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SUFFICIENCY FOR GAUSSIAN HYPERGEOMETRIC FUNCTIONS TO BE UNIFORMLY CONVEX

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ABSTRACT. Let $F(a,b;c;z)$ be the classical hypergeometric function and f be a normalized analytic functions defined on the unit disk $\mathcal{U}$. Let an operator $I_{a,b;c}(f)$ be defined by $[I_{a,b;c}(f)](z) = zF(a,b;c;z) * f(z)$. In this paper the authors identify two subfamilies of analytic functions $\mathcal{F}_1$ and $\mathcal{F}_2$ and obtain conditions on the parameters a,b,c such that $f \in \mathcal{F}_1$ implies $I_{a,b;c}(f) \in \mathcal{F}_2$.

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1. Introduction. Let $\mathcal{A}$ denote the class of all normalized analytic functions $f(z)$ in the unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. For $\beta < 1$ and real η, we let

$$R_\eta(\beta) := \left\{f \in \mathcal{A} : \operatorname{Re}\left[e^{i\eta}(f'(z) - \beta)\right] > 0,\, z \in \mathcal{U}\right\}. \tag{1.2}$$

It is a well-known result that when $\beta \ge 0$ we have $R_\eta(\beta) \subset \mathcal{S}$ and for $\beta < 0$, the functions in $R_\eta(\beta)$ need not be univalent in $\mathcal{U}$. A function f is said to be *uniformly convex* (UCV) if the image of every circular arc γ contained in $\mathcal{U}$, with center also in $\mathcal{U}$, is convex. Analytically, UCV family is characterized as follows:

$$\mathrm{UCV} = \left\{f \in \mathcal{S} : \left|\frac{zf''(z)}{f'(z)}\right| \le \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\},\, z \in \mathcal{U}\right\}, \tag{1.3}$$

see [8].

For $\alpha \ge 0$, let $\mathrm{UCT}(\alpha)$ denote the subfamily of functions $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ in $\mathcal{A}$ which satisfy the condition [11]

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \ge \alpha \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \mathcal{U}, \tag{1.4}$$

for some $\alpha \ge 0$. In this paper we are mainly interested in the Gaussian hypergeometric function $F(a,b;c;z)$ defined by

$$F(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}, \tag{1.5}$$

where $a,b,c \in \mathbb{C}$ with $c \ne 0,-1,-2,\dots$. Here $(a)_0 := 1$ for $a \ne 0$ and if n is a positive integer, $(a)_n$ denotes the ascending factorial notation $(a)_n := a(a+1)\cdots(a+n-1)$. In the exceptional case $c = -p$, where p is a positive integer, $F(a,b;c;z)$ is defined if $a = -m$ or $b = -m$, where $m = 0,1,2,\dots$ and $m \le p$. We also note that if $a = -m$, then we have $(-m)_n = 0$ for all $n \ge m+1$ and therefore in this case $F(a,b;c;z)$ becomes a polynomial of degree m. This observation is to indicate that our main results (see Theorems 2 and 3(iv)) give also geometrical information for polynomials. It is important to point out that the hypergeometric function $F(a,b;c;z)$ can be classified into three cases according to whether $\operatorname{Re}(c-a-b)$ equals zero, negative, or positive. In the last case, the function $F(a,b;c;z)$ is bounded in $\mathcal{U}$ whereas the case $\operatorname{Re}(c-a-b) \le 0$, it is unbounded in $\mathcal{U}$ as it has a pole at $z = 1$ in this case. The asymptotic behaviour of the hypergeometric function $F(a,b;c;x)$ for $\operatorname{Re}(c-a-b) \le 0$ near the singularity $x \to 1$ has been studied in detail by Ponnusamy and Vuorinen [7] and thus improving several known results in the literature, see [2] or [12, p. 299].

For $f \in \mathcal{A}$, we recall the operator $I_{a,b;c}(f)$ of Hohlov [3], which maps $\mathcal{A}$ into itself, defined by

$$[I_{a,b;c}(f)](z) = zF(a,b;c;z) * f(z), \tag{1.6}$$

where $*$ denotes the usual *Hadamard product* (convolution) of power series. When $f(z)$ equals the convex function $z/(1-z)$, then the operator $I_{a,b;c}(f)$ in this case becomes $zF(a,b;c;z)$. We also note that the operator $I_{a,b;c}(f)$ is a natural choice for studying the geometric properties of it because of its interaction with geometric function theory for the special operator popularly known as Bernardi operator. In fact the Bernardi operator is a special case of the convolution operator $I_{a,b;c}(f)$ when $a = 1$, $b = 1+\gamma$, $c = 2+\gamma$ with $\operatorname{Re}\gamma > -1$:

$$[I_{1,1+\gamma;2+\gamma}(f)](z) := B_f(z) = \frac{1+\gamma}{z^\gamma}\int_0^z t^{\gamma-1} f(t)\,dt. \tag{1.7}$$

Here $I_{1,1;2}(f)$ and $I_{1,2;3}(f)$ are known as Alexander and Liberia operators, respectively. In [3] Hohlov determined the conditions to guarantee that $I_{a,b;c}(f)$ is univalent in $\mathcal{U}$ for a function f in $\mathcal{S}$.

In this paper, we consider the following problem: for a given a,b,c such that $\operatorname{Re}(c-a-b) > 0$, we find conditions such that $zF(a,b;c;z) \in \mathrm{UCV}$ or $\mathrm{UCT}(\alpha)$. We also find conditions such that $I_{a,b;c}(f) \in \mathrm{UCV}$ if $f \in R_\eta(\beta)$.

2. Preliminary results.

By using the Gauss summation theorem [9, p. 19, eq. (20)], we immediately have Lemmas 1 and 2

Lemma 1 [6, Lem. 3.2]. *Let $a \in \mathbb{C}\setminus\{0\}$, $c > \max\{0, 2+2\operatorname{Re} a\}$, and*

$$S = \sum_{n=0}^{\infty} \frac{(n+1)^2|(a)_n|^2}{(c)_n(1)_n}. \tag{2.1}$$

Then we have

$$S = \frac{\Gamma(c-2\operatorname{Re} a)\Gamma(c)}{\Gamma(c-a)\Gamma(c-\bar{a})}\left[1 + \frac{|(a)_2|^2}{(c-2-2\operatorname{Re} a)_2} + \frac{3|a|^2}{c-1-2\operatorname{Re} a}\right]. \tag{2.2}$$

Lemma 2 [6, Lem. 3.3]. *Let $a,b \in \mathbb{C}\setminus\{0\}$, $c>0$. Then we have the following:*
(i) *For $a,b>0$, $c>a+b+1$,*

$$\sum_{n=0}^{\infty}\frac{(n+1)(a)_n(b)_n}{(c)_n(1)_n}=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{ab}{c-1-a-b}+1\right]. \tag{2.3}$$

(ii) *For $a,b>0$, $c>a+b+2$,*

$$\sum_{n=0}^{\infty}\frac{(n+1)^2(a)_n(b)_n}{(c)_n(1)_n}=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{(a)_2(b)_2}{(c-2-a-b)_2}+\frac{3ab}{c-1-a-b}\right]. \tag{2.4}$$

(iii) *For $a\neq 1$, $b\neq 1$, and $c\neq 1$ with $c>\max\{0,a+b-1\}$,*

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}=\frac{1}{(a-1)(b-1)}\left[\frac{\Gamma(c+1-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}-(c-1)\right]. \tag{2.5}$$

(iv) *For $a\neq 1$ and $c\neq 1$ with $c>\max\{0,2\operatorname{Re}a-1\}$,*

$$\sum_{n=0}^{\infty}\frac{|(a)_n|^2}{(c)_n(1)_{n+1}}=\frac{1}{|a-1|^2}\left[\frac{\Gamma(c+1-2\operatorname{Re}a)\Gamma(c)}{\Gamma(c-a)\Gamma(c-\bar{a})}-(c-1)\right]. \tag{2.6}$$

(v) *For $a,b>0$, $c>a+b+3$,*

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(n+1)^3(a)_n(b)_n}{(c)_n(1)_n}\\&=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(a)_3(b)_3}{(c-3-a-b)_3}+\frac{6(a)_2(b)_2}{(c-2-a-b)_2}+\frac{7ab}{c-1-a-b}+1\right].\end{aligned} \tag{2.7}$$

(vi) *For $c>\max\{0,2\operatorname{Re}a+3\}$,*

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(n+1)^3|(a)_n|^2}{(c)_n(1)_{n+1}}\\&=\frac{\Gamma(c-2\operatorname{Re}a)\Gamma(c)}{\Gamma(c-a)\Gamma(c-\bar{a})}\left[\frac{|(a)_3|^2}{(c-3-2\operatorname{Re}a)_3}+\frac{6|(a)_2|^2}{(c-2-2\operatorname{Re}a)_2}+\frac{7|a|^2}{c-1-2\operatorname{Re}a}+1\right].\end{aligned} \tag{2.8}$$

Our main results rely on the following lemmas:

Lemma 3 [11, Thm. 1]. *If $\sum_{n=2}^{\infty}n(2n-1)|a_n|\le 1$, then the function of the form (1.1) is in UCV.*

Lemma 4. *Let the function $f(z)$ be of the form (1.1). Then a sufficient condition for f to satisfy $\operatorname{Re} e^{i\eta}(f(z)/z-\beta)>0$ in $\mathcal{U}$ is*

$$\sum_{n=2}^{\infty}|a_n|\le(1-\beta)\cos\eta\quad\left(|\eta|<\frac{\pi}{2},\beta<1\right). \tag{2.9}$$

The condition (2.9) is also necessary if $\eta=0$, and $a_n<0$ for all $n\ge 2$.

Proof. The proof is immediate because

$$\begin{aligned}\operatorname{Re} e^{i\eta}\left(\frac{f(z)}{z}-\beta\right) &= (1-\beta)\cos\eta+\operatorname{Re}\left(e^{i\eta}\sum_{n=2}^{\infty}a_n z^{n-1}\right)\\ &\geq (1-\beta)\cos\eta-\sum_{n=2}^{\infty}|a_n||z|^{n-1}\\ &\geq (1-\beta)\cos\eta-\sum_{n=2}^{\infty}|a_n|\geq 0.\end{aligned} \tag{2.10}$$

The necessary part (with $\eta = 0$) follows upon taking $z \to 1$, since $a_n < 0$ in this case. □

3. Main results.

We begin by proving the following theorem:

Theorem 1. *Let $a, b \in \mathbb{C}\setminus\{0\}$, and $c > |a|+|b|+2$. Then a sufficient condition for the function $zF(a,b;c;z)$ belong to UCV is that*

$$\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+\frac{2(|a|)_2(|b|)_2}{(c-2-|a|-|b|)_2}+\frac{5|ab|}{c-|a|-|b|-1}\right]\leq 2. \tag{3.1}$$

Proof. Set $f(z) = zF(a,b;c;z)$. Then, by using Lemma 3, it suffices to show that

$$T := \sum_{n=1}^{\infty}(n+1)(2n+1)\left|\frac{(a)_n(b)_n}{(c)_n(1)_n}\right|\leq 1. \tag{3.2}$$

From the fact that $|(a)_n| \leq (|a|)_n$, we observe that

$$\begin{aligned} T &\leq \sum_{n=1}^{\infty}(n+1)(2n+1)\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}\\ &= 2\sum_{n=0}^{\infty}(n+1)^2\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}-\sum_{n=0}^{\infty}(n+1)\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}-1\\ &= \frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+\frac{2(|a|)_2(|b|)_2}{(c-2-|a|-|b|)_2}+\frac{5|ab|}{c-|a|-|b|-1}\right]-1,\end{aligned} \tag{3.3}$$

by (i) and (ii) of Lemma 2. Hence, by the condition (3.1), T is less than 1. This completes the proof. □

If, in the proof of Theorem 1, we start with $b = \bar{a}$, then we have the following theorem under a weaker condition on c:

Theorem 2. *Let $a \in \mathbb{C}\setminus\{0\}$, $c > \max\{2+2\operatorname{Re} a, 0\}$, and*

$$\frac{\Gamma(c-2\operatorname{Re} a)\Gamma(c)}{\Gamma(c-a)\Gamma(c-\bar{a})}\left[1+\frac{2|(a)_2|^2}{(c-2-2\operatorname{Re} a)_2}+\frac{5|a|^2}{c-1-2\operatorname{Re} a}\right]\leq 2. \tag{3.4}$$

Then $zF(a,\bar{a};c;z) \in UCV$.

Proof. The proof of this theorem follows in the similar lines of proof of Theorem 1 if we use Lemma 1 and therefore we omit the details. □

Example 1. If we take $a = -2$ in Theorem 2, then the conditions on c in Theorem 2 become $c \geq ((23+\sqrt{745})/2)$. This observation gives the following conclusion: for $c \geq ((23+\sqrt{745})/2)$, the function

$$zF(-2,-2;c;z) = z + \frac{4}{c}z^2 + \frac{2}{c(c+1)}z^3 \tag{3.5}$$

is in the class UCV.

In this way one can easily construct a higher order polynomial function lying in the UCV class.

By using Lemma 4, we obtain

Theorem 3. *Suppose that a,b,c and $\beta < 1$ are related by any one of the following conditions:*

(i) $a,b > 0$, $c > a+b$, *and*

$$\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} - 1 \leq (1-\beta)\cos\eta; \tag{3.6}$$

(ii) $-1 < a < 0$, $b > 0$, $c > b$, *and*

$$\beta \leq 1 - \frac{1}{\cos\eta}\left[1 - \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\right]; \tag{3.7}$$

(iii) $a,b \in \mathbb{C}\setminus\{0\}$, $c > |a|+|b|$, *and*

$$\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1 \leq (1-\beta)\cos\eta; \tag{3.8}$$

(iv) $a \in \mathbb{C}\setminus\{0\}$, $b = \bar{a}$, $c > \max\{0, 2\operatorname{Re} a\}$, *and*

$$\frac{\Gamma(c-2\operatorname{Re} a)\Gamma(c)}{\Gamma(c-a)\Gamma(c-\bar{a})} - 1 \leq (1-\beta)\cos\eta. \tag{3.9}$$

Then we have $\operatorname{Re} e^{i\eta}(F(a,b;c;z)-\beta) > 0$ *for* $z \in \mathcal{U}$.

Proof. By Lemma 4, it suffices to show that

$$\sum_{n=1}^{\infty}\left|\frac{(a)_n(b)_n}{(c)_n(1)_n}\right| \leq (1-\beta)\cos\eta. \tag{3.10}$$

First, we recall the well-known formula (cf. [9, p. 19, eq. (20)])

$$F(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \quad (\operatorname{Re}(c-a-b) > 0), \tag{3.11}$$

If $a,b > 0$ and $c > a+b$, then from using this formula we have

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} - 1. \tag{3.12}$$

Similarly, if $-1 < a < 0$, $b > 0$ and $c > b$ then, by Lemma 2 and the formula (3.11), we obtain that

$$\sum_{n=1}^{\infty}\left|\frac{(a)_n(b)_n}{(c)_n(1)_n}\right| = \frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} = 1 - \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}. \tag{3.13}$$

Thus under the conditions (3.6) and (3.7), the conclusion follows.

On applying the ideas of the proofs of Theorems 1 and 2, we can obtain the required conclusion by assuming the conditions (iii) and (iv), respectively. Therefore we complete the proof. □

Remark 1. From Lemma 4, we observe that the condition (3.7) for $\eta = 0$ is necessary and sufficient for $\operatorname{Re} F(a,b;c;z) > \beta$.

Now, we consider the incomplete Beta function $\phi(a,c;z)$ which is defined by

$$\phi(a,c;z) := zF(1,a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (c \neq 0,-1,-2,\dots, z \in \mathcal{U}). \tag{3.14}$$

Corresponding to the function $\phi(a,c;z)$, Carlson and Shaffer [1] defined a linear operator $\mathcal{L}(a,c)$ on $\mathcal{A}$ by the convolution [1, p. 738 eq. (2.2)]:

$$\mathcal{L}(a,c)f(z) = \phi(a,c;z) * f(z) \quad (f \in \mathcal{A}). \tag{3.15}$$

Clearly,

$$\mathcal{L}(a,c) = I_{a,1;c}. \tag{3.16}$$

We note that $\phi'(a,c;z) = F(a,2;c;z)$. Therefore if we take $b = 2$ in Theorem 3, then by using this observation we have the following corollary:

Corollary 1. *Suppose that a, c, and $\beta < 1$ are related by any one of the following conditions:*

(i) $a > 0$, $c > a+2$, *and*

$$\frac{(c-1)(c-2)}{(c-a-1)(c-a-2)} \leq 1 + (1-\beta)\cos\eta; \tag{3.17}$$

(ii) $-1 < a < 0$, $c > 2$, *and*

$$\beta \leq 1 - \frac{1}{\cos\eta}\left[1 - \frac{(c-1)(c-2)}{(c-a-1)(c-a-2)}\right]; \tag{3.18}$$

(iii) $a \in \mathbb{C}\setminus\{0\}$, $c > |a|+2$, *and*

$$\frac{(c-1)(c-2)}{(c-|a|-1)(c-|a|-2)} \leq 1 + (1-\beta)\cos\eta. \tag{3.19}$$

Then we have $\operatorname{Re} e^{i\eta}(\phi'(a,c;z) - \beta) > 0$ *in* $\mathcal{U}$.

Example 2. From (ii) of Corollary 1 we obtain the following sharp result:

$$\operatorname{Re} \phi'(a,c;z) > \frac{(c-1)(c-2)}{(c-a-1)(c-a-2)} \tag{3.20}$$

for $a \in (-1,0)$ and $c > 2$. This is an improvement of a recent result in [4, Thm. 4].

Next, we establish the following corollary which deals with convolution of functions having their real parts in the half-plane.

Corollary 2. *Let anyone of conditions* (i)–(iv) *of Theorem 3 be satisfied with $\eta = 0$. Then we have*

$$f \in R_\eta(\beta_1) \Longrightarrow I_{a,b;c}(f) \in R_\eta(\beta_2), \tag{3.21}$$

where β_2 is given by

$$1-\beta_2 = 2(1-\beta_1)(1-\beta) \tag{3.22}$$

and β is as in Theorem 3.

Proof. Define $g(z) = [I_{a,b;c}(f)](z) = zF(a,b;c;z) * f(z)$. Then it is clear that

$$g'(z) = F(a,b;c;z) * f'(z). \tag{3.23}$$

By hypothesis $\operatorname{Re} e^{i\eta}(f'(z)-\beta_1) > 0$ and $\operatorname{Re} F(a,b;c;z) > \beta$. Therefore, using Lemma in [5] (see also [10] for the case $\eta = 0$), we see that $\operatorname{Re} e^{i\eta}(g'(z)-\beta_2) > 0$, where β_2 is defined by (3.22). This completes the proof. □

By Lemma 3, we obtain

Theorem 4. *If $a,b \in \mathbb{C}\setminus\{0\}$, $c > |a|+|b|+1$ satisfy the condition*

$$2(1-\beta)\cos\eta\left[\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left(\frac{2|ab|}{c-|a|-|b|-1}+1\right)-1\right] \le 1, \tag{3.24}$$

then the operator $I_{a,b;c}(f)$ maps $R_\eta(\beta)$ into UCV.

Proof. Suppose that the function $f(z)$, defined by (1.1), is in the class $R_\eta(\beta)$. Then by using the standard technique, we can directly get the coefficient estimate

$$|a_n| \le \frac{2}{n}(1-\beta)\cos\eta \tag{3.25}$$

(see [6, eq. (4.1)]). Hence, by Lemma 3, it is enough to show that

$$2(1-\beta)\cos\eta\sum_{n=1}^{\infty}(2n-1)\left|\frac{(a)_n(b_n)}{(c)_n(1)_n}\right| \le 1. \tag{3.26}$$

If we use the hypotheses, this verification is very similar to that of Theorem 3 and therefore we omit the details. □

Corollary 3. *Suppose that $a \in \mathbb{C}\setminus\{0\}$, $c > \max\{0, 1+2\operatorname{Re} a\}$, and satisfies the condition*

$$2(1-\beta)\cos\eta\left[\frac{\Gamma(c-2\operatorname{Re} a)\Gamma(c)}{\Gamma(c-a)\Gamma(c-\bar{a})}\left(\frac{2|a|^2}{c-2\operatorname{Re} a-1}+1\right)-1\right] \le 1. \tag{3.27}$$

Then the operator $I_{a,\bar{a};c}(f)$ maps $R_\eta(\beta)$ into UCV.

Making use of Lemmas 2 and 3, we obtain the following theorem:

Theorem 5. *Let $-1 < a < 0$, $b > 0$, and $c > a+b+1$. Then a necessary and sufficient condition for $zF(a,b;c;z)$ to belong to UCT(α) is that*

$$a+b+1+\left[\frac{(1+\alpha)(a+1)(b+1)}{c-a-b-2}+3+2\alpha\right]|ab| \le c. \tag{3.28}$$

Proof. We write

$$f(z) = zF(a,b;c;z) = z+\sum_{n=2}^{\infty}a_nz^n \tag{3.29}$$

so that, by Lemma 3, it suffices to prove that

$$S := \sum_{n=1}^{\infty} (n+1)\left[(1+\alpha)(n+1)-\alpha\right]|a_{n+1}| \le 1, \tag{3.30}$$

where

$$a_{n+1} = \frac{(a)_n(b)_n}{(c)_n(1)_n}. \tag{3.31}$$

Now

$$\begin{aligned} S &= \sum_{n=1}^{\infty} (n+1)\left[(1+\alpha)(n+1)-\alpha\right]\left|\frac{(a)_n(b)_n}{(c)_n(1)_n}\right| \\ &= \frac{|ab|}{c}\left[\left\{(1+\alpha)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}\right\}\right. \\ &\qquad \left. + (2+\alpha)\left\{\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}\right\} + \sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}\right]. \end{aligned} \tag{3.32}$$

Using Lemma 2((i), (iii)), and the formula (3.11), we find that the sum S can be simplified so that

$$S = \frac{\Gamma(c-a-b-1)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[\left\{\frac{(1+\alpha)(a+1)(b+1)}{c-a-b-2}+3+2\alpha\right\}|ab|-(c-a-b-1)\right]+1 \tag{3.33}$$

which, by the condition (3.28), gives $S \le 1$. This completes the proof. □

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