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GENERALIZED FRACTIONAL CALCULUS TO A SUBCLASS OF ANALYTIC FUNCTIONS FOR OPERATORS ON HILBERT SPACE

YONG CHAN KIM, JAE HO CHOI, AND JIN SEOP LEE
Department of Mathematics
Yeungnam University
Gyongsan 712-749, KOREA

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ABSTRACT. In this paper, we investigate some generalized results of applications of fractional integral and derivative operators to a subclass of analytic functions for operators on Hilbert space.

KEY WORDS AND PHRASES: Multivalent function, Fractional calculus, Riesz-Dunford integral.
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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = \sum_{n=0}^{\infty} a_{n+1} z^{n+1} \quad (a_1 := 1), \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are ***univalent*** in the unit disk $\mathcal{U}$.

Let $\mathcal{S}_0(\alpha, \beta, \gamma, p)$ denote the class of functions

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (a_{n+p} \geq 0), \tag{1.2}$$

which are analytic and p-valent in $\mathcal{U}$ and satisfy the condition

$$\left| \frac{zf'(z)}{f(z)} - p \right| < \beta \left| \alpha \frac{zf'(z)}{f(z)} + (p - \gamma) \right| \tag{1.3}$$

for $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $0 \leq \gamma < p$, $p \in \mathbb{N}$ and $z \in \mathcal{U}$. See Lee *et al* [1] for further information on them. It is easily found that $\mathcal{S}_0(\alpha, \beta, \gamma, p) \subset \mathcal{A}$ when $p = 1$.

Let $a, b,$ and c be complex numbers with $c \neq 0, -1, -2, \cdots$. Then the *Gaussian hypergeometric function* ${}_2F_1(z)$ is defined by

$$\begin{aligned} {}_2F_1(z) &\equiv {}_2F_1(a, b; c; z) \\ &:= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \end{aligned} \tag{1.4}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$\begin{aligned} (\lambda)_n &:= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \\ &= \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \cdots\}). \end{cases} \end{aligned} \tag{1.5}$$

Let A be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. For a complex valued function f analytic on a domain E of the complex plane containing the spectrum $\sigma(A)$ of A we denote $f(A)$ as Riesz-Dunford integral [2, p. 568], that is,

$$f(A) := \frac{1}{2\pi i} \int_C f(z)(zI - A)^{-1} dz, \tag{1.6}$$

where I is the identity operator on $\mathcal{H}$ and C is positively oriented simple closed rectifiable contour containing $\sigma(A)$.

Also $f(A)$ can be defined by the series $f(A) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} A^n$ which converges in the norm topology [3].

Xiaopei [4] defined $\mathcal{S}_0(\alpha, \beta, \gamma, p; A)$ by the class of functions

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (a_{n+p} \geq 0),$$

which is analytic and p-valent in $\mathcal{U}$ and satisfies the condition,

$$\|Af'(A) - pf(A)\| < \beta \|\alpha A f'(A) + (p - \gamma) f(A)\| \tag{1.7}$$

for $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $0 \leq \gamma < p$, $p \in \mathbb{N}$ and all operators A with $\|A\| < 1$ and $A \neq \theta$ (θ denotes the zero operator on $\mathcal{H}$).

Let A^* denote the conjugate operator of A.

DEFINITION 1 ([4]). The fractional integral for operator of order a is defined by

$$D_A^{-a} f(A) = \frac{1}{\Gamma(a)} \int_0^1 A^a f(tA)(1 - t)^{a-1} dt, \tag{1.8}$$

where $a > 0$ and $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin.

DEFINITION 2 ([4]). The fractional derivative for operator of order a is defined by

$$D_A^a f(A) = \frac{1}{\Gamma(1 - a)} g'(A), \tag{1.9}$$

where $g(z) = \int_0^1 z^{1-a} f(tz)(1 - t)^{-a} dt \, (0 < a < 1)$ and $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin.

Srivastava *et al.* [5] introduced a fractional integral operator $I_{0,z}^{a,b,c}$ defined by (*cf.* [6])

$$I_{0,z}^{a,b,c} f(z) = \frac{z^{-b}}{\Gamma(a)} \int_0^1 (1 - t)^{a-1} {}_2F_1(a+b, -c; a; 1 - t) f(tz) dt$$
$$(a > 0; b, c \in \mathbb{R}; f(z) \in \mathcal{A}) \tag{1.10}$$

and Owa *et al.* [7] studied the fractional operator $J_{0,z}^{a,b,c}$ defined by (see also Kim *et al.* [8])

$$J_{0,z}^{a,b,c} f(z) = \frac{\Gamma(2 - b)\Gamma(2 + a + c)}{\Gamma(2 - b + c)} z^b I_{0,z}^{a,b,c} f(z) \quad (f \in \mathcal{A}). \tag{1.11}$$

The fractional derivative operator $D_{0,z}^{a,b,c}$ is defined by (*cf.* [9])

$$D_{0,z}^{a,b,c} f(z) = \frac{d}{dz} \left(\frac{z^{-b}}{\Gamma(1 - a)} \int_0^1 (1 - t)^{-a} {}_2F_1(b - a + 1, -c; 1 - a; 1 - t) f(tz) dt \right)$$
$$(0 \leq a < 1; b, c \in \mathbb{R}; f(z) \in \mathcal{A}). \tag{1.12}$$

And we define $D_{0,z}^{n+a,b,c}$ by

$$D_{0,z}^{n+a,b,c}f(z) = \frac{d^n}{dz^n} D_{0,z}^{a,b,c}f(z). \tag{1.13}$$

For all invertible operator A, we introduce the following definition:

DEFINITION 3. The *fractional integral operator for operator* $I_{0,A}^{a,b,c}$ is defined by

$$I_{0,A}^{a,b,c}f(A) = \frac{1}{\Gamma(a)} \int_0^1 A^{-b}\, {}_2F_1(a+b-c; a; 1-t) f(tA)(1-t)^{a-1} dt, \tag{1.14}$$

where $a > 0$ and $b, c \in \mathbb{R}$.

The *fractional derivative operator for operator* $D_{0,A}^{a,b,c}$ is defined by

$$D_{0,A}^{a,b,c}f(A) = \frac{1}{\Gamma(1-a)} g'(A), \tag{1.15}$$

where

$$g(z) = \int_0^1 z^{-b}\, {}_2F_1(b-a+1, -c; 1-a; 1-t) f(tz)(1-t)^{-a} dt,$$

$0 < a < 1$ and $b, c \in \mathbb{R}$. In both (1.14) and (1.15) $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\epsilon}), \quad z \to 0,$$

where $\epsilon > \max\{0, b-c\} - 1$ and the multiplicity of $(1-t)^{a-1}$ is in (1.14) (and that of $(1-t)^{-a}$ in (1.15)) removed by requiring $\log(1-t)$ to be real when $1 - t > 0$.

We note that

$$I_{0,A}^{a,-a,c}f(A) = D_A^{-a}f(A) \quad \text{and} \quad D_{0,A}^{a,a-1,c}f(A) = D_A^a f(A). \tag{1.17}$$

The object of this paper is to prove the distortion theorems of fractional integral and derivative operators to $\mathcal{S}_0(\alpha, \beta, \gamma, p; A)$.

2. RESULTS

LEMMA 1 (Xiaopei [4, Theorem 2.1]. An analytic function $f(z)$ is in the class $\mathcal{S}_0(\alpha, \beta, \gamma, p; A)$ for all proper contraction A with $A \neq \theta$ if and only if

$$\sum_{k=1}^{\infty} \{k + \beta[p - \gamma + \alpha(k+p)]\} a_{k+p} \leq \beta(p - \gamma + \alpha p) \tag{2.1}$$

for $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $0 \leq \gamma < p$, and $p \in \mathbb{N}$.

The result is sharp for the function

$$f(z) = z^p - \frac{\beta(p-\gamma+\alpha p)}{k + \beta[p - \gamma + \alpha(k+p)]} z^{k+p} \quad (k \geq 1).$$

THEOREM 1. Let $p > \max\{b-c-1, b-1, -1-c-a\}$ and $a(p+1) > b(a+c)$. If $f(z) \in \mathcal{S}_0(\alpha, \beta, \gamma, p; A)$, then

$$\begin{aligned} \|I_{0,A}^{a,b,c}f(A)\| \leq{}& \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p+1-b)\Gamma(a+p+1+c)} \|A\|^{p-b} \\ &+ \frac{\beta(p-\gamma+\alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}\Gamma(p+1-b)\Gamma(a+p+1+c)} \|A\|^{p+1-b} \end{aligned} \tag{2.2}$$

and

$$\|I_{0,A}^{a,b,c}f(A)\| \geq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p+1-b)\Gamma(a+p+1+c)}\|A\|^{p-b} - \frac{\beta(p-\gamma+\alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}\Gamma(p+1-b)\Gamma(a+p+1+c)}\|A\|^{p+1-b} \tag{2.3}$$

for $a>0, b, c \in \mathbb{R}$ and all invertible operator A with $(A^{\frac{1}{q}})^* A^{\frac{1}{q}} = A^{\frac{1}{q}}(A^{\frac{1}{q}})^* (q \in \mathbb{N}), \|A\| < 1$ and $r_{sp}(A) r_{sp}(A^{-1}) \leq 1$, where $r_{sp}(A)$ is the radius of spectrum of A.

PROOF. Consider the function

$$\begin{aligned} F(A) &= \frac{\Gamma(p+1-b)\Gamma(a+p+1+c)}{\Gamma(p+1-b+c)\Gamma(p+1)} A^b I_{0,A}^{a,b,c} f(A) \\ &= A^p - \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1-b+c)\Gamma(p+1+k)\Gamma(p+1-b)\Gamma(a+p+1+c)}{\Gamma(k+p+1-b)\Gamma(a+k+p+1+c)\Gamma(p+1)\Gamma(p+1-b+c)} a_{k+p} A^{k+p} \\ &= A^p - \sum_{k=1}^{\infty} B_{k+p} A^{k+p}, \end{aligned} \tag{2.4}$$

where

$$B_{k+p} = \frac{\Gamma(k+p+1-b+c)\Gamma(p+1+k)\Gamma(p+1-b)\Gamma(a+p+1+c)}{\Gamma(k+p+1-b)\Gamma(a+k+p+1+c)\Gamma(p+1)\Gamma(p+1-b+c)} a_{k+p}.$$

Hence, for convenience, we put

$$\Phi(k) = \frac{\Gamma(k+p+1-b+c)\Gamma(p+1+k)\Gamma(p+1-b)\Gamma(a+p+1+c)}{\Gamma(k+p+1-b)\Gamma(a+k+p+1+c)\Gamma(p+1)\Gamma(p+1-b+c)} \quad (k \in \mathbb{N}). \tag{2.5}$$

Then, by the constraints of the hypotheses, we note that $\Phi(k)$ is non-increasing for integers $k \geq 1$ and we have $0 < \Phi(k) < 1$. So $F(z) \in \mathcal{S}_0(\alpha, \beta, \gamma, p; A)$. By Lemma 1, we get

$$\begin{aligned} \{1+\beta[p-\gamma+\alpha(p+1)]\}\sum_{k=1}^{\infty} B_{k+p} &\leq \sum_{k=1}^{\infty}\{k+\beta[p-\gamma+\alpha(k+p)]\}B_{k+p} \\ &\leq \sum_{k=1}^{\infty}\{k+\beta[p-\gamma+\alpha(k+p)]\}a_{k+p} \\ &\leq \beta(p-\gamma+\alpha p), \end{aligned} \tag{2.6}$$

which gives

$$\sum_{k=1}^{\infty} B_{k+p} \leq \frac{\beta(p-\gamma+\alpha p)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}}.$$

Therefore, in a similar way with the proof of [4, Theorem 2.3, p. 305], we obtain

$$\|I_{0,A}^{a,b,c}f(A)\| \geq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p+1-b)\Gamma(a+p+1+c)}\|A^{-b}\|\,\|A\|^{p} - \frac{\beta(p-\gamma+\alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}\Gamma(p+1-b)\Gamma(a+p+1+c)}\|A\|^{p+1}\,\|A^{-b}\| \tag{2.7}$$

and

$$\|I_{0,A}^{a,b,c}f(A)\| \leq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p+1-b)\Gamma(a+p+1+c)}\|A^{-b}\|\,\|A\|^{p} + \frac{\beta(p-\gamma+\alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}\Gamma(p+1-b)\Gamma(a+p+1+c)}\|A\|^{p+1}\,\|A^{-b}\|. \tag{2.8}$$

By equation (7) of [4, p.307],

$$\|A^b\| = \|A\|^b \quad (b > 0). \tag{2.9}$$

Since $A^*A = AA^*$, $\|A\| = r_{sp}(A)$. So

$$1 = \|AA^{-1}\| \leq \|A\|\,\|A^{-1}\| = r_{sp}(A)r_{sp}(A^{-1}) \leq 1.$$

Thus

$$\|A^{-1}\| = \|A\|^{-1}. \tag{2.10}$$

By (2.9) and (2.10),

$$\|A^b\| = \|A\|^b \tag{2.11}$$

for all real b. Therefore from (2.7), (2.8) and (2.11) we have the desired estimates.

THEOREM 2. Let $p > \max\{b-c-1, b, -2-c+a\}$, $c+1 < (p-b)(1-a+p+c)$, and $b(2-a+c) \leq (1-a)(1+p)$. If $f(z) \in \mathcal{S}_0(\alpha,\beta,\gamma,p;A)$, then

$$\begin{aligned}\|D_{0,A}^{a,b,c}f(A)\| \leq{}& \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)}\|A\|^{p-b-1}\\ &+\frac{\beta(p+1)(p-\gamma+\alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}\Gamma(p-b)\Gamma(2-a+p+c)}\|A\|^{p-b}\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\|D_{0,A}^{a,b,c}f(A)\| \geq{}& \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)}\|A\|^{p-b-1}\\ &-\frac{\beta(p+1)(p-\gamma+\alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}\Gamma(p-b)\Gamma(2-a+p+c)}\|A\|^{p-b}\end{aligned} \tag{2.13}$$

for $0 < a < 1, b, c \in \mathbb{R}$ and all invertible operator A with $(A^{\frac{1}{q}})^*A^{\frac{1}{q}} = A^{\frac{1}{q}}(A^{\frac{1}{q}})^*(q \in \mathbb{N})$, $\|A\| < 1$ and $r_{sp}(A)r_{sp}(A^{-1}) \leq 1$, where $r_{sp}(A)$ is the radius of spectrum of A.

PROOF. Consider the function

$$\begin{aligned}G(A) &= \frac{\Gamma(p-b)\Gamma(2-a+p+c)}{\Gamma(p+1-b+c)\Gamma(p+1)}A^{b+1}D_{0,A}^{a,b,c}f(A)\\ &= A^p - \sum_{k=1}^{\infty}\frac{\Gamma(k+p+1-b+c)\Gamma(p+1+k)\Gamma(p-b)\Gamma(2-a+p+c)}{\Gamma(k+p-b)\Gamma(2-a+k+p+c)\Gamma(p+1)\Gamma(p+1-b+c)}a_{k+p}A^{k+p}\\ &= A^p - \sum_{k=1}^{\infty}C_{k+p}A^{k+p},\end{aligned} \tag{2.14}$$

where

$$C_{k+p} = \frac{\Gamma(k+p+1-b+c)\Gamma(p+1+k)\Gamma(p-b)\Gamma(2-a+p+c)}{\Gamma(k+p-b)\Gamma(2-a+k+p+c)\Gamma(p+1)\Gamma(p+1-b+c)}a_{k+p}.$$

Hence, for convenience, we put

$$\Psi(k) = \frac{\Gamma(k+p+1-b+c)\Gamma(p+k)\Gamma(p-b)\Gamma(2-a+p+c)}{\Gamma(k+p-b)\Gamma(2-a+k+p+c)\Gamma(p+1)\Gamma(p+1-b+c)} \quad (k \in \mathbb{N}). \tag{2.15}$$

Then, by the constraints of the hypotheses, we note that $\Psi(k)$ is non-increasing for integers $k \geq 1$ and we have $0 < \Psi(k) < 1$, i.e.,

$$0 < \frac{\Gamma(k+p+1-b+c)\Gamma(p+1+k)\Gamma(p-b)\Gamma(2-a+p+c)}{\Gamma(k+p-b)\Gamma(2-a+k+p+c)\Gamma(p+1)\Gamma(p+1-b+c)} < k+p.$$

Also, by the relation

$$\frac{k+p}{p+1}\{1+\beta[p-\gamma+\alpha(p+1)]\} \leq k+\beta[p-\gamma+\alpha(p+k)] \quad (k \geq 1), \tag{2.16}$$

we get

$$\sum_{k=1}^{\infty} \frac{k+p}{p+1}\{1+\beta[p-\gamma+\alpha(p+1)]\}\Psi(k)a_{k+p} \leq \sum_{k=1}^{\infty}\{k+\beta[p-\gamma+\alpha(k+p)]\}\Psi(k)a_{k+p}$$
$$\leq \beta(p-\gamma+\alpha p), \tag{2.17}$$

that is,

$$\sum_{k=1}^{\infty}(k+p)\Psi(k)a_{k+p} \leq \frac{\beta(p+1)(p-\gamma+\alpha p)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}}.$$

Therefore, in the same way with the proof of Theorem 1, we obtain

$$\begin{aligned}\|D_{0,A}^{a,b,c}f(A)\| &\leq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)}\|A\|^{p-b-1} \\ &\quad + \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)}\|A\|^{p-b}\sum_{k=1}^{\infty}(k+p)\Psi(k)a_{k+p} \\ &\leq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)}\|A\|^{p-b-1} \\ &\quad + \frac{\beta(p+1)(p-\gamma+\alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}\Gamma(p-b)\Gamma(2-a+p+c)}\|A\|^{p-b}\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}\|D_{0,A}^{a,b,c}f(A)\| &\geq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)}\|A\|^{p-b-1} \\ &\quad - \frac{\beta(p+1)(p-\gamma+\alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}\Gamma(p-b)\Gamma(2-a+p+c)}\|A\|^{p-b}.\end{aligned} \tag{2.19}$$

REMARK. (i) By the proof of Theorem 1, if we put

$$F(z) \equiv J_{0,z,p}^{a,b,c}f(z) := \frac{\Gamma(p+1-b)\Gamma(a+p+1+c)}{\Gamma(p+1-b+c)\Gamma(p+1)} z^b I_{0,z}^{a,b,c}f(z), \tag{2.20}$$

then we know that $J_{0,z,p}^{a,b,c}$ is a fractional linear operator from $\mathcal{S}_0(\alpha,\beta,\gamma,p)$ to itself.

(ii) From (1.17) it is easy to see that Theorem 1 and Theorem 2 are generalizations of [4, Theorem 3.1 and Theorem 3.2].

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