ON THE EXTENSION OF POSITIVE LINEAR FUNCTIONALS

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ABSTRACT. A necessary and sufficient condition to extend a continuous linear real functionals which is positive with respect to a semi-group defined on a subspace of a linear space is discussed in this paper. The case of a closed subspace of a Banach space is also discussed.

Keywords and phrases. Solid semi-group, extension functional, positive functional, monotonic Banach functional.

2000 Mathematics Subject Classification. 46A22, 46A40.

1. Introduction. Let E be a real linear space with a linear semi-group K (we will call the set $K \subset E$ a linear semi-group if it possesses the following properties: $K + K \subset K$, and $\alpha \cdot K \subset K$ for all $\alpha \geq 0$) with $K \subset E$ (the word, linear will be omitted) and let $L \subset E$ be a subspace of E. Also, let f_0 be a real linear functional on L. A real linear functional f on E is called an extension of f_0 if $f(x) = f_0(x)$ for every $x \in L$. A linear functional f will be called positive (with respect to K) if $f(x) \geq 0$ for every $x \in K$.

In the theory of spaces with an associated semi-group, the existence of the extension of positive linear functional plays an important role. Later on, several mathematicians studied a necessary condition to extend a linear continuous positive functional, for example, M. G. Krein theorem [4] states that "if $K \neq E$ is a semi-group in a Banach space E with interior point and the subspace $L \subset E$ contains at least one interior point of K, then each linear continuous positive functional on L (i.e., $f_0 : L \to \mathbb{R}$ such that $f_0(x) \geq 0$ for every $x \in K \cap L = K_L$) can be extended to a linear continuous positive functional f defined on E (i.e., $f : E \to \mathbb{R}$ such that $f_0(x) = f(x)$ for every $x \in L$, and $f(x) \geq 0$ for every $x \in K$)." Other important theorems on the existence of positive extension of linear functionals are given in [1, 2, 3].

2. Main results. In the present work, we introduce a new condition for the existence of the extension of positive linear functionals which depend upon the definition of Banach functional [5]. Two cases will be considered separately.

CASE (A). E is a real linear space with a semi-group K.

The functional $\omega : E \to \mathbb{R}$ is called a Banach functional if

(1) $\omega(x + y) \leq \omega(x) + \omega(y)$ for every $x, y \in E$,

(2) $\omega(\lambda \cdot x) = \lambda \cdot \omega(x)$ for every $x \in E$, for every $\lambda \geq 0$.

We say that the Banach functional ω is monotonic (with respect to K), if $\omega(x + y) \geq \omega(x)$ for every $x \in E$ and for every $y \in K$.

The following theorem is close to [1, Thm. 2.3].

Theorem 2.1. *Let $L \subset E$ be a subspace of a linear space E, and $f_0: L \to \mathbb{R}$ be a linear functional such that $f_0(x) \geq 0$ for every $x \in K_L$. Then a necessary and sufficient condition for the existence of a linear functional $f : E \to \mathbb{R}$, such that $f_0(x) = f(x)$ for every $x \in L$ and $f(x) \geq 0$ for every $x \in K$, is the existence of a monotonic Banach functional ω satisfying the inequality*

$$f_0(x) \leq \omega(x) \quad \text{for every } x \in L. \tag{2.1}$$

Proof. Let the functional f be an extension of f_0. We define the functional ω by $\omega(x) = \inf\{|f(x + y)| : y \in K\}$, where $x \in E$. Obviously, ω is a monotonic Banach functional which satisfies inequality (2.1). (Another proof, f is already a monotonic Banach functional and hence we take $\omega = f$.)

Conversely, consider the existence of a monotonic Banach functional ω satisfying inequality (2.1). Then by the Hahn-Banach theorem there exists a linear functional f on E such that $f_0(x) = f(x)$ for every $x \in L$ and $f(x) \leq \omega(x)$ for every $x \in E$. It remains to prove that f is positive. Let $x \in K$, then from the fact that ω is monotonic, we get $\omega(-x) \leq 0$ and therefore $f(-x) \leq \omega(-x) \leq 0$, which implies that $f(x) \geq 0$. Hence f is a positive extended functional of f_0. This completes the proof of the theorem. $\qquad\square$

In the following theorem, the condition that the Banach functional is monotonic is dropped.

Theorem 2.2. *Let $L \subset E$ be a subspace of a linear space E, ϖ be a Banach functional (not necessarily monotonic), and $f_0 : L \to \mathbb{R}$ be a linear functional such that*

$$f_0(x) \leq \varpi(x + y) \quad \text{for every } x \in L, \text{ for every } y \in K. \tag{2.2}$$

Then f_0 has an extension functional f on E such that $f(x) \geq 0$ for every $x \in K$.

Proof. Let $\omega(x) = \inf\{\varpi(x + y) : y \in K\}$ for every $x \in E$. We start by showing that

(i) $\omega(x) > -\infty$ for every $x \in E$,

(ii) $f_0(x) \leq \omega(x)$ for every $x \in L$,

(iii) ω is a monotonic Banach functional.

Condition (i) is a consequence of inequality (2.2) for, if $x = 0$, then $\varpi(y) \geq 0$ for every $y \in K$, but ϖ is a Banach functional, then $0 \leq \varpi(y) = \varpi(x + y + (-x)) \leq \varpi(x + y) + \varpi(-x)$, which implies that $\varpi(x + y) \geq -\varpi(-x)$ for every $y \in K$ and for every $x \in E$. Therefore, $\omega(x) \geq -\varpi(-x) > -\infty$ for every $x \in E$, this proves condition (i). Since $f_0(x) \leq \varpi(x + y)$ for every $x \in L$, every $y \in K$, then $f_0(x) \leq \omega(x)$ for every $x \in E$. This proves condition (ii).

Clearly ω is a Banach functional and it remains to show that ω is a monotonic functional (i.e., $\omega(x) \leq \omega(x + y)$ for every $y \in K$ and every $x \in E$). Since $\omega(x + y) = \inf\{\varpi(x + y + y_1) : y_1 \in K\} = \inf\{\varpi(x + z) : z = y + y_1 \in K, y_1 \in K\} \geq \inf\{\varpi(x + u) : u \in K\} = \omega(x)$ for every $y \in K$ and every $x \in E$, and hence condition (iii) is satisfied. Moreover, by using Theorem 2.1, there exists an extension functional f of the functional f_0 such that $f(x) \geq 0$ for every $x \in K$. This completes the proof of the theorem. $\qquad\square$

CASE (B). *E is a Banach space with closed semi-group K.*
We introduce the following definition.

DEFINITION. We define the following functional $|x|_K = \inf\{\|x + y\| : y \in K\}$ for each $x \in E$. Obviously, $|\cdot|_K$ is a monotonic functional with respect to the semi-group K and $|\cdot|_K$ is a Banach functional, moreover, $|x|_K \leq \|x\|$ for every $x \in E$ (hence the functional $|\cdot|_K$ is a continuous functional on E). Also, from the closeness of a semi-group K, we have $|x|_K = 0$ if and only if $-x \in K$.

EXAMPLE 1. Let $E = \ell_2 = \{x = (\xi_i) : \sum_i |\xi_i|^2 < \infty\}$, $K = \{(\xi_i) \in \ell_2 : \xi_i \geq 0, i = 1, 2, \dots\}$. Obviously, for every $x = (\xi_i) \in \ell_2$, $x = x_+ - x_-$, where $x_+ = 1/2(|\xi_i| + \xi_i)$, $x_- = 1/2(|\xi_i| - \xi_i)$, then $x_+, x_- \in K$ and $|x|_K = \|x_+\|$.

The following theorem is analogous to Theorem 2.1.

THEOREM 2.3. *Let L be a closed subspace of a Banach space E with a closed semi-group K, and let $f_0 : L \to \mathbb{R}$ be a linear continuous functional such that $f_0(x) \geq 0$ for every $x \in K_L$. Then a necessary and sufficient condition that there exists a linear continuous functional $f : E \to \mathbb{R}$ such that $f_0(x) = f(x)$ for every $x \in L$ and $f(x) \geq 0$ for every $x \in K$ is that there exist numbers $\alpha > 0$, $\beta > 0$ and a monotonic Banach functional ω that satisfies the inequalities*

$$f_0(x) \leq \alpha \cdot \omega(x) \quad \text{for every } x \in L, \tag{2.3}$$

$$\omega(x) \leq \beta \cdot \|x\| \quad \text{for every } x \in L. \tag{2.4}$$

PROOF. For the proof of the theorem, we need the following lemma.

LEMMA 2.1. *The linear continuous functional f is positive with respect to a semi-group K if and only if there exists a number $C > 0$ such that*

$$f(x) \leq C \cdot |x|_K \quad \text{for every } x \in E. \tag{2.5}$$

PROOF. If f is positive, then $f(y) \geq 0$ for every $y \in K$. Let $x \in E$, then $x \leq x + y$ for every $y \in K$. This implies that $f(x + y) = f(x) + f(y) \geq f(x)$ and therefore $f(x) \leq f(x + y) \leq |f(x + y)| \leq \|f\| \|x + y\|$. By taking the infimum of both sides of the previous inequality, we get $C = \|f\|$, $f(x) \leq C \cdot |x|_K$. Conversely, let f be a linear continuous functional and there exists a number $C > 0$ such that $f(x) \leq C \cdot |x|_K$ for every $x \in E$. In particular if $y \in K$, then $-y \in -K$ and we obtain that $-f(y) = f(-y) \leq C|-y|_K = 0$. Hence, $f(y) \geq 0$ for every $y \in K$. $\square$

Now we return to the proof of Theorem 2.3.

Let the linear continuous functional f_0 on L has a linear continuous positive extension functional f on E. Then by Lemma 2.1, there exists a number $C > 0$ such that $f(x) \leq C \cdot |x|_K$ for every $x \in E$. Put $\omega(x) = |x|_K$, $\alpha = C$ and $\beta = 1$. Hence $f_0(x) \leq C \cdot \omega(x)$ and $\omega(x) \leq \|x\|$ for every $x \in L$. Conversely, let f_0 be a continuous linear functional on L which satisfies inequalities (2.3) and (2.4). Hence by the Hahn-Banach theorem there exists an extension functional f of f_0 on E such that $f(x) \leq \alpha \cdot \omega(x)$ for every $x \in E$. Hence from the fact that ω is monotonic, we obtain that $f(x) \geq 0$ for every $x \in K$, and from the following inequality:

$$f(x) \leq \alpha \cdot \omega(x) \leq \alpha \cdot \beta \cdot \|x\|, \tag{2.6}$$

we obtain that f is a linear continuous functional on E. This completes the proof of the theorem.   □

COROLLARY 2.1. *The linear continuous positive functional f_0 on a closed subspace L of a Banach space E has a linear continuous positive extension functional f on E if and only if there exists a number $C > 0$ such that*

$$f_0(x) \le C \cdot |x|_K \quad \text{for every } x \in L. \tag{2.7}$$

PROOF. This follows directly from Theorem 2.3.   □

In the following section, we discuss the relationship between the two functionals $|x|_K$ and $|x|_{K_L}$, where $|x|_{K_L} = \inf\{\|x + y\| : y \in K_L\}$ for each $x \in L$.

It is clear that $|x|_K \le |x|_{K_L}$ for every $x \in L$. From Lemma 2.1, if f_0 is a linear continuous functional on a closed subspace L in a Banach space E such that $f_0(x) \ge 0$ for every $x \in K_L$, then there exists a number $C > 0$ such that $f_0(x) \le C \cdot |x|_{K_L}$ for every $x \in L$.

THEOREM 2.4. *Given $C > 0$ such that $|x|_{K_L} \le C \cdot |x|_K$ for every $x \in L$. Then every continuous linear positive (with respect to a closed semi-group K_L) functional f_0 on a closed subspace L has a linear continuous positive (with respect to a closed semi-group K) extension functional f on E.*

PROOF. The proof follows directly from Lemma 2.1 and Corollary 2.1.   □

We have to note that M. G. Krein theorem can be obtained directly from Theorem 2.4 and the following lemma.

LEMMA 2.2. *Let $K \ne E$ be a semi-group in a Banach space E with interior point (i.e., K is a solid semi-group), and let the closed subspace $L \subset E$ contains at least one interior point of K. Then there exists a number $C > 0$ such that $|x|_{K_L} \le C \cdot |x|_K$ for every $x \in L$.*

PROOF. Let $u_0 \in L$ be an interior point of K, then there exists $\delta_0 > 0$ such that $T(u_0, \delta_0) \subset K$, where $T(u_0, \delta_0) = \{u : \|u - u_0\| \le \delta_0\}$. It is well known that $u_\alpha = \alpha \cdot u_0 + (1 - \alpha). y, 0 < \alpha < 1$ are the interior points of K with radius $\alpha \cdot \delta_0$ (i.e., $T(u_\alpha, \alpha \cdot \delta_0) \subset K$) for every $y \in K$. Now, we suppose that $x \in L$ and $\|x\| = 1$ then $Z_\alpha = (\alpha \cdot u_0 - (1 - \alpha) \cdot x) \in L$ and $\|Z_\alpha - u_\alpha\| = (1 - \alpha)\|x + y\|$. Therefore, if we choose $\alpha = \|x + y\| \cdot (\delta_0 + \|x + y\|)^{-1}$, then $Z_\alpha \in K$ and

$$
\begin{aligned}
|x|_{K_L} &\le \|x + Z_\alpha\| = \alpha \cdot |u_0 + x\| \\
&\le (\|u_0\| + 1) \cdot (\delta_0 + \|x + y\|)^{-1}\|x + y\| \\
&\le (\|u_0\| + 1)\delta_0^{-1}\|x + y\| \\
&= C \cdot \|x + y\|,
\end{aligned}
\tag{2.8}
$$

where $C = (\delta_0)^{-1}(\|u_o\| + 1)$. By taking the infimum of both sides of the last inequality at $y \in K$, we get

$$|x|_{K_L} \le C \cdot |x|_K \quad \text{for every } x \in L. \tag{2.9}$$

□

The following example shows that, if the semi-group K does not contain an interior point, then M. G. Krein theorem cannot be applied but our work can be applied.

EXAMPLE 2.   Back to Example 1, it is known that K is not solid semi-group in $E = \ell_2$. If $x_1 = (1, 1/2, \ldots, 1/n, \ldots)$, $x_2 = (1, 1/2^2, \ldots, 1/n^2, \ldots)$, and $L = \{\alpha x_1 + \beta x_2 : \alpha, \beta \in \mathbb{R}\}$ is a subspace of E, then $K_L = K \cap L = \{\lambda x_1 + \mu x_2 : \lambda \geq 0, \lambda + \mu \geq 0\}$. Let $f_0 : L \to \mathbb{R}$ be defined by $f_0(x) = f_0(\alpha x_1 + \beta x_2) = \alpha C_1 + \beta C_2$, where C_1 and C_2 are real numbers. Then if $C_1 - C_2 > 0$, $C_2 > 0$, the functional f_0 satisfies inequality (2.7). Therefore, from Corollary 2.1, the linear continuous positive functional f_0 on L has a linear continuous positive extension functional f on E. We remark here that if $C_2 = 0$ and $C_1 > 0$, then clearly, the positive functional f_0 has no positive functional extension on E.

REFERENCES

[1]   C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Pure and Applied Mathematics, vol. 119, Academic Press, Inc., Orlando, 1985. MR 87h:47086. Zbl 608.47039.

[2]   R. E. Edwards, *Functional Analysis. Theory and Applications*, Holt, Rinehart and Winston, New York, Toronto, London, 1965. MR 36#4308. Zbl 182.16101.

[3]   M. A. Krasnosel'skij, Je. A. Lifshits, and A. V. Sobolev, *Positive Linear Systems: The Method of Positive Operators*, Sigma Series in Applied Mathematics, vol. 5, Heldermann Verlag, Berlin, 1989. MR 91f:47051. Zbl 674.47036.

[4]   M. G. Krein and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Trans. Amer. Math. Soc. (1950), no. 26, 128. MR 12,341b.

[5]   V. Pta'k, *On a theorem of Mazur and Orlicz*, Studia Math. **15** (1956), 365–366. MR 18,320b. Zbl 071.10801.

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