COMMUTATIVITY RESULTS FOR SEMIPRIME RINGS WITH DERIVATIONS

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ABSTRACT. We extend a result of Herstein concerning a derivation d on a prime ring R satisfying $[d(x),d(y)] = 0$ for all $x,y \in R$, to the case of semiprime rings. An extension of this result is proved for a two-sided ideal but is shown to be not true for a one-sided ideal. Some of our recent results dealing with U^*- and U^{**}- derivations on a prime ring are extended to semiprime rings. Finally, we obtain a result on semiprime rings for which $d(xy) = d(yx)$ for all x,y in some ideal U.

KEY WORDS AND PHRASES: Semiprime ring, derivation, commutator, and central ideal.
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1. INTRODUCTION

In his note on derivations, Herstein [1] showed that if a prime ring R of characteristic not 2 admits a nonzero derivation d such that $[d(x),d(y)] = 0$ for all x,y in R, then R is commutative. Here, we give an easy but elegant extension of this result in the case when R is semiprime. Moreover, by making use of a more recent result of Bell and Martindale [2], we can get a more general theorem for a semiprime ring, which requires the condition $[d(x),d(y)] = 0$ to hold only on some ideal of R. We notice that a one-sided ideal would not work in this new theorem, the example given by Bell and Daif [3] is a counter-example.

Recently, Bell and Daif [3] introduced the notions of U^*- and U^{**}- derivations d on a prime ring R, where U is a nonzero right ideal of R. If d is a derivation on R such that $d(x)d(y) + d(xy) = d(y)d(x) + d(yx)$ for all $x,y \in U$, we say that d is a U^*- derivation; and if $d(x)d(y) + d(yx) = d(y)d(x) + d(xy)$ for all $x,y \in U$, we call d a U^{**}- derivation. We proved that if d is a nonzero U^*- or U^{**}- derivation, then either R is commutative or $d^2(U) = d(U)d(U) = \{0\}$. This result yielded a result of Bell and Kappe [4]. We also studied derivations d satisfying $d(xy) = d(yx)$ for all $x,y \in U$. For formal reasons, we call d a U^{***}- derivation if it satisfies this condition. In this note, we extend these results to the semiprime case. We will show for a nonzero U^*- or U^{**}- derivation d that d(U) centralizes [U,U]. In the event that U is a two-sided ideal, we show that R contains a nonzero central ideal. The same conclusion is obtained when R admits a U^{***}- derivation which is nonzero on U.

For the ring R, Z will denote the center of R. For elements $x,y \in R$, the commutator $xy - yx$ will be written as $[x,y]$; and for a subset U of R, the set of all commutators of elements of U will be written as [U,U]. We will make extensive use of the familiar commutator identities $[x,yz] = y[x,z] + [x,y]z$ and $[xy,z] = x[y,z] + [x,z]y$.

To achieve our purposes, we mention the following results.

(A) [1, Theorem 1] Let R be any ring and d a derivation of R such that $d^3 \neq 0$. Then the subring of R generated by all d(r), $r \in R$, contains a nonzero ideal of R.

(B) [2, Theorem 3] Let R be a semiprime ring and U a nonzero left ideal. If R admits a derivation d which is nonzero on U and centralizing on U, then R contains a nonzero central ideal.

(C) [5, Lemma 1] Let R be a semiprime ring and U a nonzero two-sided ideal of R. If $x \in R$ and x centralizes [U,U], then x centralizes U.

2. EXTENSIONS OF HERSTEIN'S THEOREM

THEOREM 2.1. Let R be a semiprime ring and d a derivation of R with $d^3 \neq 0$. If $[d(x),d(y)] = 0$ for all $x,y \in R$, then R contains a nonzero central ideal.

PROOF. By (A), the subring generated by d(R) contains a nonzero ideal U of R. By our hypothesis, U is commutative; hence $U^2 \subseteq Z$. But R is semiprime, hence $U \neq \{0\}$ implies $U^2 \neq \{0\}$, which completes the proof.

Now we aim to extend the theorem of Herstein in the situation when the ring is semiprime and the condition $[d(x),d(y)] = 0$ is merely satisfied on an ideal of the ring.

THEOREM 2.2. Let R be a two-torsion-free semiprime ring and U a nonzero two-sided ideal of R. If R admits a derivation d which is nonzero on U and $[d(x),d(y)] = 0$ for all $x,y \in U$, then R contains a nonzero central ideal.

PROOF. We are given that

$$[d(x),d(y)] = 0 \text{ for all } x,y \in U. \tag{2.1}$$

Replacing y by yz, we therefore obtain

$$d(y)[d(x),z] + [d(x),y]d(z) = 0 \text{ for all } x,y,z \in U. \tag{2.2}$$

Putting z = zr where $z \in U$ and $r \in R$, we now get

$$d(y)z[d(x),r] + [d(x),y]zd(r) = 0 \text{ for all } x,y,z \in U,\ r \in R. \tag{2.3}$$

Now substitute $r = d(t)$, $t \in U$, to get

$$[d(x),y]z\,d^2(t) = 0 \text{ for all } x,y,z,t \in U. \tag{2.4}$$

Let $\{P_\alpha : \alpha \in \Lambda\}$ be a family of prime ideals of R such that $\bigcap_\alpha P_\alpha = \{0\}$. Now (2.4) yields $[d(x),y]zR\,d^2(t) = \{0\}$ for all $x,y,z,t \in U$; hence for each P_α, we either have

(a) $[d(x),y]U \subseteq P_\alpha$ for all $x,y \in U$,

or

(b) $d^2(U) \subseteq P_\alpha$.

Call P_α an (a)-prime ideal or (b)-prime ideal according to which of these conditions is satisfied.

Note that $[d(x),y]RU \subseteq P_\alpha$ for each (a)-prime P_α, so either $[d(x),y] \in P_\alpha$ for all $x,y \in U$ or $U \subseteq P_\alpha$. In either event,

$$[d(x),y] \in P_\alpha \text{ for all } x,y \in U \text{ and all (a)-prime } P_\alpha. \tag{2.5}$$

Now consider (b)-prime ideals. Taking $x,y \in U$, we have $d^2(xy) = d^2(x)y + xd^2(y) + 2d(x)d(y) \in P_\alpha$, so $2d(x)d(y) \in P_\alpha$ for all $x,y \in U$. Replacing y by zy shows that

$$2d(x)zd(y) \in P_\alpha \text{ for all } x,y,z \in U; \tag{2.6}$$

hence

$$2d(x)Rzd(y) \subseteq P_\alpha \text{ and } 2d(x)zRd(y) \subseteq P_\alpha \text{ for all } x,y,z \in U. \tag{2.7}$$

It follows that either $d(U) \subseteq P_\alpha$, or $2d(x)y$ and $2yd(x) \in P_\alpha$ for all $x,y \in U$. In either case,

$$2[d(x),y] \in P_\alpha \text{ for all } x,y \in U \text{ and (b)-prime } P_\alpha. \tag{2.8}$$

Thus, for all $x,y \in U$ we have (by (2.5) and (2.8)) that $2[d(x),y] \in \bigcap_\alpha P_\alpha = \{0\}$; and since R is 2-torsion-free, $[d(x),y] = 0$ for all $x,y \in U$. In particular, $[d(x),x] = 0$ for all $x \in U$, so the theorem follows by (B).

REMARK. We notice that Theorem 2.2 is not true in the case when U is one-sided. Let R be the ring of all 2x2 matrices over a field F; let $U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R$. Let d be the inner derivation given by $d(x) = x\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$ for all $x \in R$. For any two elements x and y in U, we have that $[d(x),d(y)] = 0$, but the conclusion of the theorem is not true.

3. EXTENDING RESULTS ON U*- AND U**- DERIVATIONS

THEOREM 3.1. Let R be a semiprime ring and U a nonzero right ideal of R. If R admits a nonzero U*-derivation d, then d(U) centralizes [U,U].

PROOF. The condition that d is a U*- derivation yields

$$[d(x),d(y)] = [d(y),x] + [y,d(x)] \text{ for all } x,y \in U. \tag{3.1}$$

Proceeding exactly as in [3], we see that

$$[d(x),x]UR(d(x) + d^2(x)) = \{0\} \text{ for all } x \in U. \tag{3.2}$$

Since R is semiprime, it must have a family $\{P_\alpha : \alpha \in \Lambda\}$ of prime ideals such that $\bigcap_\alpha P_\alpha = \{0\}$. Let P be a typical one of these. By (3.2) we see that for each $x \in U$, either $[d(x),x]U \subseteq P$ or $d(x) + d^2(x) \in P$. We now use the kind of argument employed in the proof of Theorem 2.2, in effect performing the calculations of [3] modulo P; we arrive at the conclusion that

$$\text{either } d(U)U \subseteq P \text{ or } [x + d(x), R] \subseteq P \text{ for all } x \in U. \tag{3.3}$$

In the first case, we can again employ the argument of [3] modulo P, obtaining the result that

$$\text{either } U \subseteq P \text{ or } [d(x),d(t)] \in P \text{ for all } x,t \in U. \tag{3.4}$$

Returning to the second possibility in (3.3), we assume that $[x + d(x), R] \subseteq P$. We then have $[x,d(t)] + [d(x),d(t)] \in P$ for all $x,t \in U$. But from (3.1) we have $[d(x),d(t)] + [x,d(t)] = [t,d(x)]$, hence we have

$$[t,d(x)] \in P \text{ for all } x,t \in U. \tag{3.5}$$

Putting $t = td(y)$ and using (3.5), we get

$$t[d(y),d(x)] \in P \text{ for all } x,y,t \in U. \tag{3.6}$$

From (3.6) we have $UR[d(y),d(x)] \subseteq P$ for all $x,y \in U$. Consequently, either $U \subseteq P$ or $[d(x),d(t)] \in P$ for all $x,t \in U$, which are the same alternatives as in (3.4).

If we consider the case $U \subseteq P$, then from (3.1) we get $[d(x),d(t)] \in P$ for all $x,t \in U$. Therefore, we always have $[d(x),d(t)] \in P$ for all $x,t \in U$. Now using the fact that $\bigcap_{\alpha} P_{\alpha} = \{0\}$, we conclude that $[d(x),d(t)] = 0$ for all $x,t \in U$. From our hypothesis, we have $d(xt) = d(tx)$ for all $x,t \in U$. This means that $d([x,t]) = 0$ for all $x,t \in U$. But $d([x,t]z) = d(z[x,t])$, hence $[x,t]d(z) = d(z)[x,t]$ for all $x,z,t \in U$. Thus $d(U)$ centralizes $[U,U]$ as required.

Similar conclusions as in the proof of Theorem 3.1 lead us to the same conclusion in the case that d is a U^{**}- derivation. Therefore, we have

THEOREM 3.2. Let R be a semiprime ring and U a nonzero right ideal of R. If R admits a nonzero U^{**}- derivation, then $d(U)$ centralizes $[U,U]$.

COROLLARY. Let R be a semiprime ring and U a nonzero two-sided ideal of R. If R admits a U^{*}- or U^{**}- derivation d which is nonzero on U, then R contains a nonzero central ideal.

PROOF. By Theorems 3.1 and 3.2, $d(U)$ centralizes $[U,U]$. By (C), we get that $d(U)$ centralizes U. The result now follows by (B).

THEOREM 3.3. Let R be a semiprime ring and U a nonzero two-sided ideal of R. If R admits a U^{***}- derivation d which is nonzero on U, then R contains a nonzero central ideal.

PROOF. Since $d(xy) = d(yx)$ for all $x,y \in U$, the argument at the end of the proof of Theorem 3.1 shows that $d(U)$ centralizes $[U,U]$. The result now follows as in the proof of the Corollary.

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