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RINGS WITH MANY IDEMPOTENTS

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ABSTRACT. We introduce a new stable range condition and investigate the structures of rings with many idempotents. These are also generalizations of corresponding results of J. Stock and H. P. Yu.

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In this paper, we examine the properties of rings satisfying idempotent 1-stable range and give one large class of such rings. We show that many useful exchange rings belong to the new class of rings. As an application, we also give a new element-wise characterization of strongly π-regular rings. These are generalizations of many known results.

Throughout, R is an associative ring with identity. $M_n(R)$ denotes the ring of $n\times n$ matrices over R. Let $M_n(R)$ has an identity I_n, and let its group of units be the general linear group $\mathrm{GL}_n(R)$. Set

$$B_{ij}(x) = I_2 + xe_{ij} \quad (i \neq j, 1 \leq i, j \leq 2), \tag{1}$$
$$[\alpha,\beta] = \alpha e_{11} + \beta e_{22},$$

where e_{11}, e_{22} and e_{ij} $(i \neq j, 1 \leq i, j \leq 2)$ are all matrix units.

DEFINITION 1. A ring R is said to satisfy idempotent 1-stable range provided that for any $a,b \in R, aR+bR = R$ implies there exists an idempotent $e \in R$ such that $a+be$ is left invertible in R.

PROPOSITION 2. *The following are equivalent:*
(1) *R satisfies idempotent 1-stable range.*
(2) *For any $a,b \in R, aR+bR = R$ implies there exists an idempotent $e \in R$ such that $a+be \in U(R)$.*

PROOF. (2)⇒(1) is trivial.
(1)⇒(2) Given $aR+bR = R$. Then there exists an idempotent $e \in R$ such that $a+be = u$ is left invertible in R. Assume that $vu = 1$ for some $v \in R$. Then $vR+0R = R$. Thus, we can find an idempotent $f \in R$ such that $v+0\cdot f = v$ is left invertible in R. So v is a unit, and then $a+be$ is a unit. □

COROLLARY 3. *The following are equivalent:*
(1) *R satisfies idempotent 1-stable range.*

(2) *For any $a, b \in R, aR + bR = R$ implies there exists an idempotent $e \in R$ such that $a + be$ is right invertible in R.*

Proof. (1)⇒(2) is clear from Proposition 2.

(2)⇒(1) Given $aR + bR = R$, then there exists an idempotent $e \in R$ such that $a + be = u$ is right invertible. Assume that $uv = 1$ for some $v \in R$. Since

$$vR + (1 - vu)R = R, \tag{2}$$

we can find an idempotent $f \in R$ such that

$$v + (1 - vu)f = w \tag{3}$$

is right invertible in R. Obviously,

$$uw = u(v + (1 - vu)f) = 1. \tag{4}$$

This implies that w is a unit. So $a + be$ is a unit, as required. □

Now we investigate elements in 2-dimensional general linear groups over rings satisfying idempotent 1-stable range. As an application, we shall give an element-wise characterization of such rings.

Theorem 4. *The following are equivalent:*

(1) *R satisfies idempotent 1-stable range.*

(2) *For any $A \in \mathrm{GL}_2(R)$, there exists an idempotent $e \in R$ such that*

$$A = [*, *]B_{21}(*)B_{12}(*)B_{21}(-e). \tag{5}$$

Proof. (1)⇒(2) Given any $A = (a_{ij}) \in \mathrm{GL}_2(R)$. Then we have

$$a_{11}R + a_{12}R = R. \tag{6}$$

So we can find an idempotent $f \in R$ such that

$$a_{11} + a_{12}f = u \in U(R). \tag{7}$$

It is easy to verify that

$$B_{21}\big(-(a_{21} + a_{22}f)u^{-1}\big)AB_{21}(f)B_{12}(-u^{-1}a_{12}) = [u, a_{22} - (a_{21} + a_{22}f)u^{-1}a_{12}]. \tag{8}$$

So

$$A = [*, *]B_{21}(*)B_{12}(*)B_{21}(-f). \tag{9}$$

Let $e = -f$. Thus the result follows.

(2)⇒(1) Given $aR + bR = R$. Then $ax + by = 1$ for some $x, y \in R$. It is easy to verify that

$$\begin{pmatrix} a & by \\ 1 & -x \end{pmatrix} = B_{12}(a)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}B_{12}(-x) \in \mathrm{GL}_2(R). \tag{10}$$

So we can find an idempotent $e \in R$ such that

$$\begin{pmatrix} a & by \\ 1 & -x \end{pmatrix} = [*,*]B_{21}(*)B_{12}(*)B_{21}(-e). \tag{11}$$

Thus, $a+bye = u \in U(R)$. So we can verify the following.

$$\begin{pmatrix} a & b \\ ye & 1 \end{pmatrix} = B_{12}(b)\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} B_{21}(ye) \in \mathrm{GL}_2(R). \tag{12}$$

Consequently, there is an idempotent $f \in R$ such that

$$\begin{pmatrix} a & b \\ ye & 1 \end{pmatrix} = [*,*]B_{21}(*)B_{12}(*)B_{21}(-f), \tag{13}$$

and then

$$\begin{pmatrix} a & b \\ ye & 1 \end{pmatrix} B_{21}(f) = [*,*]B_{21}(*)B_{12}(*). \tag{14}$$

Therefore $a+bf \in U(R)$, as desired. □

Theorem 5. *The following are equivalent:*
(1) *R satisfies idempotent 1-stable range.*
(2) *For any $x,y \in R$, there exists an idempotent $e \in R$ such that $xy-xe+1 \in U(R)$.*

Proof. (1)⇒(2) For any $x,y \in R$,

$$(1+xy)R+(-x)R = R. \tag{15}$$

So we can find an idempotent $e \in R$ such that

$$xy-xe+1 = (1+xy)+(-x)e \in U(R). \tag{16}$$

(2)⇒(1) Given $xy+b=1$, there exists an idempotent $e \in R$ such that

$$(-y)x-(-y)e+1 \in U(R). \tag{17}$$

Let $x-e=a$. Then

$$1-ya = u \in U(R). \tag{18}$$

Clearly, we have

$$x(1-ya)-ba = x-(xy+b)a = x-a = e. \tag{19}$$

So

$$x-ba(1-ya)^{-1} = e(1-ya)^{-1}. \tag{20}$$

From $xy+b=1$, we have

$$\left(x-ba(1-ya)^{-1}\right)y+b\left(1+a(1-ya)^{-1}y\right) = 1. \tag{21}$$

Hence,

$$e(1-ya)^{-1}y+b(1+a(1-ya)^{-1}y) = 1. \tag{22}$$

So

$$e(1-ya)^{-1}y(1-e)+b(1+a(1-ya)^{-1}y)(1-e) = 1-e, \tag{23}$$

and then

$$e+b(1+a(1-ya)^{-1}y)(1-e) = 1-e(1-ya)^{-1}y(1-e). \tag{24}$$

Clearly,

$$1-e(1-ya)^{-1}y(1-e)=\big(1+e(1-ya)^{-1}y(1-e)\big)^{-1}\in U(R). \tag{25}$$

So

$$\begin{aligned} x+b\big(-a(1-ya)^{-1}+(1+a(1-ya)^{-1}y)(1-e)(1-ya)^{-1}\big) \\ &= x-ba(1-ya)^{-1}+b\big(1+a(1-ya)^{-1}y\big)(1-e)(1-ya)^{-1} \\ &= e(1-ya)^{-1}+b\big(1+a(1-ya)^{-1}y\big)(1-e)(1-ya)^{-1} \\ &= \big(1-e(1-ya)^{-1}y(1-e)\big)(1-ya)^{-1}\in U(R). \end{aligned} \tag{26}$$

Therefore R has stable range one.

Given any $A=(a_{ij})\in \mathrm{GL}_2(R)$, there are $h,k\in R$ such that

$$a_{11}h+a_{12}k=1. \tag{27}$$

Since R has stable range one, there exists a $z\in R$ such that

$$a_{11}+a_{12}z=q\in U(R). \tag{28}$$

It is easy to verify that

$$B_{21}\big(-(a_{21}+a_{22}z)q^{-1}\big)AB_{21}(z)B_{12}\big(-q^{-1}a_{12}\big)=\big[q,a_{22}-(a_{21}+a_{22}z)q^{-1}a_{12}\big]. \tag{29}$$

Obviously,

$$a_{22}-(a_{21}+a_{22}z)q^{-1}a_{12}\in U(R), \tag{30}$$

and then we have $m,n\in R$ such that

$$A=[*,*]B_{21}(*)B_{12}(m)B_{21}(n). \tag{31}$$

So there is an idempotent $f\in R$ such that

$$1+m(n+1-f)=v\in U(R). \tag{32}$$

Let $e=1-f$ and $n=e+s$, then $n=-e$. Consequently, we see that

$$A=[*,*]B_{21}(*)B_{12}(m)B_{21}(s)B_{21}(-e). \tag{33}$$

Since $1+ms\in U(R)$, one can verify

$$B_{12}(m)B_{21}(s)=[1+ms,1]B_{21}(s)B_{12}(m)\Big[1,(1+sm)^{-1}\Big], \tag{34}$$

whence

$$A=[*,*]B_{21}(*)B_{12}(*)B_{21}(-e). \tag{35}$$

According to Theorem 4, we complete the proof. □

As an immediately consequence, we now derive the following result which shows that idempotent 1-stable range property is left-right symmetric.

Corollary 6. *The following are equivalent:*
(1) *R satisfies idempotent 1-stable range.*
(2) *For any $a, b \in R$, $Ra + Rb = R$ implies there exists an idempotent $e \in R$ such that $a + eb \in U(R)$.*

Proof. R satisfies idempotent 1-stable range if and only if for any $x, y \in R$, there exists an idempotent $e \in R$ such that

$$xy - xe + 1 = 1 + x(y - e) = u \in U(R). \tag{36}$$

Then a direct computation gives

$$\begin{aligned}&(1 + (y - e)x)(1 - (y - e)(x + (u^{-1} - 1)x))\\&\quad = (1 - (y - e)(x + (u^{-1} - 1)x))(1 + (y - e)x) = 1,\end{aligned} \tag{37}$$

whence we can verify that

$$xy - xe + 1 = 1 + x(y - e) \in U(R) \tag{38}$$

if and only if

$$1 + (y - e)x \in U(R) \tag{39}$$

if and only if

$$x^0 y^0 - x^0 e^0 + 1^0 = 1^0 + x^0(y^0 - e^0) \in U(R^0). \tag{40}$$

Consequently, from Theorem 5, we see that R satisfies idempotent 1-stable range if and only if so does the opposite ring R^0. Hence the result follows. □

Corollary 7. *The following are equivalent:*
(1) *R satisfies idempotent 1-stable range.*
(2) *For any $A \in \mathrm{GL}_2(R)$, there exists an idempotent $e \in R$ such that*

$$A = [*, *]B_{12}(*)B_{21}(*)B_{12}(e).$$

Proof. Replacing A by its inverse A^{-1}, we know that condition (2) can be seen to be equivalent to the following condition: for any $A \in \mathrm{GL}_2(R)$, there exists an idempotent $e \in R$ such that the transpose $A^t = B_{12}(-e)B_{21}(*)B_{12}(*)[*, *]$. In view of Theorem 4, we show that condition (2) is equivalent to the opposite ring R^0 satisfies idempotent 1-stable range. Using Corollary 6, we obtain the result. □

Corollary 8. *The following are equivalent:*
(1) *R satisfies idempotent 1-stable range.*
(2) *Given $ax + b = 1$ in R. Then there exists an idempotent $e \in R$ such that $ae + b \in U(R)$.*
(3) *Given $ax + b = 1$ in R. Then there exists an idempotent $e \in R$ such that $ex + b \in U(R)$.*

Proof. (1)⇒(2) Given $ax + b = 1$ in R. Then $bR + aR = R$. So there exists an idempotent $e \in R$ such that $ae + b \in U(R)$, as asserted.

(2)⇒(1) For any $x, y \in R$, we have

$$(-x)y + (1 + xy) = 1. \tag{41}$$

So we can find an idempotent $e \in R$ such that

$$(-x)e + (1 + xy) \in U(R). \tag{42}$$

That is,

$$xy - xe + 1 \in U(R). \tag{43}$$

Therefore the result follows from Theorem 5.

(1)⇔(3) is obvious by the left-right symmetry of idempotent 1-stable range condition. □

Theorem 9. *The following are equivalent:*

(1) *R satisfies idempotent 1-stable range.*

(2) *$R/J(R)$ satisfies idempotent 1-stable range and idempotents can be lifted modulo $J(R)$.*

Proof. (1)⇒(2) Given any $x + J(R), y + J(R) \in R/J(R)$. Since R satisfies idempotent 1-stable range, by virtue of Theorem 5, there is an idempotent $e \in R$ such that $xy - xe + 1 \in U(R)$. Thus we have

$$(x + J(R))(y + J(R)) - (x + J(R))(e + J(R)) + (1 + J(R)) \in U\left(\frac{R}{J(R)}\right) \tag{44}$$

with

$$e + J(R) = (e + J(R))^2 \in \frac{R}{J(R)}. \tag{45}$$

Using Theorem 5, we show that $R/J(R)$ satisfies idempotent 1-stable range.

Given any $a \in R$. We have $aR + (-1)R = R$. So there exists an idempotent $e \in R$ such that $a - e = u$, and then $a = e + u$. Thus R is a clean ring. By [11, Prop. 1.8, Thm. 1.1], R is exchange. Using [11, Cor. 1.3], we see that idempotents can be lifted modulo $J(R)$.

(2)⇒(1) Given $aR + bR = R$. Then we have

$$(a + J(R))\left(\frac{R}{J(R)}\right) + (b + J(R))\left(\frac{R}{J(R)}\right) = \frac{R}{J(R)}. \tag{46}$$

Since $R/J(R)$ satisfies idempotent 1-stable range, there is an idempotent

$$e + J(R) \in \frac{R}{J(R)} \tag{47}$$

such that

$$(a + J(R)) + (b + J(R))(e + J(R)) \in U\left(\frac{R}{J(R)}\right). \tag{48}$$

As idempotents can be lifted modulo $J(R)$, we may assume $e = e^2 \in R$. On the other hand, there is some $v \in R$ such that

$$v(a + be) - 1 \in J(R). \tag{49}$$

Hence $a + be$ is left invertible, as desired. □

Example 10. Every local ring satisfies idempotent 1-stable range.

Proof. Since R is local, $R/J(R)$ is a division ring. Let

$$S = \frac{R}{J(R)}. \tag{50}$$

Given $aS + bS = S$ with $a, b \in S$. If $a = 0$, then $bS = S$. So $a + b \cdot 1 = b$ is right invertible in S. If $a \neq 0$, then $a + b \cdot 0 = a$ is a unit in S. By virtue of Corollary 3, we show that $S = R/J(R)$ satisfies idempotent 1-stable range. Since R is a local ring, idempotents can be lifted modulo $J(R)$. From Theorem 9, the result follows. □

In general, every ring satisfying idempotent 1-stable range has stable range one, but the converse is not true as the following shows.

Example 11. Let $R = \{m/n \in \mathbb{Q} \mid 2 \nmid n \text{ and } 3 \nmid m (m/n \text{ in lowest terms})\}$. Then R is a semilocal ring, while idempotents do not lift modulo $J(R)$. So R has stable range one, but R does not satisfy idempotent 1-stable range from Theorem 9.

Let R be an associative ring with identity 1. Right R-module A is said to have finite exchange property if for every right R-module K and any two decompositions,

$$K = M \oplus N = \bigoplus_{i \in I} A_i, \tag{51}$$

where $M_R \cong A$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that

$$K = M \oplus \left(\bigoplus_{i \in I} A_i' \right). \tag{52}$$

We call a ring R is a (right) weakly P-exchange ring if every right R-module has finite exchange property (cf. [12]). It is well known that regular rings, right perfect rings and weakly right perfect rings are all weakly P-exchange, while there still exist weakly P-exchange rings which belong to none of the above classes ([12, Ex. 4.6]). R is called to be exchange if right R-module R has finite exchange property. We know that regular rings, π-regular rings, unital C^*-algebras of real rank zero, semiperfect rings, left or right continuous rings and clean rings are all exchange. In [16], H. P. Yu proved that every exchange ring with all idempotents central has stable range one. Now we generalize this result as follows.

Theorem 12. *Let R be a ring with all idempotents central. Then the following are equivalent:*

(1) *R satisfies idempotent 1-stable range.*
(2) *R is a clean ring.*
(3) *R is an exchange ring.*

Proof. (1)⇒(2) Given any $a \in R$. From $aR + (-1)R = R$, we have an idempotent $e \in R$ such that $a + (-1)e = u$, and then $a = e + u$. So R is clean.

(2)⇒(3) is clear from [11, Prop. 1.8, Thm. 2.1].

(3)⇒(1) Assume that R does not satisfies idempotent 1-stable range. By Proposition 2, there exist $a, b \in R$ with $aR + bR = R$, while $a + bp \notin U(R)$ for any $p = p^2 \in R$.

Let $\Omega = \{A \mid A$ is a two-sided ideal of R such that $a + bq$ is not a unit modulo A for any $q = q^2 \in R\}$. It is easy to check that Ω is a nonempty inductive set. By using Zorn's lemma, we have a two-sided ideal Q of R such that it is maximal in Ω.

By the maximality of Q, we show that R/Q is indecomposable as a ring. Given any $x \in R/Q$. Since R is exchange, so is R/Q. By [11, Thm. 1.1], there an idempotent $e \in R/Q$ such that

$$e \in x\left(\frac{R}{Q}\right), \qquad 1-e \in (1-x)\left(\frac{R}{Q}\right), \tag{53}$$

and an idempotent $f \in R/Q$ such that

$$f \in \left(\frac{R}{Q}\right)x, \qquad 1-f \in \left(\frac{R}{Q}\right)(1-x). \tag{54}$$

Since idempotents in R/Q can be lifted modulo Q, we may assume that e and f are both central idempotents in R/Q. So $e = 0$ or $e = 1$ and $f = 0$ or $f = 1$. Thus we see that x or $1-x$ is right invertible in R/Q. Similarly, x or $1-x$ is left invertible.

Assume that $x \in R/Q$ is not invertible. If x is not left invertible in R/Q, then rx is not left invertible for any $r \in R/Q$. Thus $1-rx$ is left invertible, whence rx is left quasi-regular. This shows that $x \in J(R/Q)$. If x is not right invertible in R/Q, similarly to the discussion above, we have $x \in J(R/Q)$. So $J(R/Q) = \{x \in R/Q \mid x$ is not invertible in $R/Q\}$. This implies that R/Q is local. By virtue of Example 10, we claim that R/Q satisfies idempotent 1-stable range, a contradiction. Hence the result follows. □

Theorem 12 shows that exchange rings with all idempotents central satisfy idempotent 1-stable range. Now we give an exchange ring R with noncentral idempotents, while it indeed satisfy idempotent 1-stable range.

Example 13. Let

$$R = \begin{pmatrix} \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}. \tag{55}$$

By [17, Ex. 3.10], R is an exchange ring with noncentral idempotents. According to Theorem 5, we directly verify that R satisfies idempotent 1-stable range.

Recall that a ring R is said to be strongly π-regular if every descending chain of right ideals of the form

$$aR \supseteq a^2R \supseteq a^3R \supseteq \cdots, \quad a \in R \tag{56}$$

becomes stationary. It is well known that every strongly π-regular ring is clean. Now we generalize this fact as follows.

Corollary 14. *Let R be a strongly π-regular ring. If $x, y \in R$ with $xy = yx$, then there exists an idempotent $e \in R$ such that*

$$xy + xe + 1 \in U(R). \tag{57}$$

Proof. Given any $x, y \in R$ with $xy = yx$. Let S be an additive subgroup generated by the set

$$\{x^m y^n \mid m, n \geq 0\}. \tag{58}$$

Then S is a commutative subring of R. By virtue of [3, Cor. 1.10], we can find a commutative strongly π-regular subring T of R which contains S.

By Theorem 12, T satisfies idempotent 1-stable range with $x, y \in T$. Thus we can find

$$f = f^2 \in T \subseteq R \tag{59}$$

such that

$$x(y+1) - xf + 1 \in U(T) \subseteq U(R). \tag{60}$$

Let $e = 1 - f$. Then we have idempotent $e = e^2 \in R$ such that $xy + xe + 1 \in U(R)$, as desired. □

A ring R is said to be right (left) quasi-duo if every maximal right (left) ideal is two-sided. By an argument of H. P. Yu, every weakly P-exchange ring with all idempotents central is right (left) quasi-duo. In general, the converse is not true such as

$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}, \quad \text{where } F \text{ is a field.} \tag{61}$$

Now we give a theorem which guarantees the existence of one large class of rings satisfying idempotent 1-stable range.

Theorem 15. *Let R be a right or left quasi-duo weakly P-exchange ring. Then R satisfies idempotent 1-stable range.*

Proof. By [15, Prop. 2.1(1)], right primitive right quasi-duo rings are division. So every right or left quasi-duo ring has primitive factors artinian. Let Q be a prime ideal of R. Since R is a weakly P-exchange ring, so is R/Q. Similarly to [12, Prop. 4.1(2)], the finite exchange property of $R^{(\mathbb{N})}$ forces $J(R/Q)$ to be T-nilpotent. Assume that

$$0 \neq a \in J\left(\frac{R}{Q}\right). \tag{62}$$

Then there exist $x_1, x_2, \ldots, x_n, \ldots \in R/Q$ such that

$$ax_1a \neq 0, \qquad ax_2ax_1a \neq 0, \ldots, \qquad ax_n \cdots ax_1a \neq 0, \ldots, \tag{63}$$

a contradiction. Thus $J(R/Q) = 0$. So R/Q is an indecomposable exchange ring with primitive factors artinian and $J(R/Q) = 0$. Using [17, Lem. 3.7], we claim that R/Q is simple artinian. Thus R is an exchange ring with prime factors artinian, so it is strongly π-regular. Using [17, Thm. 3.8], we see that $R/J(R)$ is a regular ring with all idempotents central. From Theorem 12, $R/J(R)$ satisfies idempotent 1-stable range. As idempotents can be lifted modulo $J(R)$, we obtain the result from Theorem 9. □

Recall that $p(a) = a$, $p(a,b) = 1 + ab$ and $p(a,b,c) = a + c + abc$ for any $a, b, c \in R$. $W(R)$ denotes the subgroup of $U(R)$ generated by

$$\{p(a,b,c)p(c,b,a)^{-1} \mid p(a,b,c) \in U(R), a, b, c \in R\}, \tag{64}$$

and $V(R)$ denotes the subgroup of $U(R)$ generated by

$$\{p(a,b)p(b,a)^{-1} \mid p(a,b) \in U(R), a,b \in R\}. \tag{65}$$

It is easy to verify that

$$p(a,b,c) = p(a,b)c + p(a), p(a,b,c)p(b,a) = p(a,b)p(c,b,a) \tag{66}$$

and

$$\begin{pmatrix} * & * \\ p(a,b,c) & * \end{pmatrix} = B_{21}(a)B_{12}(b)B_{21}(c). \tag{67}$$

We end this note by investigating Whitehead groups of rings with many idempotents.

Theorem 16. *Let R satisfy idempotent 1-stable range. Then*

$$K_1(R) \cong \frac{U(R)}{V(R)}. \tag{68}$$

Proof. For any $a,b,c \in R$ with $p(a,b,c) \in U(R)$, we see that $p(c,b,a) \in U(R)$. By virtue of Theorem 5, there exists an idempotent $e \in R$ such that $1 + b(c-e) \in U(R)$. Let $c - e = t$. Then $c = t + e$ and $1 + bt \in U(R)$. Observing that

$$\begin{aligned}
\begin{pmatrix} * & * \\ p(a,b,c) & * \end{pmatrix} &= B_{21}(a)B_{12}(b)B_{21}(c) \\
&= \big(B_{21}(a)B_{12}(b)B_{21}(t)\big)B_{21}(e) \\
&= B_{21}(a)[1+bt,1]B_{21}(t)B_{12}(b)\big[1,(1+tb)^{-1}\big]B_{21}(e) \\
&= [1+bt,1]B_{21}(a+t+abt)B_{12}(b)\big[1,(1+tb)^{-1}\big]B_{21}(e) \\
&= \big[1+bt,(1+tb)^{-1}\big]B_{21}\big((1+tb)(a+t+abt)\big)B_{12}\big(b(1+tb)^{-1}\big)B_{21}(e) \\
&= \begin{pmatrix} * & * \\ (1+tb)^{-1}p\big((1+tb)(a+t+abt),b(1+tb)^{-1},e\big) & * \end{pmatrix}.
\end{aligned} \tag{69}$$

Thus we have

$$p(a,b,c) = (1+tb)^{-1}p\big((1+tb)(a+t+abt),b(1+tb)^{-1},e\big). \tag{70}$$

Analogously to [10, Thm. 1.6], we know that

$$\begin{aligned}
p(a,b,c) &\equiv (1+tb)^{-1}p\big(e,b(1+tb)^{-1},(1+tb)(a+t+abt)\big)\big(\operatorname{mod} V(R)\big) \\
&= (1+tb)^{-1}\big(p\big(e,b(1+tb)^{-1}\big)(1+tb)(a+t+abt)+p(e)\big) \\
&= (1+tb)^{-1}\big(p\big(e,b(1+tb)^{-1}\big)p(t,b)p(a,b,t)+p(e)\big) \\
&= (1+tb)^{-1}\big(p\big(e,b(1+tb)^{-1}\big)p(t,b,a)p(b,t)+p(e)\big).
\end{aligned} \tag{71}$$

Similarly, we can verify that

$$\begin{aligned}
p(c,b,a) &= p\big(e,(1+bt)^{-1}b,(t+a+tba)(1+bt)\big)(1+bt)^{-1} \\
&= \big(p\big(e,(1+bt)^{-1}b\big)(t+a+tba)(1+bt)+p(e)\big)(1+bt)^{-1} \\
&= \big(p\big(e,(1+bt)^{-1}b\big)p(t,b,a)p(b,t)+p(e)\big)(1+bt)^{-1}.
\end{aligned} \tag{72}$$

It is easy to check that

$$b(1+tb)^{-1} = (1+bt)^{-1}b. \tag{73}$$

Consequently, we have

$$(1+tb)p(a,b,c) \equiv p(c,b,a)(1+bt)\big(\operatorname{mod} V(R)\big). \tag{74}$$

Thus

$$p(a,b,c)\big(p(c,b,a)\big)^{-1} \in V(R). \tag{75}$$

Therefore we conclude that

$$K_1(R) \cong \frac{U(R)}{W(R)} \cong \frac{U(R)}{V(R)}, \tag{76}$$

as asserted. □

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