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ON FUNCTIONAL REPRESENTATION OF LOCALLY m-PSEUDOCONVEX ALGEBRAS

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ABSTRACT. Functional representation of a topological algebra (A,T) has been studied in many papers under various assumptions for the topology T on A. Usually the image $\hat{A}$ of the Gelfand map has been equipped with the compact-open topology. This leads, in several cases, to such kind of difficulties as, for instance, that the Gelfand map is not necessarily continuous or that the compact-open topology is not of the same type as the topology T. In this paper, we study locally m-pseudoconvex algebras and provide $\hat{A}$ with such kind of topology that the above two claims are fulfilled. By using this representation the description of the closed ideals of (A,T) is studied.

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1. Introduction. Let A be a commutative locally m-pseudoconvex topological algebra over the complex numbers. Let $\mathfrak{Q} = \{q_\lambda \mid \lambda \in \Lambda\}$ be a family of multiplicative k_λ-homogeneous seminorms defining a Hausdorff topology $T(\mathfrak{Q})$ on A (k_λ-homogeneity means that $q_\lambda(\alpha x) = |\alpha|^{k_\lambda} q_\lambda(x)$ for all $x \in A$ and $\alpha \in \mathbb{C}$). If $k_\lambda = 1$, for all $\lambda \in \Lambda$, then $(A, T(\mathfrak{Q}))$ is a locally m-convex algebra. If A has unit element e, we assume that $q_\lambda(e) = 1$, for all $\lambda \in \Lambda$. If A does not have unit and $A_e = s\{(x,\alpha) \mid x \in A, \alpha \in \mathbb{C}\}$ is the corresponding algebra with adjoint unit, we can define for each $\lambda \in \Lambda$ the k_λ-homogeneous seminorm Q_λ on A_e by $Q_\lambda(x,\alpha) = q_\lambda(x) + |\alpha|^{k_\lambda}$, $(x,\alpha) \in A_e$. Denote by $T(\mathfrak{Q}_e)$ the topology on A_e defined by these seminorms. Now, $Q_\lambda(x,0) = q_\lambda(x)$ for all $x \in A$ and $(A, T(\mathfrak{Q}))$ can be considered as a closed maximal ideal of $(A_e, T(\mathfrak{Q}_e))$. It must be noted that if the seminorms q_λ satisfy some condition (for example they can be square preserving), then the seminorms Q_λ defined above do not necessarily satisfy this condition. In those cases (if it is possible), we define the seminorms Q_λ so that the seminorms Q_λ satisfy this additional condition and $Q_\lambda(x,0) = q_\lambda(x)$ for all $x \in A$.

Let $\Delta(A)$ be the set of all nontrivial continuous complex homomorphisms on A. We assume that $\Delta(A)$ is nonempty. If $x \in A$ is given, then its Gelfand transform is defined by

$$\hat{x}(\tau) = \tau(x), \quad \tau \in \Delta(A). \tag{1}$$

We equip the space $\Delta(A)$ with the weak topology generated by the functions $\hat{A} = \{\hat{x} \mid x \in A\}$. This is called the Gelfand topology. The set $\Delta(A)$ can also be equipped with the so called hull-kernel topology. (See [9] or [15].)

If $q_\lambda \in \mathcal{Q}$, then we can define a mapping p_λ on A by

$$p_\lambda(x) = \left[q_\lambda(x)\right]^{1/k_\lambda}, \quad x \in A. \tag{2}$$

Let p be any (1-homogeneous) seminorm on A and let $k \in (0,1]$ be fixed. If we define a mapping q on A by $q(x) = [p(x)]^k$, $x \in A$, we can see that q is a k-homogeneous seminorm on A. However, the converse of this is not true in general. There are locally m-pseudoconvex algebras $(A, T(\mathcal{Q}))$ such that $p_\lambda = q_\lambda^{1/k_\lambda}$ is not a seminorm on A. Namely, the triangle inequality is not necessarily valid for p_λ. But there are many interesting pseudoconvex algebras for which $p_\lambda = q_\lambda^{1/k_\lambda}$ is a seminorm for each $\lambda \in \Lambda$. We say that $(A, T(\mathcal{Q}))$ has the property (LC) if p_λ defined in (2) is a seminorm for every $\lambda \in \Lambda$. We show later that if, for example, each q_λ is square preserving, then $(A, T(\mathcal{Q}))$ has the property (LC). Suppose, now, that $(A, T(\mathcal{Q}))$ has the property (LC). Let $T(\mathcal{P})$ be a topology on A defined by a family $\mathcal{P} = \{p_\lambda \mid \lambda \in \Lambda\}$ of seminorms on A. For any net $\{x_\nu\}$ on $(A, T(\mathcal{Q}))$, we have $x_\nu \to x$, for some $x \in A$, if and only if $x_\nu \to x$ with respect to the topology $T(\mathcal{P})$. Thus, these two topologies on A are equivalent. This means that we have $\Delta(A, T(\mathcal{Q})) = \Delta(A, T(\mathcal{P})) = \Delta(A)$. If $q_\lambda \in \mathcal{Q}$, we denote $N_\lambda = \ker q_\lambda = \{x \in A \mid q_\lambda(x) = 0\}$. Then N_λ is a closed ideal of $(A, T(\mathcal{Q}))$ for each $\lambda \in \Lambda$. Obviously, $N_\lambda = \ker p_\lambda$ and thus, q_λ and p_λ have the same kernel.

Let $A_\lambda = A/N_\lambda$ be the quotient algebra of A modulo N_λ. We denote $x_\lambda = x + N_\lambda$, $x \in A$, $\lambda \in \Lambda$. A_λ is k_λ-normed algebra with a k_λ-norm defined by $\dot{q}_\lambda(x_\lambda) = q_\lambda(x)$, $x_\lambda \in A_\lambda$.

We can define a partial ordering on Λ, as usual, by setting $\lambda \le \mu$ if and only if $p_\lambda \le p_\mu (p_\lambda(x) \le p_\mu(x)$ for all $x \in A)$. If we assume that $\mathcal{P}$ is closed under taking maxima of two of its members, then Λ is a directed set. Note that the condition $\lambda \le \mu$ does not necessarily imply that $q_\lambda \le q_\mu$ as the following example shows.

Example 1. Let $A = C(\mathbb{R})$ and define a family of pseudonorms on A by $\{q_n \mid n \in \mathbb{N}\}$, where $q_n = [\sup_{t\in[-n,n]} |x(t)|]^{1/n}$, $x \in A$ and $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. If we, now, take a function $x \in A$ for which $|x(s)| \le \sup_{t\in[-n,n]} |x(t)|$ for all $s \in [-(n+1), n+1]\backslash[-n,n]$, then we have $n < n+1$, but $q_n(x) > q_{n+1}(x)$.

In this paper, we study the Gelfand representation and ideal structure of $(A, T(\mathcal{P}))$ when the topology $T(\mathcal{P})$ is locally m-pseudoconvex. Functional representation of topological algebras have been considered in many papers (started with Banach algebras and then extended to the more general ones). Usually the image $\hat{A}$ of the Gelfand mapping has been equipped with the compact-open topology (or in some cases with Michael's topology). See, for example, [9, 12, 14, 17, 16]. However, there is some difficulties with the continuity of the Gelfand mapping or these topologies are not of the same type as the original topology of A. We use, in this paper, such kind of topology for $\hat{A}$ that there is no such kind of problems. We do not use the projective limits at all either and, therefore, the assumption that the family $\mathcal{P}$ is directed is not necessary. More important is the assumption that $T(\mathcal{P})$ is a Hausdorff topology. Also, note that if A is without unit, then the role of the complex homomorphism τ_∞, where $\tau_\infty(x) = 0$ for all $x \in A$, is more complicated here than in the normed case. What kind of difficulties in this case, has been described in [10].

2. Basic results. Now, we study the structure of the carrier space $\Delta(A)$.

Lemma 1. *Suppose that a locally m-pseudoconvex algebra $(A, T(\mathcal{Q}))$ does not have unit and that it satisfies the property (LC). Then there is a family $\mathcal{Q}_e$ of seminorms on A_e such that $(A_e, T(\mathcal{Q}_e))$ has the property (LC).*

Proof. For each $q_\lambda \in \mathcal{Q}$, we can define a k_λ-homogeneous seminorm Q_λ on A_e by

$$Q_\lambda(x,\alpha) = \left(p_\lambda(x) + |\alpha|\right)^{k_\lambda}, \quad (x,\alpha) \in A_e, \tag{3}$$

where $p_\lambda(x) = [q_\lambda(x)]^{1/k_\lambda}$. Then we can see that the mapping P_λ defined by $P_\lambda(x,\alpha) = [Q_\lambda(x,\alpha)]^{1/k_\lambda}$, $(x,\alpha) \in A_e$, is a seminorm on A_e for each $\lambda \in \Lambda$. So, if we take $\mathcal{Q}_e$ the family of seminorms defined in (3), then $(A_e, T(\mathcal{Q}_e))$ has the property (LC). □

Let $\mathcal{Q}_e = \{Q_\lambda \mid \lambda \in \Lambda\}$ be the family of seminorms on A_e defined by (3) and let $\mathcal{P}_e = \{P_\lambda \mid \lambda \in \Lambda\}$. Since $Q_\lambda = P_\lambda^{k_\lambda}$, we can see that the topologies $T(\mathcal{Q}_e)$ and $T(\mathcal{P}_e)$ of A_e are equivalent and we have $\Delta(A_e, T(\mathcal{Q}_e)) = \Delta(A, T(\mathcal{P}_e)) = \Delta(A_e)$. Note that we also have $Q_\lambda(x, 0) = q_\lambda(x)$ for all $x \in A$ and $\lambda \in \Lambda$.

Lemma 2. *Let $(A, T(\mathcal{Q}))$ be a locally m-pseudoconvex algebra without unit. Let I be a closed (proper) regular ideal of $(A, T(\mathcal{Q}))$. Then there is a unique closed ideal I_e of $(A_e, T(\mathcal{Q}_e))$ such that $I = I_e \cap A$ and $I_e \not\subset A$.*

Proof. Let u be identity in A modulo I. If we, now, take $I_e = \{y \in A_e \mid uy \in I\}$, then $I = I_e \cap A$, I_e is unique, and $I_e \not\subset A$. (See [13].) Let (y_ν) be a net in I_e for which $y_\nu \to y$ for some $y \in A_e$. Then $uy_\nu \to uy$. Since each $uy_\nu \in I$, we can see that $uy \in \operatorname{cl}(I) = I$ and, thus, I_e is closed. □

Corollary 1. *Let $(A, T(\mathcal{Q}))$ be as in Lemma 2. Then for each $\tau \in \Delta(A)$ there is a unique $\tau_e \in \Delta(A_e)$ such that $\tau_{e|A} = \tau$.*

If $(A, T(\mathcal{Q}))$ does not have unit, then, by Corollary 1, for each $\tau \in \Delta(A)$ there is a unique extension τ_e on A_e. So, the mapping $\tau \mapsto \tau_e, \tau \in \Delta(A)$, is a bijection from $\Delta(A)$ onto $\Delta(A)_e = \{\tau_e \mid \tau \in \Delta(A)\}$. If $\tau \in \Delta(A)$, then we clearly have $\tau_e(x,\alpha) = \tau(x) + \alpha$, $(x,\alpha) \in A_e$. Now, if we identify each $\tau \in \Delta(A)$ with its extension τ_e, we can formally write $\Delta(A) \subset \Delta(A_e)$. Let τ_∞ be an element of $\Delta(A_e)$ for which $\tau_\infty(x,\alpha) = \alpha$, $(x,\alpha) \in A_e$. If $\omega \in \Delta(A_e)$ is given, then either $\omega_{|A} \in \Delta(A)$ or $\omega = \tau_\infty$. Thus, we have $\Delta(A_e) = \Delta(A) \cup \{\tau_\infty\}$. (To be more exact each element $\omega \in \Delta(A_e)$ is either an extension of some $\tau \in \Delta(A)$ or $\omega = \tau_\infty$.) If (τ_ν) is a given net on $\Delta(A)$ for which $\tau_\nu \to \tau$ for some $\tau \in \Delta(A)$, then $\hat{x}(\tau_\nu) \to \hat{x}(\tau)$ for all $x \in A$. Thus, $(x,\alpha)\hat{}((\tau_\nu)_e) = \hat{x}(\tau_\nu) + \alpha \to \hat{x}(\tau) + \alpha = (x,\alpha)\hat{}(\tau_e)$ for all $(x,\alpha) \in A_e$. This means that $\Delta(A)$ is homeomorphic to $\Delta(A)_e$. Thus, $\Delta(A)$ and $\Delta(A)_e$ can be identified as topological spaces. So, we can see that $\Delta(A_e) = \Delta(A)_e \cup \{\tau_\infty\} = \Delta(A) \cup \{\tau_\infty\}$ within a homeomorphism. Note that $\Delta(A) \cup \{\tau_\infty\}$ is not a one-point compactification of $\Delta(A)$. To see more about the structure of the carrier space $\Delta(A_e)$, see [10] where a locally m-convex case without unit has been studied.

Let I be an ideal of A. The set $h(I) = \{\tau \in \Delta(A) \mid \hat{x}(\tau) = 0, x \in I\}$ is called the hull of I. The kernel of a nonempty subset E of $\Delta(A)$ is defined by $k(E) = \{x \in A \mid \hat{x}(\tau) = 0, \tau \in E\}$ and for the empty set, we define $k(\emptyset) = A$.

If $(A,T(\mathcal{P}))$ is a commutative locally m-convex algebra with unit, it is known (see [4]) that the family $\{h(N_\lambda) \mid \lambda \in \Lambda\}$ is a compact cover of $\Delta(A)$, which is closed under finite unions. Obviously, this result holds also for locally m-pseudoconvex algebras with unit and with the property (LC). Suppose $(A,T(\mathcal{Q}))$ is without unit and has the property (LC). Let $M_\lambda = \{(x,\alpha) \in A_e \mid Q_\lambda(x,\alpha) = 0\}$. From the definition of Q_λ, it follows that $(x,\alpha) \in M_\lambda$ if and only if $q_\lambda(x) = 0$ and $\alpha = 0$. Thus, $M_\lambda = \{(x,0) \in A_e \mid x \in N_\lambda\}$. Denote by h_e the hull-operation on $\Delta(A_e)$. Now, $h_e(M_\lambda) = \{\omega \in \Delta(A_e) \mid (x,\alpha)\hat{\ }(\omega) = 0,\ (x,\alpha) \in M_\lambda\} = \{\tau \in \Delta(A) \cup \{\tau_\infty\} \mid \hat{x}(\tau) = 0,\ x \in N_\lambda\} = h(N_\lambda) \cup \{\tau_\infty\}$. Since $h_e(M_\lambda)$ is compact for each $\lambda \in \Lambda$, we can see that $h(N_\lambda)$ is either a locally compact or a compact subset of $\Delta(A)$ depending on whether τ_∞ is an isolated point of $h(N_\lambda) \subset \Delta(A) \cup \{\tau_\infty\}$ or not. Thus, we can write.

Lemma 3. *Let $(A,T(\mathcal{Q}))$ be a commutative locally m-pseudoconvex algebra with the property (LC). If A has unit, then $h(N_\lambda)$ is compact for each $\lambda \in \Lambda$. If A does not have unit, then each $h(N_\lambda)$, $\lambda \in \Lambda$, is locally compact and $h(M_\lambda)$ is a one-point compactification of $h(N_\lambda)$ for each $\lambda \in \Lambda$. If τ_∞ is an isolated point of $h(N_\lambda)$, then $h(N_\lambda)$ is compact.*

Note that $h(N_\lambda) = \{\tau \in \Delta(A) \mid \tau \text{ is } p_\lambda\text{-continuous}\}$. Now, we can prove the main result of this section.

Theorem 1. *Let $(A,T(\mathcal{Q}))$ be a commutative locally m-pseudoconvex algebra with the property (LC). Then the family $\{h(N_\lambda) \mid \lambda \in \Lambda\}$ forms a locally compact cover of $\Delta(A)$ which is closed under finite unions. Furthermore, if $\tau \in h(N_\lambda)$, then $|\hat{x}(\tau)|^{k_\lambda} \le q_\lambda(x)$ for all $x \in A$.*

Proof. For each $\lambda \in \Lambda$, take $p_\lambda = q_\lambda^{1/k_\lambda}$. If $\tau \in \Delta(A)$, then there is a $\lambda \in \Lambda$ and some positive constant M such that $|\tau(x)| \le Mp_\lambda(x)$ for all $x \in A$. Thus, if $x \in N_\lambda$ then $|\tau(x)| \le Mp_\lambda(x) = 0$ and we can see that $\tau \in h(N_\lambda)$. This shows that $\Delta(A) = \cup_{\lambda\in\Lambda} h(N_\lambda)$. Each $h(N_\lambda)$ is locally compact by Lemma 3. Furthermore, the family $\{h(N_\lambda) \mid \lambda \in \Lambda\}$ is closed under finite unions since we assumed that the family $\mathcal{P}$ is directed. If $\tau \in h(N_\lambda)$, then we have, by [4], $|\tau(x)| \le p_\lambda(x)$ for all $x \in A$. So, $|\hat{x}(\tau)|^{k_\lambda} = |\tau(x)|^{k_\lambda} \le [p_\lambda(x)]^{k_\lambda} = q_\lambda(x)$ for all $x \in A$. □

By Theorem 1, $\{h(N_\lambda) \mid \lambda \in \Lambda\}$ forms a locally compact cover of the carrier space $\Delta(A)$. In the literature, the corresponding cover in the locally m-convex case with unit has been constructed by using the polars of the neighborhoods of zero. (See [16, 15] or [9].) But it is important to notice that the role of the element τ_∞ differs in the locally m-pseudoconvex case if we compare it with the normed case.

3. On locally m-pseudoconvex function algebras. Let X be a completely regular space. The algebra $C(X)$ of all continuous complex-valued functions can be equipped by several kinds of topologies. Usually, the so called compact-open topology is defined by the family $\mathcal{P}(\mathcal{K}(X)) = \{p_K \mid K \in \mathcal{K}(X)\}$ of seminorms, where $p_K(x) = \sup_{t\in K} |x(t)|$ for each $x \in C(X)$ and $K \in \mathcal{K}(X)$ with $\mathcal{K}(X)$ the family of all compact subsets of X. For our purposes, it is, however, better to consider a more general topology on $C(X)$. Let $\mathcal{K} \subset \mathcal{K}(X)$ be a compact cover of X which is closed under finite unions. Let $\mathcal{P}(\mathcal{K}) =$

$\{p_K \mid K \in \mathcal{K}\}$. Suppose that for each $K \in \mathcal{K}$ there is a fixed $r_K \in (0,1]$ and let $\mathfrak{Q}(\mathcal{K}) = \{q_K \mid K \in \mathcal{K}\}$, where q_K is defined by $q_K = [\sup_{t\in K}|x(t)|]^{r_K}$, $x \in C(X)$. Denote by $T(\mathfrak{Q})$ (correspondingly $T(\mathcal{P})$) the topology on $C(X)$ defined by the family $\mathfrak{Q}(\mathcal{K})$ (correspondingly by $\mathcal{P}(\mathcal{K})$). Then $(C(X),T(\mathfrak{Q}))$ is a locally m-pseudoconvex topological algebra and, correspondingly, $(C(X),T(\mathcal{P}))$ is a locally m-convex algebra. Note that compact-open and point-open topologies of $C(X)$ are special cases of the topology $T(\mathfrak{Q})$. Now, we give some properties of the algebra $(C(X),T(\mathfrak{Q}))$.

Lemma 4. *Let X be a completely regular space. Then*
(i) $\Delta(C(X),T(\mathfrak{Q})) = \{\tau_t \mid t \in X\}$, *where* $\tau_t = x(t)$, $x \in C(X)$.
(ii) *If I is a closed ideal of $(C(X),T(\mathfrak{Q}))$, then $k(h(I)) = I$.*

Proof. These results can be shown like the corresponding results for the algebra $(C(X),T(\mathcal{P}))$. See [4, Lem. 2.1]. □

The condition (ii) of Lemma 4 means that, for each closed ideal I of $(C(X),T(\mathfrak{Q}))$, there is some closed subset E of X such that $I = k(E) = \{x \in C(X) \mid x(t) = 0,\ t \in E\}$.

Lemma 5. *Let B be a symmetric subalgebra of $C(X)$. If B separates the points of X, then* $\mathrm{cl}(B) = C(X)$ *or* $\mathrm{cl}(B) = I_{t_0}$ *for some $t_0 \in X$. (By* $\mathrm{cl}(B)$*, we mean the closure of B with respect to the topology $T(\mathfrak{Q})$).*

Proof. This result can be proved like the corresponding result for the normed or compact-open topologies. See [18]. □

Let t_0 be a given point of X. Denote by $X_0 = X\setminus\{t_0\}$. Furthermore, let $C_\infty(X_0) = \{g|_{X_0} \mid g \in C(X), g(t_0) = 0\}$. Let $\mathcal{K}_0 = \{K\setminus\{t_0\} \mid K \in \mathcal{K}\}$, where $\mathcal{K}$ is a compact cover of X which is closed under finite unions. We denote $K_0 = K\setminus\{t_0\}$, $K \in \mathcal{K}$. So, each $K_0 \in \mathcal{K}_0$ is locally compact and, thus, $\mathcal{K}_0$ forms a locally compact cover of X_0 which is closed under finite unions. If $x \in C_\infty(X_0)$, then $x|_{K_0} \in C_0(K_0)$ = the set of all bounded continuous complex valued functions on X_0 vanishing at infinity. Note the difference between $C_\infty(X_0)$ and $C_0(K_0)$. The algebra $C_\infty(X_0)$ can also contain unbounded functions and the space X_0 is only completely regular but not locally compact. Obviously, K is a one point compactification of K_0 for each $K_0 \in \mathcal{K}_0$. Note that K_0 is compact if and only if t_0 is not an element of K. Now, we provide the algebra $C_\infty(X_0)$ with a topology given by the following family of seminorms $\mathfrak{Q}_0 = \{q_{K_0} \mid K_0 \in \mathcal{K}_0\}$, where $q_{K_0}(x) = [\sup_{t\in K_0}|x(t)|]^{r_{K_0}}$, $x \in C_\infty(X_0)$. For each $K_0 \in \mathcal{K}_0$, $r_{K_0} \in (0,1]$ is fixed. Denote this topology by $T(\mathfrak{Q}_0)$. Now, we give some properties of the algebra $(C_\infty(X_0),T(\mathfrak{Q}_0))$.

The following properties of the algebra are easy to verify.

Lemma 6. $\Delta(C_\infty(X_0)) = \{\tau_t \mid t \in X_0\}$. *Furthermore,* $\Delta(C_\infty(X_0))$ *and X_0 are homeomorphic.*

Lemma 7. *Let B be a subalgebra of $C_\infty(X_0)$. If B is symmetric and for each $t \in X_0$ there is $x \in B$ such that $x(t) \neq 0$, then* $\mathrm{cl}(B) = C_\infty(X_0)$.

Lemma 8. *Let I be a closed (proper) ideal of $(C_\infty(X_0),T(\mathfrak{Q}_0))$. Then there is a closed subset E of X_0 such that $I = k(E) = \{x \in C_\infty(X_0) \mid x(t) = 0,\ t \in E\}$. Furthermore, I is regular if and only if t_0 is an isolated point of E.*

Proof. Let I be a closed ideal of $(C_\infty(X_0),T(\mathfrak{Q}_0))$. Let $K_0 \in \mathcal{K}_0$ be arbitrary. Denote by $I_{K_0} = \{x|_{K_0} \mid x \in I\}$. We show that I_{K_0} is an ideal of $(C_0(K_0), T(q_{K_0}))$. Note that q_{K_0} defines a r_{K_0}-homogeneous norm on $C_0(K_0)$. So, let $g \in I_{K_0}$ and $f \in C(K_0)$ be given. Now, f can also be considered as a continuous function on K if we define $f(t_0) = 0$. Since K is compact, there is an extension, say $y \in C(X)$, such that $y|_{X_0} \in C_\infty(X_0)$ and $y|_{K_0} = f$. Since I is an ideal of $C_\infty(X_0)$, we have $gy \in I$ and, thus, $gf = (gy)|_{K_0} \in I_{K_0}$. Obviously, I_{K_0} is a subspace of $C_0(K_0)$. Thus, I_{K_0} is an ideal of $C_0(K_0)$. Let $E = \cap_{f \in I} Z(f)$, where $Z(f)$ designates the zero set of f. It can be shown that $\text{cl}(I_{K_0}) = k(E \cap K_0)$, where cl is a closure in $C_0(K_0)$ with respect to the topology $T(q_{K_0})$. (See the proof of [15, Lem. 1.5, p. 221–222].) Now, it is easy to see that, for each $x \in k(E)$ and $K_0 \in \mathcal{K}_0$ and given $\epsilon > 0$, there is some $y \in I$ such that $q_{K_0}(x - y) < \epsilon$. This implies that $k(E) \subset I$. Since we trivially have $I \subset k(E)$, it follows that $I = k(E)$. Suppose that I is regular. Now, I can be considered as a closed ideal of $(C(X), T(\mathfrak{Q}))$. By Lemma 2, there is a closed ideal I_1 of $(C(X), T(\mathfrak{Q}))$ such that $I = I_1 \cap C_\infty(X_0)$ and $I_1 \not\subset C(X_0)$. By Lemma 4, I is of the form $I = \{x \in C(X) \mid x(t) = 0, t \in E\}$ for some closed subset E of X. Since $I_1 \not\subset C_\infty(X_0)$, it follows that $t_0 \notin E$. Because E is closed, it follows that t_0 is an isolated point of E. If t_0 is an isolated point of E, then there is an element $u \in C_\infty(X_0)$ such that $0 \le u(t) \le 1$, for every $t \in X_0$, $u(t) = 1$, $t \in E$, and $u(t_0) = 0$. Now, u is identity in $C_\infty(X_0)$ modulo I and, thus, I is regular. □

By Lemma 6, each closed ideal I of $(C_\infty(X_0), T(\mathfrak{Q}_0))$ is of the form $I = k(E) = \{x \in C_\infty(X_0) \mid x(t) = 0, t \in E\}$ for some closed subset E of X_0. Obviously, $C_\infty(X_0)$ can be considered as a maximal closed ideal of $(C(X), T(\mathfrak{Q}))$. Now, we give an example of proper closed subalgebra B of some $(C(X), T(\mathfrak{Q}))$, such that B is not an ideal of $C(X)$, B does not have unit and $\Delta(B) = \Delta(C(X))$.

Example 2. Let $\mathbb{R}$ be the set of reals, equipped with the usual topology, and let $B = \{x \in C(\mathbb{R}) \mid \lim_{t\to\infty} x(t) = 0\}$. We can define the topology on $C(\mathbb{R})$ and B by the sequence $\mathfrak{Q} = \{q_n \mid n \in \mathbb{N}\}$ of seminorms, where $q_n(x) = [\sup_{t\in[-n,n]} |x(t)|]^{1/n}$, $x \in C(\mathbb{R})$ or B. Obviously, B is a proper subalgebra of $C(\mathbb{R})$ which is not an ideal and B does not have unit. It is easy to see that $\Delta(C(\mathbb{R})) = \Delta(B) = \{\tau_t \mid t \in \mathbb{R}\}$. Note that we could have also provided B with an equivalent topology defined by the family $Q' = \{q'_n \mid n \in \mathbb{N}\}$ of seminorms, where $q'_n(x) = [\sup_{[-n,\infty)} |x(t)|]^{1/n}$, $x \in B$. If $N'_n = \ker q'_n$, then $B/N'_n = C_0([-n,\infty))$ within a $1/n$-homogeneous isometrical isomorphism. Obviously, $h(N'_n) = \{\tau_t \mid t \in [-n,\infty)\}$. Note that Stone-Weierstrass theorem holds for $(B, T(\mathfrak{Q})) = (B, T(\mathfrak{Q}'))$. So, if B' is a symmetric subalgebra of B that separates the points of X and, for each $t \in X$, there is $x \in B'$ such that $x(t) \ne 0$, then $\text{cl}(B') = B$. Clearly, $(B, T(\mathfrak{Q}'))$ has the property of nuclear hullity. Furthermore, B_e is the functions of B plus all the constant functions. The carrier space $\Delta(B_e)$ is homeomorphic to $\mathbb{R} \cup \{\infty\}$. Thus, $B_e \ne C(\mathbb{R})$.

4. On Gelfand representation of locally m-pseudoconvex algebras. There are two basic methods to study the structure of locally m-convex algebras. These are projective limits and functional representation. For projective (or inverse) limits of topological algebras, see [1, 9, 15], or [16]. In this paper, we study only functional representation. Functional representation of a commutative topological algebra (A, T) has been studied in several papers under various assumptions with the topology T. See, for example, [2, 4, 7, 9, 11, 14, 15, 17, 16, 19] or [21]. Usually (at least in the case where T is given by the family of submultiplicative seminorms), the image $\hat{A}$ of the Gelfand mapping has been endowed either with a compact-open topology or with a topology of compact convergence on equicontinuous subsets of $\Delta(A)$ (this is the so called Michael's topology; see, e.g., [15]). The problem is that the Gelfand mapping is not necessarily continuous with respect to these topologies. Now, when we provide the image $\hat{A}$ with a topology, we will require two properties for this topology. First, it must be of the same type as the topology of A. Second, the Gelfand mapping must be continuous with respect to this topology. In [4], A was endowed with a locally m-convex topology and the algebra $\hat{A}$ was equipped with the topology of compact convergence on the hulls of the kernels of the seminorms defining the topology on A. The use of hulls suits for our topology better since by the well-known result for normed algebras, we have $(A/N_\lambda)\hat{}\subset C_0(h(N_\lambda))$. It must be noted that the Gelfand mapping is automatically continuous with respect to this topology on $\hat{A}$. This topology is also useful in describing the ideal structure of A. See, for example, [7], where the corresponding topology has been used for the vector valued function algebras. In this paper, we extend the results obtained in [4] in such a way that A does not necessarily have unit element and the topology on A is locally m-pseudoconvex. Functional representation of the so called p-Banach algebras has been studied in [21]. Even though the case where $T(\mathcal{P})$ has the property (LC) could be treated also as the case where $T(\mathcal{P})$ is locally m-convex, we study the Gelfand representation in locally m-pseudoconvex form to get the exact description of these type of algebras. Note that if the seminorms defining the topology on A are not submultiplicative, then functional representation of (A,T) is more complicated. Some particular cases of such kind of algebras have been studied in [9, 11, 19]. Also, in these structures, the use of hulls to get the Gelfand representation is very useful.

Let $(A,T(\mathfrak{Q}))$ be a commutative locally m-pseudoconvex algebra with the property (LC). Then by Theorem 1, $\Delta(A)=\cup\{h(N_\lambda)\mid \lambda\in\Lambda\}$ and if $\tau\in h(N_\lambda)$, then $|\hat{x}(\tau)|^{k_\lambda}\le q_\lambda(x)$ for each $x\in A$. Let $\lambda\in\Lambda$. We can define a k_λ-homogeneous seminorm $\hat{q}_\lambda$ on $\hat{A}$ by $\widehat{q_\lambda}(\hat{x})=[\sup_{\tau\in h(N_\lambda)}|\hat{x}(\tau)|]^{k_\lambda}$, $x\in A$. Denote by $T(\hat{\mathfrak{Q}})$ the topology on $\hat{A}$ defined by the family $\hat{\mathfrak{Q}}=\{\hat{q}_\lambda\mid\lambda\in\Lambda\}$. Now, we can easily prove

Theorem 2. *Let $(A,T(\mathfrak{Q}))$ be a commutative locally m-pseudoconvex algebra with the property (LC). Then $\hat{q}_\lambda(\hat{x})\le q_\lambda(x)$ for each $x\in A$ and $\lambda\in\Lambda$ and the Gelfand mapping $x\mapsto\hat{x}$, $x\in A$, is a continuous homomorphism from $(A,T(\mathfrak{Q}))$ onto $(\hat{A},T(\hat{\mathfrak{Q}}))$.*

By Theorem 2, each commutative locally m-pseudoconvex algebra with the property (LC) can be considered as a subalgebra of some locally m-pseudoconvex function algebra.

If A does not have unit, then $\hat{A}\subset C_\infty(\Delta(A))=\{g|_{\Delta(A)}\mid g\in C(\Delta(A_e)),\, g(\tau_\infty)=0\}$. If A has unit, then $\hat{A}\subset C(\Delta(A))$. We say that A is full if $\hat{A}=C_\infty(\Delta(A))$ (or $\hat{A}=C(\Delta(A))$ in the case A has unit).

Let $(A,T(\mathfrak{Q}))$ be a commutative locally pseudoconvex algebra. If $q_\lambda(x^2)=q_\lambda(x)^2$ for all $x\in A$ and $\lambda\in\Lambda$, then we say that $(A,T(\mathfrak{Q}))$ is a square algebra.

It can be shown that each square preserving k_λ-homogeneous seminorm is automatically submultiplicative. See [5, 6].

Lemma 9. *If $(A,T(\mathfrak{Q}))$ is a commutative locally pseudoconvex square algebra, then it has the property (LC).*

Proof. Let $\lambda \in \Lambda$ be arbitrary. Now, the quotient algebra $A_\lambda = A/N_\lambda$ is a commutative k_λ-homogeneous normed algebra with a norm $\dot{q}_\lambda$, where $\dot{q}_\lambda$ is defined by $\dot{q}_\lambda(x_\lambda) = q_\lambda(x)$, $x_\lambda \in A_\lambda$. By [21, Thm. 4.8] we have

$$\left[\sup_{\tau_\lambda \in \Delta(A_\lambda)} |\hat{x}_\lambda(\tau_\lambda)|\right]^{k_\lambda} = \lim_{n\to\infty} \sqrt[2^n]{\dot{q}_\lambda(x_\lambda^{2^n})} = \dot{q}_\lambda(x_\lambda) = q_\lambda(x) \quad \text{for all } x \in A. \tag{4}$$

If $\tau \in h(N_\lambda)$, then we can define an element τ_λ of $\Delta(A_\lambda)$ by $\tau_\lambda(x_\lambda) = \tau(x)$, $x_\lambda \in A_\lambda$. The mapping $\tau \mapsto \tau_\lambda, \tau \in h(N_\lambda)$, is a homeomorphism from $h(N_\lambda)$ onto $\Delta(A_\lambda)$. (See [15].) This implies that

$$\left[\sup_{\tau_\lambda \in \Delta(A_\lambda)} |\hat{x}(\tau_\lambda)|\right]^{k_\lambda} = \left[\sup_{\tau \in h(N_\lambda)} |\hat{x}(\tau)|\right]^{k_\lambda} = \hat{q}_\lambda(\hat{x}). \tag{5}$$

Thus, we can see that $q_\lambda(x) = \hat{q}_\lambda(\hat{x})$ for all $x \in A$. Since $\hat{q}_\lambda^{1/k_\lambda}$ is a seminorm, we can see that $(A,T(\mathfrak{Q}))$ also has the property (LC). □

Furthermore, we have

Lemma 10. *Let $(A,T(\mathfrak{Q}))$ be a commutative locally pseudoconvex square algebra without unit. Then there is a family $\mathfrak{Q}_e = \{Q_\lambda \mid \lambda \in \Lambda\}$ of pseudonorms on A_e such that $(A_e,T(\mathfrak{Q}_e))$ is a square-algebra and $Q_\lambda(x,0) = q_\lambda(x)$ for all $x \in A$.*

Proof. For a given $q_\lambda \in \mathfrak{Q}$, define Q_λ on A_e by

$$Q_\lambda(x,\alpha) = \sup_{q_\lambda(y)\le 1} q_\lambda(xy+\alpha y) \quad \text{for all } (x,\alpha) \in A_e. \tag{6}$$

Now, it is easy to see that each such Q_λ is square preserving k_λ-homogeneous seminorm on A_e. Furthermore, $Q(x,0) = \sup_{q_\lambda(y)\le 1} q_\lambda(xy)$. So, to see that Q_λ is an extension of q_λ, we have to show that $q_\lambda(x) = \sup_{q_\lambda(y)\le 1} q_\lambda(xy)$ for all $x \in A$ and $\lambda \in \Lambda$. If $q_\lambda(x) = 0$, then the right side of the equation is also zero. So, we have equality. If $q_\lambda(x) \ne 0$, then

$$\sup_{q_\lambda(y)\le 1} q_\lambda(xy) \ge q_\lambda\left(x\frac{x}{q_\lambda(x)}\right) = \frac{1}{q_\lambda(x)} q_\lambda(x^2) = q_\lambda(x). \tag{7}$$

So, we have $q_\lambda(x) \le \sup_{q_\lambda(y)\le 1} q_\lambda(xy)$. The inequality in the other direction is trivial, since q_λ is submultiplicative. So, Q_λ satisfies the required conditions.

When $(A,T(\mathfrak{Q}))$ is a locally convex square algebra without unit, we always provide A_e with the topology defined in Lemma 10. Note that if we denote $M_\lambda = \ker Q_\lambda$, then we have $h_e(M_\lambda) = h(N_\lambda) \cup \{\tau_\infty\}$. Since each Q_λ is square preserving, we have, in this case,

$$Q_\lambda(x,\alpha) = \sup_{\tau \in h(N_\lambda)\cup\{\tau_\infty\}} |\hat{x}(\tau)+\alpha|^{k_\lambda} \quad \text{for all } (x,\alpha) \in A_e \text{ and } \lambda \in \Lambda. \tag{8}$$

If $(A,T(\mathfrak{Q}))$ is a locally pseudoconvex square algebra, then $(A,T(\mathfrak{Q}))$ and $(\hat{A},T(\hat{\mathfrak{Q}}))$ can be identified as topological algebras. Thus, the only locally pseudoconvex square algebras are subalgebras of the function algebra $(C(X), T(\mathfrak{Q}))$ for some completely regular space X. □

The properties of locally m-convex (= locally convex) square algebras have been studied in [14, 8, 5, 6].

Let $(A, T(\mathfrak{Q}))$ be a commutative locally pseudoconvex algebra with an involution $x \mapsto x^*$, $x \in A$. We say that $(A, T(\mathfrak{Q}))$ is a star algebra if

$$q_\lambda(xx^*) = q_\lambda(x)^2 \quad \text{for all } x \in A \text{ and } \lambda \in \Lambda. \tag{9}$$

It is easy to see that a pseudoconvex star algebra is also a square algebra.

Lemma 11. *Let $(A, T(\mathfrak{Q}))$ be a locally pseudoconvex star algebra without unit. Then there is a family $\mathfrak{Q}_e$ of seminorms on A_e such that $(A_e, T(\mathfrak{Q}_e))$ is a star algebra and $Q_\lambda(x, 0) = q_\lambda(x)$ for all $x \in A$ and $\lambda \in \Lambda$.*

Proof. This result can be shown similarly to the proof of Lemma 10. Also, we can apply the proof of [12, Thm. 2.3]. □

If $(A, T(\mathfrak{Q}))$ is a locally convex star algebra without unit, we always provide A_e with the topology defined in Lemma 10. It can be shown that if $(A, T(\mathfrak{Q}))$ is complete, then $(A_e, T(\mathfrak{Q}_e))$ is also complete. See [12].

Theorem 3. *Let $(A, T(\mathfrak{Q}))$ be a commutative locally pseudoconvex star algebra. Then $\mathrm{cl}(\hat{A}) = C_0(\Delta(A))$, where cl means the closure with respect to the topology $T(\hat{\mathfrak{Q}})$. In particular, if $(A, T(\mathfrak{Q}))$ is complete, then $\hat{A} = C_\infty(\Delta(A))$. (Note that $C_\infty(\Delta(A)) = C(\Delta(A))$ if A has unit.)*

Proof. The functions of $\hat{A}$ separate the point of $\Delta(A)$ and it follows from condition (9) that $\hat{A}$ is a symmetric subset of $C(\Delta(A))$. Obviously, for each $\tau \in \Delta(A)$, there is $x \in A$ such that $\hat{x}(\tau) \neq 0$. Now, we can apply either Lemma 6 or Lemma 8 to show that $\mathrm{cl}(\hat{A}) = C(\Delta(A))$ (if A has unit) or $\mathrm{cl}(\hat{A}) = C_\infty(\Delta(A))$ (in the case A is without unit). If $(A, T(\mathfrak{Q}))$ is complete, then $(\hat{A}, T(\hat{\mathfrak{Q}}))$ is complete too. Thus, $\hat{A}$ is, in this case, a closed subset of $(C_\infty(\Delta(A)), T(\hat{\mathfrak{Q}}))$ from which it follows that A is full. □

Theorem 3 is a generalization of the corresponding result for locally convex star algebras. See [16] or [12]. Note that in both of these papers, the projective limits were used to prove this result. Also, see [4].

Theorem 4. *Suppose that $(A, T(\mathfrak{Q}))$ is full. Then $(A_e, T(\mathfrak{Q}_e))$ is also full.*

Proof. Suppose that $\hat{A} = C_\infty(\Delta(A))$. Let $g \in C(\Delta(A_e)) = C(\Delta(A) \cup \{\tau_\infty\})$ be given. Now, we have $g(\tau_\infty) < \infty$. Thus, if we define a function s on $\Delta(A_e)$ by $s(\tau) = g(\tau) - g(\tau_\infty)$, $\tau \in \Delta(A_e)$, then $s|_{\Delta(A)} \in C_\infty(\Delta(A))$. Since A is full, there is $x \in A$ such that $\hat{x} = s|_{\Delta(A)}$. Now, if we take $\alpha = g(\tau_\infty)$, we can see that $(x, \alpha)\hat{} = g$. □

Corollary 2. *Let $(A, T(\mathfrak{Q}))$ be a full locally pseudoconvex star algebra. Then the quotient algebra $(A_\lambda, T(\{\dot{q}_\lambda\}))$ is complete, for each $\lambda \in \Lambda$.*

Proof. We show that the mapping $x_\lambda \mapsto \hat{x}|_{h(N_\lambda)}$, $x_\lambda \in A_\lambda$, is an isometric isomorphism from $(A_\lambda, T(\{\dot{q}_\lambda\}))$ onto $(C_0(h(N_\lambda)), T(\{\hat{q}_\lambda\}))$. Since $q_\lambda(x) = \hat{q}_\lambda(\hat{x})$ for each $x \in A$, we can see that the mapping $x_\lambda \mapsto \hat{x}|_{h(N_\lambda)}$, $x_\lambda \in A_\lambda$, is isometric. We show that it is a surjection. Let $g \in C_0(h(N_\lambda))$ be arbitrary. We can consider g also as a continuous function on $h(N_\lambda) \cup \{\tau_\infty\}$ if we define $g(\tau_\infty) = 0$. Since $h(N_\lambda) \cup \{\tau_\infty\}$ is compact, there is a function $G \in C(\Delta(A_e))$ such that $G|_{h(N_\lambda)\cup\{\tau_\infty\}} = g$. From the conditions $G(\tau_\infty) = g(\tau_\infty) = 0$ and $\hat{A}_e = C(\Delta(A_e))$, it follows that there is $x \in A$ such that $\hat{x} = G$. So $\hat{x}|_{h(N_\lambda)} = g$ which proves the surjectivity. Now, our result follows from the fact that $(C_0(h(N_\lambda)), T(\{\hat{q}_\lambda\}))$ is, as a k_λ-Banach algebra, complete. □

Example 3. Let X be a completely regular space and let t_0 be a given point of X. Let $(C(X), T(Q))$ and $(C_\infty(X_0), T(Q_0))$ be as in Lemmas 4 and 6. Let $M_K = \{x \in C(X) \mid q_K(x) = 0\}$ and $N_{K_0} = \{x \in C_\infty(X_0) \mid q_{K_0}(x) = 0\}$. By Lemmas 4 and 6, we have $\Delta(C(X)) = \{\tau_t \mid t \in K\}$, $\Delta(C_\infty(X_0)) = \{\tau_t \mid t \in K_0\}$, $h(M_K) = \{\tau_t \mid t \in K\}$, and $h(N_{K_0}) = \{\tau_t \mid t \in K_0\}$. Obviously, both of the algebras above are square algebras. Let $g \in C_\infty(\Delta(C_\infty(X_0)))$ be arbitrary. Now, each $\tau \in \Delta(C_\infty(X_0))$ is of the form $\tau = \tau_t$ for some $t \in X_0$. So, we can define a function x on X_0 by $x(t) = g(\tau) = g(\tau_t)$, $t \in X_0$. The function x is continuous and we have $\hat{x} = g$. Thus, $C_\infty(X_0)\hat{} = C_\infty(\Delta(C_\infty(X_0)))$. Similarly, we get $C(X)\hat{} = C(\Delta(C(X)))$. Note that we did not assume that algebra $(C_\infty(X_0), T(\mathfrak{Q}_0))$ (or $(C(X), T(\mathfrak{Q})))$ is complete. Thus, it may happen that $\hat{A} = C_\infty(\Delta(A))$ without the assumption that $(A, T(\mathfrak{Q}))$ is complete. It is easy to see that $C(X)/M_K$ is isometrically isomorphic to $C(K)$ and, correspondingly, $C_\infty(X_0)/N_{K_0}$ is isometrically isomorphic to $C_0(K_0)$ (topologies in these two algebras are defined by r_K-homogeneous supnorm). Thus, those two quotient algebras are complete.

Next, we study the ideal structure of locally convex star algebras.

Theorem 5. *Let $(A, T(\mathfrak{Q}))$ be a commutative full locally pseudoconvex star algebra. Then $k(h(I)) = I$ for all closed ideal of $(A, T(\mathfrak{Q}))$. Furthermore, I is regular if and only if τ_∞ is an isolated point of $h(I)$.*

Proof. We can apply Lemmas 5 or 8. □

Corollary 3. *Let $(A, T(\mathfrak{Q}))$ be a complete locally pseudoconvex star algebra. Then $k(h(I)) = I$ for all closed ideal I of $(A, T(\mathfrak{Q}))$.*

We say that a locally m-pseudoconvex algebra is normal if the functions of $\hat{A}$ separate any two disjoint closed subsets of $\Delta(A)$. (This means that, for each pair E_1 and E_2 of disjoint closed subsets of $\Delta(A)$, there is $x \in A$ such that $\hat{x}(\tau) = 1, \tau \in E_1$ and $\hat{x}(\tau) = 0, \tau \in E_2$.)

Lemma 12. *Suppose that $(A, T(\mathfrak{Q}))$ is a commutative normal m-pseudoconvex algebra without unit. Then $(A_e, T(\mathfrak{Q}_e))$ is also normal.*

Proof. Let E_1 and E_2 be two closed disjoint subsets of $\Delta(A_e)$. Now, $E_i \cap \Delta(A) = E_i \setminus \{\tau_\infty\} = F_i, i = 1,2$ is a pair of closed disjoint subsets of $\Delta(A)$. Note that $F_i = E_i$ if $\tau_\infty \notin E_i$. Since $(A, T(\mathfrak{Q}))$ is normal, there is $x \in A$ such that $\hat{x}(\tau) = 1$ if $\tau \in F_1$ and $\hat{x}(\tau) = 0$ if $\tau \in F_2$. This means that $(x, 0) \in A_e$ separates the sets E_1 and E_2. Note that we must have $\hat{x}(\tau) = 0, \tau \in E_i$ if $\tau_\infty \in E_i$. So, as above, we must assume that $\tau_\infty \notin E_1$. □

Now, it is easy to see that the following lemma is valid.

Lemma 13. *Suppose that $(A,T(\mathfrak{Q}))$ is a commutative normal m-pseudoconvex algebra. Then $\Delta(A)$ and $\Delta(A_e)$ are normal topological spaces.*

Next, we prove a result which is known for locally convex algebras with unit (see [4]) and for B^*-algebras (see [3]).

Theorem 6. *Let $(A,T(\mathfrak{Q}))$ be a commutative full normal pseudoconvex star algebra. If I_1 and I_2 are closed ideals of $(A,T(\mathfrak{Q}))$, then I_1+I_2 is either a closed ideal of $(A,T(\mathfrak{Q}))$ or $I_1+I_2=A$.*

Proof. We study only the case where A does not have unit. It suffices to show that $k(h(I_1+I_2))\subset I_1+I_2$. Let $x\in k(h(I_1+I_2))$ be arbitrary. We have $h(I_1+I_2)=h(I_1)\cap h(I_2)$. Let g be a function on $E=h(I_1)\cup h(I_2)\cup\{\tau_\infty\}$ defined by

$$g(\tau)=\begin{cases}\hat{x}(\tau), & \text{if } \tau\in h(I_1),\\ 0, & \text{if } \tau\in h(I_2)\cup\{\tau_\infty\}.\end{cases}\tag{10}$$

Now, g is continuous on the closed set $E\subset\Delta(A_e)=\Delta(A)\cup\{\tau_\infty\}$. By Lemma 12, $\Delta(A_e)$ is a normal topological space. So, by Tietze extension theorem, there is a function $G\in C(\Delta(A_e))$ such that $G|_E=g$. By Theorem 4, we have $\hat{A}_e=C(\Delta(A_e))$. So, there is $(y,\alpha)\in A_e$ such that $(y,\alpha)\hat{}=G$. Since $0=g(\tau_\infty)=\hat{y}(\tau_\infty)+\alpha=\alpha$, we can see that $g=\hat{y}|_E$. Thus, $(x-y)\hat{}(\tau)=\hat{x}(\tau)-\hat{y}(\tau)=\hat{x}(\tau)-g(\tau)=0$ for all $\tau\in h(I_1)$. This implies that $x-y\in k(h(I_1))=I_1$. Similarly, we can see that $y\in k(h(I_2))=I_2$. So, $x=(x-y)+y\in I_1+I_2$. This implies that I_1+I_2 is a closed ideal of $(A,T(\mathfrak{Q}))$. If $h(I_1)\cap h(I_2)=\emptyset$, then $I_1+I_2=k(\emptyset)=A$. □

Thus, we get

Corollary 4. *Let $(A,T(\mathfrak{Q}))$ be as in Theorem 6. If I_1 and I_2 are closed ideals of $(A,T(\mathfrak{Q}))$ for which $h(I_1)\cap h(I_2)=\emptyset$, then $I_1+I_2=A$.*

Corollary 5. *Let $(A,T(\mathfrak{Q}))$ be as in Theorem 6. Then, for each closed ideal $I\subset A$, we have $I=\cap\{I+N_\lambda\mid\lambda\in\Lambda_0\}$, where $\Lambda_0=\{\lambda\in\Lambda\mid h(I)\cap h(N_\lambda)\neq\emptyset\}$.*

Proof. If I is a closed ideal of $(A,T(\mathfrak{Q}))$, then, for each $\lambda\in\Lambda$, we have $I+N_\lambda=k(h(I+N_\lambda))=k(h(I)\cap h(N_\lambda))$. Now, our result follows from Lemma 12. □

By Theorem 3, each complete locally pseudoconvex star algebra is full. On the other hand, there are also noncomplete full locally convex star algebras (by Example 5). Therefore, the assumption that A is full is more general than the assumption that $(A,T(\mathfrak{Q}))$ is complete.

5. On quotient algebras. Let $(A,T(\mathfrak{Q}))$ be a commutative locally m-pseudoconvex algebra with the property (LC). If I is a (proper) closed ideal of $(A,T(\mathfrak{Q}))$, then the quotient algebra A/I is also a locally m-pseudoconvex algebra, if we define the topology on A/I by the family $\dot{\mathfrak{Q}}=\{\dot{q}_\lambda\mid\lambda\in\Lambda\}$ of pseudonorms, where $\dot{q}_\lambda$ is defined by $\dot{q}_\lambda(x+I)=\inf_{y\in I}q_\lambda(x+y)$ for $x+I\in A/I$ and $\lambda\in\Lambda$. Denote this topology by $T(\dot{\mathfrak{Q}})$.

Furthermore, let $\dot{N}_\lambda = \ker \dot{q}_\lambda$. We can define for each $\omega \in \Delta(A/I)$ the mapping τ_ω on A by $\tau_\omega(x) = \omega(x+I)$, $x \in A$. It is easy to see that the mapping $\omega \mapsto \tau_\omega$, $\omega \in \Delta(A/I)$, is a homeomorphism from $\Delta(A/I)$ onto $h(I)$.

The following lemma is easy to prove.

Lemma 14. *Suppose that $(A, T(\mathfrak{Q}))$ has the property (LC) and let I be a closed ideal of $(A, T(\mathfrak{Q}))$. Then also $(A/I, T(\dot{\mathfrak{Q}}))$ has the property (LC).*

Theorem 7. *Let $(A, T(\mathfrak{Q}))$ be a commutative locally m-pseudoconvex algebra with the property (LC) and let I be a closed ideal of $(A, T(\mathfrak{Q}))$ for which $h(I) \neq \emptyset$. Then*

$$\{\tau_\omega \mid \omega \in h(\dot{N}_\lambda)\} = h(I) \cap h(N_\lambda). \tag{11}$$

Proof. Let ω be an arbitrary element of $h(\dot{N}_\lambda)$. By Theorem 1 and Lemma 14, we have $|\omega(x+I)|^{k_\lambda} \leq \dot{q}_\lambda(x+I) \leq q_\lambda(x)$ for each $x \in A$. Thus, if $u \in I$ and $v \in N_\lambda$ are given, then $|\tau_\omega(u+v)|^{k_\lambda} = |\omega(u+v+I)|^{k_\lambda} = |\omega(v+I)|^{k_\lambda} \leq q_\lambda(v) = 0$ which shows that $\tau_\omega \in h(I+N_\lambda) = h(I) \cap h(N_\lambda)$. Thus, $\{\tau_\omega \mid \omega \in h(\dot{N}_\lambda)\} \subset h(I) \cap h(N_\lambda)$.

To prove the converse, let $\tau \in h(I) \cap h(N_\lambda)$ be arbitrary. Now, $\tau \in h(I)$ and, thus, there is some $\omega \in \Delta(A/I)$ such that $\tau = \tau_\omega$. We must show that $\omega \in h(\dot{N}_\lambda)$. Let $x+I \in \dot{N}_\lambda$ be arbitrary. Then for each $\epsilon > 0$, there is some $y_0 \in I$ such that $q_\lambda(x+y_0) < \epsilon$. Now,

$$|\omega(x+I)|^{k_\lambda} = |\tau_\omega(x)|^{k_\lambda} = |\tau(x)|^{k_\lambda} = |\tau(x+y_0)|^{k_\lambda} \leq q_\lambda(x+y_0) < \epsilon. \tag{12}$$

This proves that $h(I) \cap h(N_\lambda) \subset \{\tau_\omega \mid \omega \in h(\dot{N}_\lambda)\}$.

Note that it may happen that $h(I) \cap h(N_\lambda) = \emptyset$. □

Corollary 6. *Let $(A, T(\mathfrak{Q}))$ and I be as in Theorem 6. Then the mapping $\omega \mapsto \tau_\omega$, $\omega \in h(\dot{N}_\lambda)$, is a homeomorphism from $h(\dot{N}_\lambda)$ onto $h(I) \cap h(N_\lambda)$.*

Next, we consider the functional representation of the commutative locally m-pseudoconvex algebra $(A/I, T(\dot{\mathfrak{Q}}))$. The Gelfand function $(x+I)^\wedge$ on $\Delta(A/I)$ satisfies the equation

$$(x+I)^\wedge(\omega) = \hat{x}(\tau_\omega), \quad \omega \in \Delta(A/I). \tag{13}$$

Since $h(I) = \{\tau_\omega \mid \omega \in \Delta(A/I)\}$, we can see that $(x+I)^\wedge = \hat{x}|_{h(I)}$ for each $x+I \in A/I$. Thus, $(A/I)^\wedge \subset C_\infty(h(I))$. Let $E_\lambda = h(I) \cap h(N_\lambda)$. Now, we can define the topology on $(A/I)^\wedge$ by using the family $\hat{\dot{\mathfrak{Q}}} = \{\hat{\dot{q}}_\lambda \mid \lambda \in \Lambda\}$ of seminorms, where $\hat{\dot{q}}_\lambda$, is defined by

$$\hat{\dot{q}}_\lambda = \sup_{\tau \in E_\lambda} |\hat{x}(\tau)|^{k_\lambda}, \quad x \in A \text{ and } \lambda \in \Lambda. \tag{14}$$

We, obviously, have

Theorem 8. *Let $(A, T(\mathfrak{Q}))$ be a commutative locally m-pseudoconvex algebra and let I be a closed ideal of $(A, T(\mathfrak{Q}))$ for which $h(I) \neq \emptyset$. Then the Gelfand mapping*

$x+I \mapsto (x+I)\hat{} = \hat{x}|_{h(N_\lambda)}$, $x+I \in A/I$, is a continuous homomorphism from $(A/I, T(\hat{\mathfrak{Q}}))$ into $(C_\infty(h(I)), T(\hat{\hat{\mathfrak{Q}}}))$.

It is easy to see that the Gelfand mapping of A/I is an injection if and only if $k(h(I)) = I$. Now, we give a sufficient condition for the property $(A/I)\hat{} = C_\infty(h(I))$.

Theorem 9. *Let $(A, T(\mathfrak{Q}))$ be a normal locally pseudoconvex full star algebra and let I be a closed ideal of $(A, T(\mathfrak{Q}))$. Then the Gelfand mapping of A/I has the properties*
(i) $(A/I)\hat{} = C_\infty(h(I))$.
(ii) $\dot{q}_\lambda(x+I) = \hat{\hat{q}}_\lambda(\hat{x})$ *for each $x+I \in A/I$ and $\lambda \in \Lambda$.*

Proof. To prove (i), let $g \in C_\infty(h(I))$ be arbitrary. We can consider g also as a continuous function on $h(I) \cup \{\tau_\infty\}$ if we define $g(\tau_\infty) = 0$. Now, $h(I) \cup \{\tau_\infty\}$ is a closed subset of the normal space $\Delta(A_e) = \Delta(A) \cup \{\tau_\infty\}$. By Tietze theorem, there is a function $G \in C(\Delta(A_e))$ such that $G|_{h(I)\cup\{\tau_\infty\}} = g$. Since A is full, we have $\hat{A}_e = C(\Delta(A_e))$. So, there is an element $(x,\alpha) \in A_e$ such that $(x,\alpha)\hat{} = G$. From the condition $g(\tau_\infty) = 0$, we get $\alpha = 0$. Thus, there is $x \in A$ for which $\hat{x}|_{h(I)} = g$.

To prove (ii), we first assume that A has unit. Let $x \in A$ and $y \in I$ be arbitrary. Now, $\hat{y}(\tau) = 0$ for all $\tau \in h(I) \cap h(N_\lambda)$. Thus, we get

$$q_\lambda(x+y) = \hat{q}_\lambda(\hat{x}+\hat{y}) \geq \hat{\hat{q}}_\lambda(\hat{x}+\hat{y}) = \hat{\hat{q}}_\lambda(\hat{x}). \tag{15}$$

This implies that

$$\dot{q}_\lambda(x+I) = \inf_{y\in I} q_\lambda(x+y) \geq \hat{\hat{q}}_\lambda(\hat{x}) \quad \text{for all } x+I \in A/I \text{ and } \lambda \in \Lambda. \tag{16}$$

To prove the converse inequality let $x \in A$, $\lambda \in \Lambda$, and $\epsilon > 0$ be given. Let $U_\lambda = \{\tau \in \Delta(A) \mid |\hat{x}(\tau) - \hat{x}(\tau')|^{k_\lambda} < \epsilon$ for some $\tau' \in E_\lambda\}$. Then U_λ is an open subset of $\Delta(A)$ and, obviously, $E_\lambda \subset U_\lambda$. Now, for each $\tau \in U_\lambda$, there is $\tau' \in E_\lambda$ such that $|\hat{x}(\tau)|^{k_\lambda} < |\hat{x}(\tau')|^{k_\lambda} + \epsilon$. This follows from the definition of U_λ and from the obvious fact that $||\hat{x}(\tau)|^{k_\lambda} - |\hat{x}(\tau')|^{k_\lambda}| \leq |\hat{x}(\tau) - \hat{x}(\tau')|^{k_\lambda}$. Similarly, we can get an open neighborhood V of $h(I)$ such that, for each $\tau \in V$, there is some $\tau' \in h(I)$ such that $|\hat{x}(\tau)|^{k_\lambda} < |\hat{x}(\tau')|^{k_\lambda} + \epsilon$. By Urysohn lemma, there is an element $y \in A$ such that $0 \leq \hat{y}(\tau) \leq 1$ for every $\tau \in \Delta(A)$ and $\hat{y}(\tau) = 1$ for each $\tau \in h(I)$ and $\hat{y}(\tau) = 0$, for each $\tau \in \Delta(A)\backslash V$. Let $V_\lambda = V \cap U_\lambda$. Now, we can see that $(xy)\hat{}(\tau) = \hat{x}(\tau)$ for all $\tau \in h(I)$ and, therefore, $x - xy \in k(h(I)) = I$. So, $x + I = xy + I$ and we get

$$\begin{aligned}\dot{q}_\lambda(x+I) = \dot{q}_\lambda(xy+I) \leq q_\lambda(xy) = \hat{q}_\lambda(\hat{x}\hat{y}) = \sup_{\tau\in V_\lambda} |\hat{x}(\tau)\hat{y}(\tau)|^{k_\lambda} \\ = \sup_{\tau\in V_\lambda} |\hat{x}(\tau)|^{k_\lambda} \leq \sup_{\tau\in E_\lambda} |\hat{x}(\tau)|^{k_\lambda} + \epsilon = \hat{\hat{q}}_\lambda(\hat{x}) + \epsilon.\end{aligned} \tag{17}$$

Thus, $\dot{q}_\lambda(x+I) \leq \hat{\hat{q}}_\lambda(x)$, $x + I \in A/I$. Suppose that A does not have unit. Let $I_e = \{(x,0) \in A_e \mid x \in I\}$. Now, we have $(A_e/I_e)\hat{} = C(h(I) \cup \{\tau_\infty\})$. Furthermore, let $F_\lambda = h(I) \cup h(N_\lambda) \cup \{\tau_\infty\}$. Then

$$\dot{q}_\lambda(x+I) = \dot{Q}_\lambda\big((x,0) + I_e\big) = \sup_{\tau\in F_\lambda} |\hat{x}(\tau)|^{k_\lambda} = \sup_{\tau\in E_\lambda} |\hat{x}(\tau)|^{k_\lambda} = \hat{\hat{q}}_\lambda(\hat{x}). \tag{18}$$

□

Theorem 9 is a generalization of the corresponding results for B^*-algebras, (see

[18, Ch. III, Cor. 10] or [20, Thm. 4.2.4]) and for locally convex star algebras with unit (see [4]). There seems to be a mistake in [20] in the proof of Theorem 4.2.4. Namely, it is not possible, in general, to take an element of A such that $\hat{u}(\tau) = 1$ for every $\tau \in h(I)$. This is possible if either I is regular or A has unit.

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