus \{q\}$ is open which implies that q is a cut point of the space Z. Since $q \in Z \setminus K$ it follows that either

$q = f(x)$ or $q \neq f(x)$. If $q = f(x)$ we consider again the component M of y in $X \setminus f^{-1}(C_{f(x)})$, and let $a \in \overline{M} \cap f^{-1}(C_{f(x)})$. Then $f(M) \subseteq K$, the point $f(a)$ is a limit point of K and $f(a) \in C_{f(x)}$. That is $f(a) = q$. But then there exists an open connected neighbourhood U_a of a such that $f(U_a) \subseteq U_{f(x)}$ which implies that $f(U_a) \subseteq C_{f(x)}$. Hence $U_a \subseteq f^{-1}(C_{f(x)})$ which is impossible because $U_a \cap M \neq \emptyset$. If $q \neq f(x)$ then obviously $q \in E(f(x), f(y))$.

Finally, observing that case (3) is reduced to case (1) or (2) we conclude that $E(f(x), f(y)) \neq \emptyset$, which is impossible by Proposition 1.

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