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ON THE REPRESENTATION OF m AS $\sum_{k=-n}^{n}\epsilon_k k$

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ABSTRACT. Let $A(n,m)$ be the number of solutions of $\sum_{k=-n}^{n}\epsilon_k k = m$ where each $\epsilon_k \in \{0,1\}$. We determine the asymptotic behavior of $A(n,m)$ for $m = o(n^{3/2})$, extending results of van Lint and of Entringer.

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For a nonnegative integer n and an integer m, let

$$A(n,m) = \#\left\{(\epsilon_{-n},\dots,\epsilon_0,\dots,\epsilon_n) \in \{0,1\}^{2n+1} : \sum_{k=-n}^{n}\epsilon_k k = m\right\}. \tag{1}$$

van Lint [2] answered a question of Erdös by determining the asymptotic behavior of $A(n,0)$. Entringer [1] used this result and induction to determine the asymptotic behavior of $A(n,m)$ for $m = O(n)$. In this note, we give a further extension by showing that

$$A(n,m) \sim \left(\frac{3}{\pi}\right)^{1/2} 2^{2n+1} n^{-3/2} \quad \text{as } n \to \infty, \tag{2}$$

for $m = o(n^{3/2})$. We estimate the integral below, as in [2], though our analysis is more involved. It is immediately seen that $A(n,m)$ is the coefficient of z^m in $\prod_{k=-n}^{n}(1+z^k)$ and, hence,

$$\begin{aligned} A(n,m) &= \frac{1}{2\pi i}\oint_C \frac{\prod_{k=-n}^{n}(1+z^k)}{z^{m+1}}\,dz \\ &= \frac{2^{2n+2}}{\pi}\int_0^{\pi/2} \cos 2mx \prod_{k=1}^{n}\cos^2 kx\,dx, \end{aligned} \tag{3}$$

upon parameterizing the unit circle C (see [1, 2]). Note that $A(n,m) = A(n,-m)$ and $A(n,m) = 0$ if and only if $|m| > \binom{n+1}{2}$. Hence, we assume that m is a nonnegative integer. We denote the nonnegative integers by $\mathbb{N}$; the integers by $\mathbb{Z}$; and the real numbers by $\mathbb{R}$.

We use the following Taylor series approximations which are valid for all $x \in \mathbb{R}$.

$$\sin x = x - \frac{x^3}{6} + r(x); \quad |r(x)| \le \frac{x^4}{24} \quad \text{for } x \in \mathbb{R} \quad \text{and} \quad r(x) \ge 0 \quad \text{for } x \in [0,\pi]; \tag{4}$$

$$\cos x = 1 + s(x); \quad |s(x)| \le |x| \quad \text{for } x \in \mathbb{R}; \tag{5}$$

$$\cos^2 x = 1 - x^2 + t(x); \quad |t(x)| \le \frac{2|x|^3}{3} \quad \text{for } x \in \mathbb{R}; \tag{6}$$

$$e^{-x} = 1 - x + u(x); \quad 0 \le u(x) \le \frac{x^2}{2} \quad \text{for } x \in [0, \infty). \tag{7}$$

Of course, r, s, t, and u are all infinitely-differentiable functions on $\mathbb{R}$. We also use the following standard inequalities:

$$e^{x-x^2} \le 1 + x \le e^{x-x^2/6} \quad \text{for } x \in [-0.68, 0.68]; \tag{8}$$

$$1 - x \le e^{-x} \qquad \text{for } x \in \mathbb{R}. \tag{9}$$

For all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$ with $\sin x \ne 0$, (4) gives

$$\frac{\sin nx}{\sin x} = n - \frac{n^3 - n}{6} x^2 + v(n,x), \tag{10}$$

where

$$v(n,x) = \frac{-((n^3-n)/36)x^5 + ((n^3-n)/6)x^2 r(x) + r(nx) - nr(x)}{x - (x^3/6) + r(x)}, \tag{11}$$

so that

$$|v(n,x)| \le \frac{n^4 x^4/23}{5x/6} = \frac{6}{115} n^4 x^3 \quad \text{for } x \in [0,1] \text{ and } n \ge 20. \tag{12}$$

(Naturally, we define $\sin nx / \sin x = n$ when $x = 0$ to remove that discontinuity.) We require the following result (see [2] for a statement of a version of (a)).

Lemma. (a) *For* $(\pi/2n) \le x \le \pi/2$ *and* $n \ge 4$,

$$\left| \frac{\sin nx}{\sin x} \right| \le \frac{2n}{3}. \tag{13}$$

(b) *For* $0 \le x \le (\pi/2n)$ *and* $n \ge 20$,

$$\left| \frac{\sin nx}{\sin x} \right| \le n - \frac{n^3 x^2}{12}. \tag{14}$$

Proof. (a) First, (4) gives $\sin(\pi/2n) \ge (\pi/2n) - (\pi^3/48n^3) \ge (3/2n)$ for $n \ge 4$. Hence,

$$\left| \frac{\sin nx}{\sin x} \right| = \frac{|\sin nx|}{\sin x} \le \frac{1}{\sin(\pi/2n)} \le \frac{2n}{3}. \tag{15}$$

(b) Next, (10) gives $n - ((n^3-n)/6)x^2 + v(n,x) \le n - n^3 x^2((1/6) - (1/6n^2) - (6/115)nx) \le n - (n^3x^2/12)$ for $n \ge 20$. Hence,

$$\left| \frac{\sin nx}{\sin x} \right| = \frac{\sin nx}{\sin x} \le n - \frac{n^3 x^2}{12}. \tag{16}$$

□

For all $x \in \mathbb{R}$ and $n \ge 1$, (9) gives (see [2])

$$0 \le \prod_{k=1}^{n} \cos^2 kx = \prod_{k=1}^{n} (1 - \sin^2 kx) \le \exp\left(-\sum_{k=1}^{n} \sin^2 kx\right)$$
$$= \exp\left(-\frac{n}{2} + \frac{\sin nx \cos(n+1)x}{2\sin x}\right) \le \exp\left(-\frac{n}{2} + \frac{1}{2}\left|\frac{\sin nx}{\sin x}\right|\right). \tag{17}$$

Hence, for all $m \in \mathbb{N}$ and $n \ge 20$, the lemma and (17) now give

$$\left| \int_{\pi/2n}^{\pi/2} \cos 2mx \prod_{k=1}^{n} \cos^2 kx \, dx \right| \le 2e^{-n/6}, \tag{18}$$

and, for all $0 \le c \le n^{1/2}$,

$$\left| \int_{cn^{-3/2}}^{\pi/2n} \cos 2mx \prod_{k=1}^{n} \cos^2 kx \, dx \right| \le \int_{cn^{-3/2}}^{\pi/2n} e^{-n^3x^2/24} \, dx \le e^{-c^2/24}. \tag{19}$$

If $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, (6) and (7) give

$$\cos^2 kx = e^{-k^2x^2}(1 + w(k,x)), \tag{20}$$

where

$$w(k,x) = e^{k^2x^2}(t(kx) - u(k^2x^2)) \tag{21}$$

is infinitely-differentiable on $\mathbb{R}$ for each integer k and, for $1 \le k \le n$, $0 \le x \le n^{-1}$,

$$|w(k,x)| \le 4k^3x^3. \tag{22}$$

Now, for $0 \le x \le an^{-1} \le 0.5n^{-1}$ and $n \ge 7$,

$$\sum_{k=1}^{n} \left(|w(k,x)| + |w(k,x)|^2\right) \le 6x^3 \sum_{k=1}^{n} k^3 \le 2n^4x^3 \le 2a^3 n, \tag{23}$$

so that (8) gives

$$e^{-2a^3n} \le \prod_{k=1}^{n} (1 + w(k,x)) \le e^{2a^3n}. \tag{24}$$

Hence, for all $m \in \mathbb{N}$, $0 \le b \le 0.5n^{1/2}$, $n \ge 7$, (20) and (24) give with $\sigma = \sigma(n) = n(n+1)(2n+1)/6$,

$$\left| \int_0^{bn^{-3/2}} \cos 2mx \prod_{k=1}^{n} \cos^2 kx \, dx \right| \le \int_0^{bn^{-3/2}} e^{-\sigma x^2 + 2b^3 n^{-1/2}} \, dx \le \frac{be^{2b^3n^{-1/2}}}{n^{3/2}}. \tag{25}$$

For $0 \le bn^{-3/2} \le cn^{-3/2} \le 0.5n^{-1}$, $n \ge 7$, $t \in \mathbb{Z}$, (20) and (24) give

$$e^{-2c^3n^{-1/2}} \int_{bn^{-3/2}}^{cn^{-3/2}} x^t e^{-\sigma x^2} \, dx \le \int_{bn^{-3/2}}^{cn^{-3/2}} x^t \prod_{k=1}^{n} \cos^2 kx \, dx$$
$$\le e^{2c^3n^{-1/2}} \int_{bn^{-3/2}}^{cn^{-3/2}} x^t e^{-\sigma x^2} \, dx. \tag{26}$$

Hence,

$$\int_{bn^{-3/2}}^{cn^{-3/2}} \prod_{k=1}^{n} \cos^2 kx \, dx \sim \frac{(3\pi)^{1/2}}{2} n^{-3/2}, \tag{27}$$

and, for all $m \in \mathbb{N}$,

$$\int_{bn^{-3/2}}^{cn^{-3/2}} s(2mx) \prod_{k=1}^{n} \cos^2 kx\, dx = O(mn^{-3}), \tag{28}$$

since

$$\int_{bn^{-3/2}}^{cn^{-3/2}} e^{-\sigma x^2}\, dx \sim \frac{(3\pi)^{1/2}}{2} n^{-3/2}, \tag{29}$$

$$\int_{bn^{-3/2}}^{cn^{-3/2}} x e^{-\sigma x^2}\, dx \sim \frac{3}{2} n^{-3}, \tag{30}$$

and (26) holds for all sufficiently large n provided $b = b(n) \to 0$, $c = c(n) \to \infty$ with $c = o(n^{1/6})$ as $n \to \infty$.

Consequently, (5), (18), (19), (25), (27), and (28) give

$$\begin{aligned}\int_0^{\pi/2} \cos 2mx \prod_{k=1}^{n} \cos^2 kx\, dx &= \int_{(\ln n)^{-1/2} n^{-3/2}}^{7(\ln n)^{1/2} n^{-3/2}} \prod_{k=1}^{n} \cos^2 kx\, dx \\ &\quad + \int_{(\ln n)^{-1/2} n^{-3/2}}^{7(\ln n)^{1/2} n^{-3/2}} s(2mx) \prod_{k=1}^{n} \cos^2 kx\, dx \\ &\quad + O\big((\ln n)^{-1/2} n^{-3/2}\big) \\ &\sim \frac{(3\pi)^{1/2}}{2} n^{-3/2} \quad \text{as } n \to \infty,\end{aligned} \tag{31}$$

for all $m = m(n) = o(n^{3/2})$ (our error term being adequate for our analysis which indicates where the integral is concentrated). Hence, (3) gives

$$A(n,m) \sim \left(\frac{3}{\pi}\right)^{1/2} 2^{2n+1} n^{-3/2} \quad \text{as } n \to \infty, \tag{32}$$

for all $m = m(n) = o(n^{3/2})$. This completes the proof. □

References

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