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A NONLOCAL PARABOLIC SYSTEM WITH APPLICATION TO A THERMOELASTIC PROBLEM

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ABSTRACT. A system modeling the thermoelastic bards contacts is studied. The problem is first transformed into an equivalent nonlocal parabolic systems using a transformation, and then the existence and uniqueness of the solutions are demonstrated via the theoretical potential representation theory of the parabolic equations. Finally some realistic situations in the applications are discussed using the results obtained in this paper.

Keywords and phrases. Nonlocal parabolic systems, thermoelastic bars, contact problem, inequality, existence and uniqueness.

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1. Introduction. In this paper we extend the results obtained in [3, 8] for a nonlocal parabolic system for two dependent variables to a general system for n such variables. The result has application to the problem of thermoelastic contact of n rods. We consider first the existence, uniqueness and continuous dependence of the solutions of the nonlocal parabolic system of equations governing the temperature distribution in the rods.

Then consider the quantity

$$\theta = \left(\theta^1(x,t), \theta^2(x,t), \ldots, \theta^n(x,t)\right)^T, \tag{1.1}$$

where

$$\theta^i_t - c^i\theta^i_{xx} = \frac{b^i}{|\Omega_i|}\frac{d}{dt}\max\{I(\theta)+g, 0\}, \quad x \in \Omega_i, t \in J, \tag{1.2}$$

$$\mu_1\theta^1_x(0,t) + \nu_1\theta^1(0,t) = f_1(t), \quad t \in J, \tag{1.3}$$

$$\mu_n\theta^n_x(1,t) + \nu_n\theta^n(1,t) = f_n(t), \quad t \in J, \tag{1.4}$$

$$K_i\theta^i_x(\ell_{2i-1},t) = K_{i+1}\theta^{i+1}_x(\ell_{2i},t), \quad t \in J, \tag{1.5}$$

$$-K_i\theta^i_x(\ell_{2i-1},t) = m_i\left[\theta^i(\ell_{2i-1},t) - \theta^{i+1}(\ell_{2i},t)\right], \quad t \in J, \tag{1.6}$$

$$\theta^i(x,0) = \theta^i_0(x), \quad x \in \Omega_i, \tag{1.7}$$

where $c^i > 0$, $K_i > 0$, $m_i > 0$, $b^i > 0$, $\mu_1^2 + \nu_1^2 \neq 0$, $\mu_n^2 + \nu_n^2 \neq 0$, and $i = 1,2,3,\ldots,n$, and g, f_1, f_n are known functions. We take

$$J = (0,T), \quad T > 0 \qquad \text{and} \qquad \Omega_i = (\ell_{2i-2}, \ell_{2i-1}), \tag{1.8}$$

where

$$0 = \ell_0 < \ell_1 < \ell_2 < \cdots < \ell_{2n-2} < \ell_{2n-1} = 1. \tag{1.9}$$

The integral operator $I(\theta)$ is defined by

$$I(\theta) = \sum_{i=1}^{n} a_i \int_{\Omega_i} \theta^i(x,t)\,dx, \quad a_i > 0,\, t \in J. \tag{1.10}$$

The quantities $f_1(t)$, $f_n(t)$, $\theta_0^i(x)$ are taken to be known functions.

Definition 1.1. A vector θ is said to be a solution to the problems (1.2)–(1.10) if $\theta^i \in C^2(\Omega_i \times J) \cap L^\infty(\Omega_i \times J)$ and satisfies equation (1.2) together with the initial and boundary conditions almost everywhere.

2. An equivalent problem.

The problem described by equations (1.1)–(1.10) can be reduced to an equivalent problem by setting

$$w^i = \theta^i - \frac{b^i}{|\Omega_i|} \max\{I(\theta)+g,0\}, \quad i = 1,\dots,n. \tag{2.1}$$

Multiplying each of equations (2.1) by a_i in turn and integrating over Ω_i and summing we have

$$I(w) = I(\theta) - \left(\sum_{i=1}^{n} a_i b^i\right) \max\{I(\theta)+g,0\}, \tag{2.2}$$

where $w = (w^1, w^2, \dots, w^n)^T$. If we add g to either side we have

$$I(w)+g = I(\theta)+g - \left(\sum_{i=1}^{n} a_i b^i\right) \max\{I(\theta)+g,0\}. \tag{2.3}$$

Lemma 2.1. *If $Q = 1 - \sum_{i=1}^{n} a_i b^i > 0$ then equation (2.1) has the unique inverse*

$$\theta^i = w^i + \frac{b^i}{|\Omega_i|Q} \max\{I(w)+g,0\}. \tag{2.4}$$

Proof. Equation (2.3) implies that

$$I(w)+g > 0 \Longleftrightarrow I(\theta)+g > 0, \tag{2.5}$$

and the result follows. □

In terms of w the problem described by equations (1.1)–(1.10) may be reformulated for $i = 1,2,\dots,n$ as

$$w_t^i - c^i w_{xx}^i = 0, \quad x \in \Omega_i, \quad t \in J, \tag{2.6}$$

$$\mu_1 w_x^1(0,t) + \nu_1\left(w^1(0,t) + \frac{b_1}{|\Omega_1|Q} \max\{I(w)+g,0\}\right) = f_1(t), \quad t \in J, \tag{2.7}$$

$$\mu_n w_x^n(1,t) + \nu_n\left(w^n(1,t) + \frac{b_n}{|\Omega_n|Q} \max\{I(w)+g,0\}\right) = f_n(t), \quad t \in J, \tag{2.8}$$

$$w_x^{i+1}(\ell_{2i},t) = \frac{K_i}{K_{i+1}} w_x^i(\ell_{2i-1},t), \quad t \in J, \tag{2.9}$$

$$\begin{aligned} w_x^i(\ell_{2i-1},t) = -\frac{m_i}{K_i}\Big\{ & w^i(\ell_{2i-1},t) - w^{i+1}(\ell_{2i},t) \\ & + \frac{H_i}{Q} \max\{I(w)+g,0\}\Big\}, \quad t \in J, \end{aligned} \tag{2.10}$$

$$w^i(x,0) = \theta_0^i(x) + \frac{b_i}{|\Omega_i|Q}\max\{I(w_0)+g,0\}, \quad x \in \Omega_i. \tag{2.11}$$

Since the initial values $\theta_0^i(x)$, $i = 1,2,\dots,n$ are known we take

$$w^i(x,0) = \Phi^i(x), \quad x \in \Omega_i, \tag{2.12}$$

to be known quantities.

3. Preliminaries. In this section we list several classical results from [4] and develop solutions to the problems (2.6)–(2.12). We refer to [4] for proofs of the following lemmas:

Lemma 3.1. *If $\Phi(x) \in C[0,1]$, then*

$$V(x,t,\phi) = \int_0^1 [\Theta(x+\xi,t)+\Theta(x-\xi,t)]\Phi(\xi)\,d\xi \tag{3.1}$$

solves the problems

$$\begin{aligned} V_t &= V_{xx}, \quad 0<x<1,\ t>0,\\ V(x,0) &= \Phi(x), \quad 0<x<1,\\ V_x(0,t) &= V_x(1,t) = 0, \quad t>0. \end{aligned} \tag{3.2}$$

Here we have defined

$$K(x,t) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/4t}, \quad t>0, \tag{3.3}$$

$$\Theta(x,t) = \sum_{n=-\infty}^{\infty} K(x+2n,t), \quad t>0. \tag{3.4}$$

Lemma 3.2. *Let $w(x,t)$ be the solution of*

$$\begin{aligned} w_t &= w_{xx}, \quad 0<x<1,\ t>0,\\ w(x,0) &= \Phi(x), \quad 0<x<1,\\ w(0,t) &= F(t), \qquad w_x(1,t) = G(t), \quad t>0, \end{aligned} \tag{3.5}$$

where $F,G \in C[0,T]$, $T>0$, $\Phi \in C[0,1]$. Then

$$w(x,t) = V(x,t,\Phi) - 2\int_0^t K_x(x,t-s)\psi_1(s)\,ds + 2\int_0^t K(x-1,t-s)\psi_2(s)\,ds, \tag{3.6}$$

where ψ_1,ψ_2 are the unique solutions in $C[0,T]$ of the Volterra system

$$F(t) = V(0,t,\Phi) + \psi_1(t) + 2\int_0^t K(-1,t-s)\psi_2(s)\,ds, \tag{3.7}$$

$$G(t) = -2\int_0^t K_{xx}(1,t-s)\psi_1(s)\,ds + \psi_2(t). \tag{3.8}$$

Note. (i) If F,G are piecewise continuous and bounded, then Lemma 3.2 holds with ψ_1,ψ_2 piecewise continuous and bounded.

(ii) If F and G have a singularity at $t=0$ with $F(t) = G(t) = O(t^{-\alpha})$, $0<\alpha<1$, then ψ_1,ψ_2 have the same singularity at $t=0$.

(iii) If F and $G \in L^1(0,T)$, then $\psi_1,\psi_2 \in L^1(0,T)$ and the solution $w \in L^1[(0,1)\times(0,T)]$.

Lemma 3.3. *Let $w(x,t)$ be the solution of*

$$\begin{aligned} w_t &= w_{xx}, \quad 0<x<1,\ t>0, \\ w(x,0) &= \Phi(x), \quad 0<x<1, \\ w_x(0,t) &= H(t), \qquad w_x(1,t) = J(t), \quad t>0, \end{aligned} \tag{3.9}$$

where $F,G \in C[0,T]$, $\Phi \in C[0,1]$. Then

$$w(x,t) = V(x,t,\Phi) - 2\int_0^t K(x,t-s)H(s)\,ds + 2\int_0^t K(x-1,t-s)J(s)\,ds. \tag{3.10}$$

Extensions. We require the following corollaries of Lemmas 3.2 and 3.3 to adapt the solutions to the intervals of interest for the problems (2.6)–(2.12).

Corollary 3.1. *With the assumptions of Lemma 3.2 the solution of*

$$\begin{aligned} w_t^1 &= c^1 w_{xx}^1, \quad x\in\Omega_1,\ t>0, \\ w^1(0,t) &= F^1(t), \qquad w_x^1(1,t) = G^1(t), \quad t>0, \\ w^1(x,0) &= \Phi^1(x), \quad x\in\Omega^1, \end{aligned} \tag{3.11}$$

is given by

$$\begin{aligned} w^1(x,t) = V^1(x,t,\Omega^1,c^1,\ell_1) &- 2\int_0^t K_x\left(\frac{x}{\ell_1},\frac{c^1}{\ell_1^2}(t-s)\right)\frac{c^1}{\ell_1}\psi_1^1(s)\,ds \\ &+ 2\int_0^t K\left(\frac{x-\ell_1}{\ell_1},\frac{c^1}{\ell_1^2}(t-s)\right)\frac{c^1}{\ell_1^2}\psi_2^1(s)\,ds, \end{aligned} \tag{3.12}$$

where ψ_1^1,ψ_1^2 are the unique solutions of the Volterra system

$$F^1(t) = V^1(0,t,\Phi_1,c^1,\ell_1) + \psi_1^1(t) + 2\int_0^t K\left(-1,\frac{c^1}{\ell_1^2}(t-s)\right)\frac{c^1}{\ell_1^2}\psi_2^1(s)\,ds, \tag{3.13}$$

$$2G^1(t) = -2c^1\int_0^t K_{xx}\left(1,\frac{c^1}{\ell_1^2}(t-s)\right)\psi_1^1(s)\,ds + \psi_2^1(t), \tag{3.14}$$

$$V^1(x,t,\Phi^1,c^1,\ell_1) = \frac{1}{\ell_1}\int_0^{\ell_1}\left\{\theta\left(\frac{x+\xi}{\ell_1},\frac{c^1}{\ell_1^2}t\right)+\theta\left(\frac{x-\xi}{\ell_1},\frac{c^1}{\ell_1^2}t\right)\right\}\Phi_1(\xi)\,d\xi. \tag{3.15}$$

Proof. Set $\hat{x} = x/\ell_1$, $\hat{t} = c^1t/\ell_1^2$ in equations (3.11), and consider Lemma 3.2 in terms of the new variables $\hat{x}$, $\hat{t}$. □

Corollary 3.2. *With the assumptions of Lemma 3.3, the solution of*

$$\begin{aligned} w_t^j &= c^j w_{xx}^j, \quad x\in\Omega_j,\ t>0, \\ w_x^j(\ell_{2j-2},t) &= H^j(t), \qquad w_x^j(\ell_{2j-1},t) = J^j(t), \quad t>0, \\ w^j(x,0) &= \Phi_j(x), \quad x\in\Omega_j \end{aligned} \tag{3.16}$$

for $j = 2, 3, \dots, n-1$ *is given by*

$$\begin{aligned} w^j(x,t) = {} & V^2(x,t,\Phi_j,c^j,\ell_{2j-2},\ell_{2j-1}) \\ & -2\int_0^t K\left(\frac{x-\ell_{2j-2}}{\ell_{2j-1}-\ell_{2j-2}}, \frac{c^j(t-s)}{|\Omega_j|^2}\right) c^j H^j(s)\,ds \\ & +2\int_0^t K\left(\frac{x-\ell_{2j-2}}{|\Omega_j|}, \frac{c^j(t-s)}{|\Omega_j|^2}\right) c^j J^j(s)\,ds, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} & V^2(x,t,\Phi_j,c^j,\ell_{2j-2},\ell_{2j-1}) \\ & \quad = \frac{1}{|\Omega_j|}\int_{\Omega_j}\left\{\theta\left(\frac{x+\xi-2\ell_{2j-2}}{|\Omega_j|}, \frac{c^j t}{|\Omega_j|^2}\right) + \theta\left(\frac{x-\xi}{|\Omega_j|}, \frac{c^j t}{|\Omega_j|^2}\right)\right\}\Phi^j(\xi)\,d\xi. \end{aligned} \tag{3.18}$$

Proof. Set $\hat{x} = (x-\ell_{2j-2})/|\Omega_j|$, $\hat{t} = c^j t/|\Omega_j|^2$ and proceed as in Corollary 3.1. $\square$

Corollary 3.3. *With the assumptions of Lemma 3.2, the solution of*

$$\begin{gathered} w_t^n = c^n w_{xx}^n, \quad x \in \Omega_n,\ t > 0, \\ w_x^n(\ell_{2n-2},t) = G^n(t), \qquad w^n(1,t) = F^n(t), \quad t > 0, \\ w^n(x,0) = \Phi_n(x), \quad x \in \Omega_n, \end{gathered} \tag{3.19}$$

is given by

$$\begin{aligned} w^n(x,t) = {} & V^3(x,t,\Phi_n,c^n,\ell_{2n-2}) \\ & +2\int_0^t K\left(\frac{x-\ell_{2n-2}}{|\Omega_n|}, \frac{c^n(t-s)}{|\Omega_n|^2}\right)\frac{c^n}{|\Omega_n|^2}\,\psi_2^n(s)\,ds \\ & +2\int_0^t K_x\left(\frac{x-\ell_{2n-2}}{|\Omega_n|}-1, \frac{c^n(t-s)}{|\Omega_n|^2}\right)\frac{c^n}{|\Omega_n|}\,\psi_1^n(s)\,ds, \end{aligned} \tag{3.20}$$

where ψ_1^n, ψ_2^n *are the unique solutions of the Volterra system*

$$F^n(t) = V^3(1,t,\Phi_n,c^n\ell_{2n-2}) + \psi_1^n(t) + 2\int_0^t K\left(1, \frac{c^n(t-s)}{|\Omega_n|^2}\right)\frac{c^n}{|\Omega_n|^2}\,\psi_2^n(s)\,ds, \tag{3.21}$$

$$|\Omega_n| G^n(t) = 2c^n\int_0^t K_{xx}\left(-1, \frac{c^n(t-s)}{|\Omega_n|^2}\right)\psi_1^n(s)ds - \psi_2^n(t), \tag{3.22}$$

$$\begin{aligned} & V^3(x,t,\Phi_n,c^n,\ell_{2n-2}) \\ & \quad = \frac{1}{|\Omega_n|}\int_{\Omega_n}\left\{\theta\left(\frac{x+\xi-2\ell_{2n-2}}{|\Omega_n|}, \frac{c^n t}{|\Omega_n|^2}\right) + \theta\left(\frac{x-\xi}{|\Omega_n|}, \frac{c^n t}{|\Omega_n|^2}\right)\right\}\Phi^n(\xi)\,d\xi. \end{aligned} \tag{3.23}$$

Proof. Set $\hat{x} = (1-x)/(1-\ell_{2n-2})$, $\hat{t} = c^n t/|\Omega_n|^2$ first, and proceed as in Corollary 3.1. $\square$

We now set

$$\psi_1^j = H^j(t), \qquad \psi_2^j(t) = J^j(t), \quad j = 2, 3, \dots, n-1. \tag{3.24}$$

Clearly, once

$$\psi = \left(\psi_1^1, \psi_2^1, \ldots, \psi_1^n, \psi_2^n\right)^T \tag{3.25}$$

is determined uniquely, the solution to our problem is known.

We will show that ψ satisfies a matrix Volterra system of the form

$$\psi(t) = G(t) + \int_0^t A(t-s)\psi(s)\,ds + M \max\left\{H(t) + \int_0^t B(t-s)\psi(s)\,ds, 0\right\}, \tag{3.26}$$

where G, H are suitable vectors and A, M, B suitable matrices. We require the following lemma:

Lemma 3.4. *Let*

$$\begin{aligned} G(t) &= (G_1, G_2, \ldots, G_N)^T \in [\mathcal{L}_1(0,T)]^N, \\ H(t) &= (H_1, H_2, \ldots, H_N)^T \in [\mathcal{L}_1(0,T)]^N, \end{aligned} \tag{3.27}$$

and let the $N \times N$ *matrices*

$$A(t) = ((a_{ij}(t))), \qquad B(t) = ((b_{ij}(t))), \qquad M(t) = ((m_{ij}(t))), \tag{3.28}$$

$i, j = 1, 2, \ldots, N$, *be such that for some constants* $C_a, C_b, C_m > 0$, $0 < \alpha < 1$,

$$\begin{aligned} \|A\| &= \max_{i,j} |a_{ij}| \le C_a t^{-\alpha}, \\ \|B\| &= \max_{i,j} |b_{ij}| \le C_b t^{-\alpha}, \quad \text{where } t > 0. \\ \|M\| &= \max_{i,j} |M_{ij}| \le C_m t^{-\alpha}, \end{aligned} \tag{3.29}$$

Then the system (3.26) has a unique solution $\psi(t) \in [L(0,T)]^N$. *In particular if* ψ_1, ψ_2 *are two solutions corresponding to data* G_1, H_1, *and* G_2, H_2 *respectively then there exists a constant* $C = C(C_a, C_b, C_m, \alpha, T) > 0$ *such that*

$$\int_0^T ||\psi_1 - \psi_2||\,dt \le C \int_0^T \left\{||G_1 - G_2|| + ||H_1 - H_2||\right\} dt, \tag{3.30}$$

where $\|\psi\| = \sum_{i=1}^N |\psi_i|$.

Proof. See [8]. □

4. Existence and uniqueness. We rewrite equations (2.7), (2.8), (2.9), and (2.10) in the form

$$w^1(0,t) = f_1(t) - \frac{b_1}{|\Omega_1| Q} S(\psi), \tag{4.1}$$

$$w^n(1,t) = f_n(t) - \frac{b_n}{|\Omega_n| Q} S(\psi), \tag{4.2}$$

$$w_x^{j+1}(\ell_{2j}, t) = \frac{K_j}{K_{j+1}} w_x^j(\ell_{2j-1}, t), \tag{4.3}$$

$$w_x^j(\ell_{2j-1}, t) = -\frac{m_j}{K_j}\left\{w^j(\ell_{2j-1}, t) - w^{j+1}(\ell_{2j}, t) + \frac{H_j}{Q^1} S(\psi)\right\} \tag{4.4}$$

for $j = 1, 2, \dots, n-1$.

Since $\mu_1^2 + \nu_1^2 \neq 0$, $\mu_n^2 + \nu_n^2 \neq 0$ we have considered the typical case

$$\mu_1 = \mu_2 = 0, \qquad \nu_1 = \nu_2 = 1. \tag{4.5}$$

The general case will follow by similar arguments. Since, as we pointed out earlier, $\Phi_j(x)$ are known for $j = 1, 2, \dots, n$ we have taken V^1, V^2, V^3 as known quantities. Further, since each element w^j, $j = 1, \dots, n$ can be expressed in terms of the appropriate elements of ψ we have written, for the moment,

$$S(\psi) = \max\{I^*(\psi) + g, 0\}, \qquad I^*(\psi) = I(w(\psi)). \tag{4.6}$$

We again note that

$$H_j = \left(\frac{b_j}{|\Omega_j|} - \frac{b_{j+1}}{|\Omega_{j+1}|}\right), \qquad \psi_1^j = H^j, \qquad \psi_2^j = J^j. \tag{4.7}$$

Equations (2.10) may now be used to determine equations for ψ_2^i, $i = 1, 2, \dots, n-1$ and the last of equations (2.9) $(i = n-1)$ to determine ψ_2^n as follows.

For $i = 1$ equations (2.10), together with equations (3.14) and (3.17) give

$$\begin{aligned}
\psi_2^1(t) = {} & \frac{m_1\ell_1}{K_1}\left\{V^2(\ell_2, t, \Phi_2, c^2, \ell_2, \ell_3) - V^1(\ell_1, t, \Phi_1, c^1, \ell_1)\right\} \\
& + 2c^1 \int_0^t K_{xx}\left(1, \frac{c^1(t-s)}{|\Omega_1|}\right)\psi_1^1(s)\,ds \\
& - \frac{m_1\ell_1}{K_1}\left\{-2\int_0^t K_x\left(1, \frac{c^1(t-s)}{|\Omega_n|^2}\right)\frac{c^1}{|\Omega_1|^2}\psi_1^1(s)\,ds\right. \\
& \qquad + 2\int_0^t K\left(0, \frac{c^1(t-s)}{|\Omega_1|^2}\right)\frac{c^1}{|\Omega_1|^2}\psi_2^1(s)\,ds \\
& \qquad + 2\int_0^t K\left(0, \frac{c^2(t-s)}{|\Omega_2|^2}\right)c^2\psi_1^2(s)\,ds \\
& \qquad \left. - 2\int_0^t K\left(-1, \frac{c^2(t-s)}{|\Omega_2|^2}\right)c^2\psi_2^2(s)\,ds\right\} + \frac{m\ell_1}{K_1 Q}H_2 S(\psi),
\end{aligned} \tag{4.8}$$

and for $j = 2, \dots, n-1$ we have, using Corollary 3.2 and equation (4.7),

$$\begin{aligned}
\psi_2^j = {} & \frac{m_j}{K_j}\left\{V^2(\ell_{2j}, t, \Phi_{j+1}, \ell_{2j}, \ell_{2j+1}) - V^2(\ell_{2j-1}, t, \Phi_j, \ell_{2j-2}, \ell_{2j-1})\right\} \\
& - \frac{m_j}{K_j}\left\{-2\int_0^t K\left(1, \frac{c^j(t-s)}{|\Omega_j|^2}\right)c^j\psi_1^j(s)\,ds\right. \\
& \qquad + 2\int_0^t K\left(0, \frac{c^j(t-s)}{|\Omega_j|^2}\right)c^j\psi_2^j(s)\,ds \\
& \qquad + 2\int_0^t K\left(0, \frac{c^{j+1}(t-s)}{|\Omega_{j+1}|^2}\right)c^{j+1}\psi_1^{j+1}(s)\,ds \\
& \qquad \left. - 2\int_0^t K\left(-1, \frac{c^{j+1}(t-s)}{|\Omega_{j+1}|^2}\right)c^{j+1}\psi_2^{j+1}(s)\,ds\right\} - \frac{m_j H_j}{K_j Q}S(\psi).
\end{aligned} \tag{4.9}$$

Next, from equations (2.9) with $i = n-1$ and equation (3.22), we have

$$\psi_2^n = -|\Omega_n| \frac{K_{n-1}}{K_n} \psi_2^{n-1}(t) - 2c^n \int_0^t K_{xx}\left(-1, \frac{c^n(t-s)}{|\Omega_n|^2}\right) \psi_1^n(s)\,ds, \tag{4.10}$$

where ψ_2^{n-1} is given in equations (4.9).

We next turn to equations (4.8), (4.9), and (4.10) for $\psi_1^j(t)$, $j = 1,2,\dots,n$. Using equations (4.1), (4.2), and equations (3.13), (3.21) we have

$$\begin{aligned}\psi_1^1(t) &= \{f_1(t) - V^1(0,t,\Phi_1,c^1,\ell_1)\}\\ &\quad - 2\int_0^t K\left(-1, \frac{c^1}{|\Omega_1|^2}(t-s)\right) \frac{c^1}{|\Omega_1|^2} \psi_2^1(s)\,ds - \frac{b_1}{|\Omega_1| Q} S(\psi),\end{aligned} \tag{4.11}$$

$$\begin{aligned}\psi_1^n(t) &= \{f_n(t) - V^3(1,t,\Phi_n,\ell_{2n-2})\}\\ &\quad - 2\int_0^t K\left(1, \frac{c^n(t-s)}{|\Omega_n|^2}\right) \frac{c^n}{|\Omega_n|^2} \psi_2^n(s)\,ds - \frac{b_n}{|\Omega_n| Q} S(\psi),\end{aligned} \tag{4.12}$$

while the first of equations (4.3), $i = 1$, gives

$$\psi_1^2(t) = \frac{K_1}{\ell_1 K_2} \left\{ \psi_2^1(t) - 2c^1 \int_0^t K_{xx}\left(1, \frac{c^1(t-s)}{|\Omega_1|^2}\right) \psi_1^1(s)\,ds \right\}, \tag{4.13}$$

where again ψ_2^1 is given in equation (4.8).

The remaining equations (4.3) for $i = 2,\dots,n-2$ give

$$\psi_1^j(t) = \frac{K_j}{K_{j+1}} \psi_2^{j-1}(t), \quad j = 3,4,\dots,n-1 \tag{4.14}$$

with the right hand sides given in equations (4.9).

Clearly the set of equations for ψ have the form given in equations (3.26) and it remains to establish suitable estimates.

Lemma 4.1. *The unique solution $\{\psi_1^i, \psi_2^i\}$, $i = 1,2,\dots,n$, exist for systems (4.8)–(4.14) and there exists $C > 0$ such that*

$$\sum_{i=1}^n \int_0^T \left(|\psi_1^i - \overline{\psi}_1^i| + |\psi_2^i - \overline{\psi}_2^i| \right) dt \le C \int_0^T \left(|g - \overline{g}| + |f_1 - \overline{f}_1| + |f_n - \overline{f}_n| \right) dt, \tag{4.15}$$

where $\{\psi_1^i, \psi_2^i\}$ and $\{\overline{\psi}_1^i, \overline{\psi}_2^i\}$ are the solutions with the data $\{g, f_1, f_n\}$ and $\{\overline{g}, \overline{f}_1, \overline{f}_n\}$, respectively.

Proof. By Lemma 3.4 it is sufficient to estimate the kernel. Clearly it is from Cannon [4] that

$$|K(\pm 1,t)| + |K_{xx}(\pm 1,t)| + |K_x(\pm 1,t)| \le C, \quad t > 0 \tag{4.16}$$

and

$$|K(0,t)| \le \frac{C}{\sqrt{t}}, \quad t > 0. \tag{4.17}$$

Thus, the assumptions of Lemma 3.4 are satisfied and Lemma 4.1 follows. □

Theorem 4.1. *The system (1.1)–(1.10) possesses a unique solution which depends continuously upon the data.*

Proof. It follows from Lemma 4.1 and the equivalence analysis in Section 2. □

5. Application to thermoelastic bars. Consider n thermoelastic bars lying along the positive x axis with the ith bar, $1 \le i \le n$, occupying the interval Ω_i. We use the notation of Section 1. The equations describing the displacements and temperature distributions are given by

$$K_i\alpha_i\theta_0(u_i)_{xt} + c_i(\theta^i)_t = k_i(\theta^i)_{xx}, \quad x \in \Omega_i, \tag{5.1}$$

$$\sigma_i = (\lambda_i + 2\mu_i)(u_i)_x - K_i\alpha_i(\theta^i - \theta_0), \quad x \in \Omega_i, \tag{5.2}$$

$$(\sigma_i)_x = 0, \quad x \in \Omega_i \tag{5.3}$$

for $i = 1,2,\dots,n$, $t \in J$. $u_i(x,t)$ denotes the displacement and $\theta^i(x,t)$ the temperature of the ith bar at position x and time t. $\sigma_i(x,t)$ represents the corresponding stress and c_i, k_i, α_i are constants, $i = 1,2,\dots,n$, denoting the heat capacity, conductivity and coefficient of thermal expansion, respectively, of the ith bar. θ_0 is a reference temperature, measured in degrees Kelvin, normally taken as the ambient temperature. It is convenient to nondimensionalize the quantities of interest and we set

$$\begin{aligned} \hat{x} &= \frac{x}{L}, \qquad \hat{u}_i = \frac{\pi_i u_i}{L}, \qquad \hat{t} = \frac{k_1 t}{c_1 L^2}, \\ \hat{\sigma}_i &= \frac{\sigma_i}{K_i}, \qquad \hat{\theta}^i = \frac{\theta^i - \theta_0}{\theta_0}, \qquad \pi_i^2 = \frac{(\lambda_i + 2\mu_i)k_1}{c_1\theta_0 k_i}, \end{aligned} \tag{5.4}$$

where

$$K_i = 3\lambda_i + 2\mu_i \tag{5.5}$$

for $i = 1,2,\dots,n$. The quantities λ_i, μ_i are the Lamé elastic constants.

The intervals $x \in \Omega_i$ are replaced with the corresponding intervals $\hat{x} \in \hat{\Omega}_i$, $i = 1,2,\dots,n$. If the above quantities are substituted in equations (5.1), (5.2), and (5.3) and subsequently the hats dropped we have the equations in the following nondimensional form

$$d_i(\theta^i)_t - (\theta^i)_{xx} = -a_i(u_i)_{xt}, \quad x \in \Omega_i, \tag{5.6}$$

$$\sigma_i = \beta_i\{(u_i)_x - a_i\theta^i\}, \quad x \in \Omega_i, \tag{5.7}$$

$$(\sigma_i)_x = 0, \quad x \in \Omega_i \tag{5.8}$$

for $i = 1,2,\dots,n$, where

$$d_i = \frac{c_i k_1}{c_1 k_i}, \qquad \beta_i = \frac{\lambda_i + 2\mu_i}{K_i \pi_i}, \qquad a_i = \frac{\alpha_i K_i k_1}{c_1 \pi_i k_i}. \tag{5.9}$$

Clearly equation (5.8) implies that

$$\sigma_i = \sigma_i(t), \quad i = 1,2,\dots,n. \tag{5.10}$$

Since (5.10) holds this implies that if one end of any of the bars is free then $\sigma_i(t) = 0$, $i = 1,2,\dots,n$ whereas if all of the bars are in contact then $\sigma_i(t) \le 0$, $i = 1,2,\dots,n$.

In addition to the governing equations we require the initial and boundary conditions. The conditions on the $\theta_i(x,t)$ are given in equations (1.3) through (1.7). For the moment we require only the initial conditions, namely,

$$\theta^i(x,0) = \theta_0^i(x), \quad x \in \Omega_i,\ i = 1,2,\dots,n, \tag{5.11}$$

together with the conditions

$$u_1(0,t) = 0, \qquad u_n(1,t) = 0. \tag{5.12}$$

There are essentially two cases to consider depending on whether all bars are in contact or not. There are then subcases depending on how the bars are grouped in contact. The difficulties are the same whether we consider n bars or three bars. The latter case simplifies and clarifies the procedure and we now confine our attention to that case. The generalization required for n bars then follows.

We consider then three bars lying along the positive x axis lying in the intervals $\Omega_1 = [0,\ell_1]$, $\Omega_2 = [\ell_2,\ell_3]$, $\Omega_3 = [\ell_4,1]$ where

$$0 < \ell_1 \le \ell_2 < \ell_3 \le \ell_4 < 1, \tag{5.13}$$

and set

$$g_1 = \ell_1 - \ell_2, \qquad g_2 = \ell_4 - \ell_3. \tag{5.14}$$

We begin by considering the initial conditions. Set

$$\Theta_1(x,t) = a_1 \int_0^x \theta^1(s,t)\,ds, \quad 0 \le x \le \ell_1, \tag{5.15}$$

$$\Theta_2(x,t) = a_2 \int_{\ell_2}^x \theta^2(s,t)\,ds, \quad \ell_2 \le x \le \ell_3, \tag{5.16}$$

$$\Theta_3(x,t) = a_3 \int_x^1 \theta^3(s,t)\,ds, \quad \ell_4 \le x \le 1. \tag{5.17}$$

From equations (5.11), $\theta_i(x,0)$, $i = 1,2,3$ are known, so that $\Theta_i(x,0)$ are known. There are two cases.

Case I. If

$$u_1(\ell_1,0) < u_2(\ell_2,0) + g_1 \tag{5.18}$$

or

$$u_2(\ell_3,0) < u_3(\ell_4,0) + g_2, \tag{5.19}$$

then $\sigma_i(0) = 0$, $i = 1,2,3$.

Using equations (5.7) and (5.12) we have

$$u_1(x,0) = \Theta_1(x,0), \quad 0 \le x \le \ell_1, \tag{5.20}$$

$$u_2(x,0) = u_2(\ell_2,0) + \Theta_2(x,0), \quad \ell_2 \le x \le \ell_3, \tag{5.21}$$

$$u_3(x,0) = -\Theta_3(x,0), \quad \ell_4 \le x \le 1. \tag{5.22}$$

If equation (5.18) does not hold but (5.19) does then

$$u_2(\ell_2,0) = \Theta_1(\ell_1,0) - g_1, \tag{5.23}$$

whereas if (5.18) holds and (5.19) does not

$$u_2(\ell_2,0) = g_2 - \Theta_2(\ell_3,0) - \Theta_3(\ell_4,0). \tag{5.24}$$

Thus if the middle bar is in contact with either of the end bars initially then the initial stresses, displacements and temperatures are known. If on the other hand the middle bar has no contact with the other two initially $u_2(\ell_2,0)$ is indeterminate and an additional initial condition must be added. If we define

$$\Omega(t) = \frac{g_1 + g_2 - \Theta_1(\ell_1,t) - \Theta_2(\ell_3,t) - \Theta_3(\ell_4,t)}{(1+\lambda+\mu)}, \tag{5.25}$$

where

$$\lambda = \frac{\beta_1(1-\ell_4)}{\beta_3\ell_1}, \qquad \mu = \frac{\beta_1(\ell_3-\ell_2)}{\beta_2\ell_1}, \tag{5.26}$$

then in all three of the above subcases

$$\Omega(0) > 0, \tag{5.27}$$

and conversely if (5.27) holds, then one of these subcases does.

Case II. If both of the conditions

$$u_1(\ell_1,0) = u_2(\ell_2,0) + g_1, \qquad u_2(\ell_3,0) = u_3(\ell_4,0) + g_2, \tag{5.28}$$

hold, then

$$\sigma_i(0) \le 0, \quad i = 1,2,3. \tag{5.29}$$

Again using equations (5.7) and (5.12) we have

$$u_1(x,0) = \Theta_1(x,0) + \frac{x\sigma_1(0)}{\beta_1}, \quad 0 \le x \le \ell_1, \tag{5.30}$$

$$u_2(x,0) = u_2(\ell_2,0) + \Theta_2(x,0) + (x-\ell_2)\frac{\sigma_2(0)}{\beta_3}, \quad \ell_2 \le x \le \ell_3, \tag{5.31}$$

$$u_3(x,0) = -\Theta_3(x,0) - (1-x)\frac{\sigma_3(0)}{\beta_3}, \quad \ell_4 \le x \le 1. \tag{5.32}$$

Then $u_i(x,0)$, $i = 1,2,3$ are known once $\sigma_i(0)$, $i = 1,2,3$ are. $u_2(\ell_2,0)$ is determined from equation (5.28). Since $\sigma_1(0) = \sigma_2(0) = \sigma_3(0)$ it follows that

$$\begin{aligned}\{u_1(\ell_1,0) - \Theta_1(\ell_1,0)\}\frac{\beta_1}{\ell_1} &= \{u_2(\ell_3,0) - u_2(\ell_2,0) - \Theta_2(\ell_3,0)\}\frac{\beta_2}{\ell_3-\ell_2} \\ &= -\{u_3(\ell_4,0) + \Theta_3(\ell_4,0)\}\frac{\beta_3}{1-\ell_4},\end{aligned} \tag{5.33}$$

and making use of equations (5.28) we find

$$u_3(\ell_4,0) = -\lambda\{u_1(\ell_1,0)-\Theta_1(\ell_1,0)\}-\Theta_3(\ell_4,0), \tag{5.34}$$

$$u_2(\ell_3,0)-u_2(\ell_2,0) = \mu\{u_1(\ell_1,0)-\Theta_1(\ell_1,0)\}+\Theta_2(\ell_3,0), \tag{5.35}$$

$$u_2(\ell_3,0)-u_2(\ell_2,0) = u_3(\ell_4,0)-u_1(\ell_1,0)+g_1+g_2, \tag{5.36}$$

where λ,μ are given in equation (5.26). Substituting from equations (5.34) and (5.35) into equation (5.36) gives

$$(1+\lambda+\mu)u_1(\ell_1,0) = g_1+g_2+(\lambda+\mu)\Theta_1(\ell_1,0)-\Theta_2(\ell_3,0)-\Theta_3(\ell_4,0). \tag{5.37}$$

Again substituting back into equations (5.30), (5.31), and (5.32) gives

$$\frac{\sigma_1(0)}{\beta_1} = \frac{\Omega(0)}{\ell_1}, \qquad \frac{\sigma_2(0)}{\beta_2} = \frac{\mu\Omega(0)}{\ell_3-\ell_2}, \qquad \frac{\sigma_3(0)}{\beta_3} = \frac{\lambda\Omega(0)}{1-\ell_4}, \tag{5.38}$$

with $\Omega(t)$ given by equation (5.25).

All initial values are now determined. Clearly, if $\sigma_i(0) \le 0$, $i = 1,2,3$, then $\Omega(0) \le 0$. Conversely, if $\Omega(0) \le 0$, then Case II holds.

The general situation for $t > 0$ may be handled in the same manner except that $\Theta_i(x,t)$, $i = 1,2,3$ are not known a priori.

Case I. Here equations (5.18) through (5.24) are replaced by the same equations with $t = 0$ replaced by the general time t. If both conditions replacing (5.18), (5.19) hold, that is

$$u_1(\ell_1,t) < u_2(\ell_2,t)+g_1, \qquad u_2(\ell_3,t) < u_3(t_4,t)+g_2, \tag{5.39}$$

then $u_2(\ell_2,t)$ is indeterminate. In order to make the problem determinate an extra physical assumption is required as to how the bar expands. The simplest such assumption is that the expansions at either end are equal in magnitude; that is $u_2(\ell_2,t) = -u_2(\ell_3,t)$, until at least two of the bars are again in contact.

Since in this case, for $i = 1,2,3$, $\sigma_i(t) = 0$ and $u_i(x,t)$ are given by the updated forms of equations (5.20) through (5.22), we may substitute in equations (5.6) to give

$$(d_i+a_i^2)(\theta^i)_t-(\theta^i)_{xx} = 0, \quad x \in \Omega_i \quad \text{for } i = 1,2,3. \tag{5.40}$$

Case II. In this case we follow the procedure of equations (5.30) through (5.32) again replacing $t = 0$ with general $t > 0$. On substituting the updated values of the stresses $\sigma_i(t)$ into the expressions for the updated values of $u_i(x,t)$ we can substitute into equations (5.6) to obtain

$$(d_i+a_i^2)(\theta^i)_t+(\theta^i)_{xx} = -\frac{a_i}{\Omega_i}\frac{d}{dt}\Omega(t), \quad x \in \Omega_i \tag{5.41}$$

for $i = 1,2,3$ where $\Omega(t)$ is given in equation (5.25).

Since, in this case, $\sigma_i(t) \le 0$, $i = 1,2,3$ then $\Omega(t) \le 0$. If $\Omega(t) > 0$ we have Case I. This allows us to combine equations (5.40) and (5.41) in the form

$$(d_i+a_i^2)(\theta^i)_t-(\theta^i)_{xx} = \frac{d_i}{\Omega_i}\frac{d}{dt}\max(\Omega,0), \quad x \in \Omega_i. \tag{5.42}$$

If we set

$$b^i = c^i = \frac{1}{(d_i + a_i^2)}, \quad i = 1,2,3,$$
$$g = -(g_1 + g_2), \tag{5.43}$$

then it is clear that equations (1.2) are a direct generalization of equations (5.42).

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