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BIORTHOGONALITY CONDITION FOR AXISYMMETRIC STOKES FLOW IN SPHERICAL GEOMETRIES

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ABSTRACT. We derive the biorthogonality condition for axisymmetric Stokes flow in a region between two concentric spheres. This biorthogonality condition is a property satisfied by the eigenfunctions and adjoint eigenfunctions, which is needed to compute the coefficients of the eigenfunction expansion solution of the corresponding creeping flow problem.

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1. Introduction. Recently, the eigenfunction expansion method has been used extensively for solving problems of Stokes flow. The method leads to the development of a set of eigenfunctions, adjoint eigenfunctions, biorthogonality conditions and an algorithm for the computation of the eigenfunction expansions. This technique was first introduced by Smith [10] in his solution of the biharmonic problem governing the bending of a semi-infinite strip clamped at its side and loaded at its top edge.

The biorthogonal series expansion method was also used by Joseph [2] in his study of the free surface on the round edge of a flowing liquid filling a torsion flow viscometer and by Joseph and Sturges [3] in the steady flow induced in a rectangular cavity by the uniform translation of a covering plate or belt. Similar biorthogonal eigenfunction expansions and biorthogonality conditions are required for the axisymmetric Stokes flow problems in a wedge shaped trench studied by Liu and Joseph [7], the axisymmetric Stokes flow in a cone studied by Liu and Joseph [8] and for the problem of Stokes flow in a trench between concentric cylinders studied by Yoo and Joseph [11].

Most recently, biorthogonality conditions were used by Khuri to solve Stokes flow in a sectorial cavity [5] and by Khuri and Wang for solving Stokes flow around a bend [6].

The previous references are just a small sample of problems arising in Stokes flow and elasticity which can be solved in biorthogonal series of eigenfunctions generated by separating variables. A list of several other problems is given in [4, 9].

In this paper, we derive the biorthogonality condition for axisymmetric Stokes flow in a spherical region by implementing a theorem proved by Khuri [5]. This biorthogonality condition is a property satisfied by the eigenfunctions and adjoint eigenfunctions, which is needed to compute the coefficients of the eigenfunction expansion solution of the corresponding creeping flow problem.

2. Biorthogonality conditions. We state a biorthogonality property satisfied by the eigenfunctions and adjoint eigenfunctions of the following fourth-order boundary value problem:

$$(P_0(r)y''(r))'' + (P_1(r;\alpha)y'(r))' + P_2(r;\alpha)y(r) = 0 \quad r \in [r_1, r_2] \tag{2.1}$$

The boundary conditions are given by

$$y(r_1) = y(r_2) = y'(r_1) = y'(r_2) = 0. \tag{2.2}$$

This biorthogonality condition, given in Theorem 2.1, which was proved by Khuri [5], gives the biorthogonality property for the boundary value problem given in equations (2.1) and (2.2) with certain restrictions imposed on the coefficients.

Theorem 2.1 (biorthogonality condition). *Consider the boundary value problem given in (2.1) and (2.2), where $P_0(r)$, $P_1''(r;\alpha)$, $P_2(r;\alpha)$ are continuous and $P_0(r) \neq 0$ on $r_1 \le r \le r_2$. P_i in equation (2.1) is a polynomial of degree at most i in the parameter α, in particular, let $P_1(r;\alpha) = p_{11}(r)\alpha + p_{12}(r)$, and we require*

$$\begin{gathered} P_1^2(r;\alpha) - 4P_0(r)P_2(r;\alpha) = p_{31}(r)\alpha + p_{32}(r), \\ p_{11}^2(r) + p_{31}^2(r) \not\equiv 0. \end{gathered} \tag{2.3}$$

Then with P_n^* defined by

$$P_n^* = \int_{r_1}^{r_2} \left[\phi_2^{(n)}(r), \phi_1^{(n)}(r)\right] B(r) \begin{bmatrix} \phi_1^{(n)}(r) \\ \phi_2^{(n)}(r) \end{bmatrix} dr, \tag{2.4}$$

we have the following biorthogonality condition:

$$\int_{r_1}^{r_2} \left[\phi_2^{(m)}(r), \phi_1^{(m)}(r)\right] B(r) \begin{bmatrix} \phi_1^{(n)}(r) \\ \phi_2^{(n)}(r) \end{bmatrix} dr = P_n^* \delta_{mn}, \tag{2.5}$$

where δ_{mn} is the Kronecker's delta,

$$B(r) = \begin{pmatrix} -\dfrac{1}{2}\dfrac{p_{11}(r)}{P_0(r)} & 0 \\ \dfrac{1}{2}p_{11}''(r) + \dfrac{1}{4}\dfrac{p_{31}(r)}{P_0(r)} & -\dfrac{1}{2}\dfrac{p_{11}(r)}{P_0(r)} \end{pmatrix} \tag{2.6}$$

with

$$\begin{gathered} \phi_1^{(n)}(r) = y_n(r), \\ \phi_2^{(n)}(r) = P_0(r)y_n''(r) + \frac{1}{2}P_1(r;\alpha_n)y_n(r). \end{gathered} \tag{2.7}$$

Here y_i is an eigenfunction of equation (2.1) corresponding to the eigenvalue α_i. Assume the eigenvalues α_i are simple.

3. Axisymmetric Stokes flow in spherical regions. In this section, the biorthogonality condition for the axisymmetrical creeping flow in a region between two concentric spheres is derived. The flow region is

$$v = \{r,\theta : 0 < r_1 \le r \le r_2,\ -\theta_1 \le \theta \le \theta_1\}. \tag{3.1}$$

The Stokes flow equation in spherical coordinates (r,θ,ϕ) in v is given by

$$E^4\Psi(r,\theta) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\right)^2\Psi(r,\theta) = 0. \tag{3.2}$$

The velocity components in the (r,θ) direction in terms of the stream function are given by

$$v_r = -\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, \qquad v_\theta = \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}. \tag{3.3}$$

Requiring the velocity to vanish on $r = r_1$, r_2, (3.3) gives

$$\Psi(r_1,\theta) = \Psi(r_2,\theta) = \frac{\partial\Psi}{\partial r}(r_1,\theta) = \frac{\partial\Psi}{\partial r}(r_2,\theta) = 0. \tag{3.4}$$

Separable solutions of (3.2) and (3.4) in the form

$$\Psi(r,\theta) \sim T(\cos\theta)y(r) \tag{3.5}$$

exist (see [1]) when $y(r)$ satisfies the following equation:

$$y^{(4)} + \frac{2}{r^2}p(1-p)y^{(2)} - \frac{4}{r^3}p(1-p)y^{(1)} + p(1-p)(2+p)(3-p)\frac{1}{r^4}y = 0 \tag{3.6}$$

and the boundary conditions

$$y(r_1) = y(r_2) = y'(r_1) = y'(r_2) = 0. \tag{3.7}$$

Seeking an eigenfunction solution in r direction it is necessary that the function $T(\eta)$, where $\eta = \cos\theta$ be required to satisfy the following equation:

$$(1-\eta^2)T''(\eta) - p(1-p)T(\eta) = 0. \tag{3.8}$$

Equation (3.8) is Gegenbauer's equation of degree $-1/2$ where p could be complex. The two independent solutions of (3.8) are $C_p^{-1/2}(\eta)$ and $D_p^{-1/2}(\eta)$ that are termed as Gegenbauer functions of the first and second kind, respectively. Clearly, equation (3.6) can be written in the following form:

$$(y'')'' + 2p(1-p)\left(\frac{1}{r^2}y'\right)' + p(1-p)(2+p)(3-p)\frac{1}{r^4}y = 0. \tag{3.9}$$

The hypothesis of Theorem 2.1 is satisfied when $\alpha_n \ne \alpha_m$ with

$$P_0(r) = 1; \qquad P_1(r;\alpha) = \frac{2}{r^2}\alpha; \qquad P_2(r;\alpha) = \frac{1}{r^4}\alpha(\alpha+6), \tag{3.10}$$

where

$$\alpha = p(1-p). \tag{3.11}$$

Since

$$P_1^2(r;\alpha) - 4P_0(r)P_2(r;\alpha) = -\frac{24}{r^4}\alpha \tag{3.12}$$

so

$$p_{31}(r) = -\frac{24}{r^4}; \qquad p_{32}(r) = 0. \tag{3.13}$$

Clearly,

$$p_{11}(r) = \frac{2}{r^2}; \qquad p_{12}(r) = 0. \tag{3.14}$$

Thus using Theorem 2.1. The biorthogonality condition is given by

$$\int_{r_1}^{r_2} \frac{-1}{r^2}\left[\psi_1^{(m)}(r), \psi_2^{(m)}(r)\right]\begin{bmatrix}\phi_1^{(n)}(r)\\ \\ \phi_2^{(n)}(r)\end{bmatrix} dr = P_n^*\delta_{mn}, \qquad p_n(1-p_n) \neq p_m(1-p_m) \tag{3.15}$$

upon using

$$B(r) = -\frac{1}{r^2}I_{2\times 2} = \begin{pmatrix} -\dfrac{1}{r^2} & 0 \\ & \\ 0 & -\dfrac{1}{r^2}\end{pmatrix}, \tag{3.16}$$

where $I_{2\times 2}$ is the identity matrix. The eigenfunctions satisfy

$$\phi_1^{(n)}(r) = y_n(r), \qquad \phi_2^{(n)}(r) = y_n''(r) + \frac{\alpha_n}{r^2}y_n(r) \tag{3.17}$$

and the adjoint eigenfunctions satisfy

$$\psi_1^{(m)}(r) = y_m''(r) + \frac{\alpha_m}{r^2}y_m(r), \qquad \psi_2^{(m)}(r) = y_m(r), \tag{3.18}$$

where

$$\alpha_n = p_n(1-p_n). \tag{3.19}$$

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