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ON THE SYSTEM OF TWO NONLINEAR DIFFERENCE EQUATIONS $x_{n+1} = A + x_{n-1}/y_n,\ y_{n+1} = A + y_{n-1}/x_n$

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ABSTRACT. We study the oscillatory behavior, the periodicity and the asymptotic behavior of the positive solutions of the system of two nonlinear difference equations $x_{n+1} = A + x_{n-1}/y_n$ and $y_{n+1} = A + y_{n-1}/x_n$, where A is a positive constant, and $n = 0, 1, \ldots$.

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1. Introduction. In [3] Kulenovic, Ladas and Sizer studied the global stability character and the periodic nature of the positive solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{\gamma x_n + \delta x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta$ are positive constants and $x_{-1}, x_0 > 0$.

In [1] Amlech, Grove, Ladas and Georgiou studied the global stability, the boundedness and the periodic nature of the positive solutions of the difference equation

$$x_{n+1} = A + \frac{x_{n-1}}{x_n}, \tag{1.2}$$

where A is a nonnegative constant and $x_{-1}, x_0 > 0$. Equation (1.2) is different from (1.1) since in this special case δ is considered equal to zero.

In this paper, we generalize the results concerning equation (1.2) to the system of two nonlinear difference equations

$$x_{n+1} = A + \frac{x_{n-1}}{y_n}, \quad y_{n+1} = A + \frac{y_{n-1}}{x_n}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where A is a nonnegative constant and x_{-1}, x_0, y_{-1}, y_0 are positive numbers.

We note that if (x_n, y_n) is a solution of (1.3) such that $x_{-1} = y_{-1}$, $x_0 = y_0$, then $x_n = y_n$, $n = -1, 0, \ldots$ and so x_n is a solution of (1.2).

Moreover, if (μ_1, μ_2) is a positive equilibrium of system (1.3), then

$$(\mu_1, \mu_2) = \begin{cases} (c, c) = (1 + A, 1 + A), & \text{if } A \neq 1, \\ \left(\mu, \dfrac{\mu}{\mu - 1}\right), \quad \mu \in (1, \infty), & \text{if } A = 1. \end{cases} \tag{1.4}$$

Observe that if $A = 1$ we have a continuous of positive equilibriums which lie on the hyperbola $\mu_1 \mu_2 = \mu_1 + \mu_2$. Moreover, if (x_n, y_n) is a solution of (1.3) eventually equal to (c,c), then $(x_n,y_n) = (c,c)$ for all $n = -1,0,\dots$. We call this solution the trivial solution.

A function $z_n : \mathbb{N} \to \mathbb{R}^+$ oscillates about $z \in \mathbb{R}^+$ if for every $\tau \in \mathbb{N}$ there exist $s,m \in \mathbb{N}$, $s \ge \tau$, $m \ge \tau$ such that $(z_s - z)(z_m - z) \le 0$. We say that a solution (x_n,y_n) of (1.3) oscillates about (μ_1,μ_2) if x_n (resp., y_n) oscillates about μ_1 (resp., μ_2).

In this paper, first we find conditions so that a positive solution (x_n,y_n) of system (1.3) oscillates about (μ_1,μ_2). Moreover we prove that system (1.3) has periodic solutions of period 2 if $A = 1$ and we find necessary and sufficient conditions so that a solution of system (1.3) is periodic of period 2. Also, we find conditions so that a positive solution of system (1.3) tends to (μ_1,μ_2) as $n \to \infty$. Furthermore, if $A < 1$ we prove that system (1.3) has unbounded solutions, if $A = 1$ every positive solution of system (1.3) tends to a period 2 solution of (1.3) and if $A > 1$ the positive equilibrium (c,c) of system (1.3) is globally asymptotically stable.

2. Main results.

Now we prove our main results. In the first proposition we study the oscillatory behavior of the positive solutions of (1.3).

Proposition 2.1. *A positive solution (x_n,y_n) of system (1.3) oscillates about (μ_1,μ_2) if there exists an $s \in \{0,1,\dots\}$ such that one of the following conditions is satisfied:*

(i) $x_s \ge \mu_1,\ y_s \ge \mu_2,\ x_{s+1} < \mu_1,\ y_{s+1} < \mu_2,$

(ii) $x_s < \mu_1,\ y_s < \mu_2,\ x_{s+1} \ge \mu_1,\ y_{s+1} \ge \mu_2,$

(iii) $x_s \ge \mu_1,\ y_s \ge \mu_2,\ x_{s+1} \ge \mu_1,\ y_{s+1} < \mu_2,\ y_s > -Ax_{s+1} + (1/(\mu_1 - A))x_{s+1}^2,$

(iv) $x_s \ge \mu_1,\ y_s < \mu_2,\ x_{s+1} \ge \mu_1,\ y_{s+1} \ge \mu_2,\ y_{s+1} > (1/(\mu_1 - A))x_s,$

(v) $x_s \ge \mu_1,\ y_s \ge \mu_2,\ x_{s+1} < \mu_1,\ y_{s+1} \ge \mu_2,\ x_s > -Ay_{s+1} + (1/(\mu_2 - A))y_{s+1}^2,$

(vi) $x_s < \mu_1,\ y_s \ge \mu_2,\ x_{s+1} \ge \mu_1,\ y_{s+1} \ge \mu_2,\ x_{s+1} > (1/(\mu_2 - A))y_s,$

(vii) $x_s < \mu_1,\ y_s < \mu_2,\ x_{s+1} \ge \mu_1,\ y_{s+1} < \mu_2,\ x_s \le -Ay_{s+1} + (1/(\mu_2 - A))y_{s+1}^2,$

(viii) $x_s < \mu_1,\ y_s < \mu_2,\ x_{s+1} < \mu_1,\ y_{s+1} \ge \mu_2,\ y_s \le -Ax_{s+1} + (1/(\mu_1 - A))x_{s+1}^2,$

(ix) $x_s \ge \mu_1,\ y_s < \mu_2,\ x_{s+1} < \mu_1,\ y_{s+1} < \mu_2,\ x_{s+1} \le (1/(\mu_2 - A))y_s,$

(x) $x_s < \mu_1,\ y_s \ge \mu_2,\ x_{s+1} < \mu_1,\ y_{s+1} < \mu_2,\ y_{s+1} \le (1/(\mu_1 - A))x_s,$

(xi) $x_s \ge \mu_1,\ y_s < \mu_2,\ x_{s+1} < \mu_1,\ y_{s+1} \ge \mu_2,\ y_{s+1} > (1/(\mu_1 - A))x_s,\ y_s \le -Ax_{s+1} + (1/(\mu_1 - A))x_{s+1}^2,$

(xii) $x_s \ge \mu_1,\ y_s < \mu_2,\ x_{s+1} < \mu_1,\ y_{s+1} \ge \mu_2,\ x_{s+1} \le (1/(\mu_2 - A))y_s,\ x_s > -Ay_{s+1} + (1/(\mu_2 - A))y_{s+1}^2,$

(xiii) $x_s < \mu_1,\ y_s \ge \mu_2,\ x_{s+1} \ge \mu_1,\ y_{s+1} < \mu_2,\ x_{s+1} > (1/(\mu_2 - A))y_s,\ x_s \le -Ay_{s+1} + (1/(\mu_2 - A))y_{s+1}^2,$

(xiv) $x_s < \mu_1,\ y_s \ge \mu_2,\ x_{s+1} \ge \mu_1,\ y_{s+1} < \mu_2,\ y_{s+1} \le (1/(\mu_1 - A))x_s,\ y_s > -Ax_{s+1} + (1/(\mu_1 - A))x_{s+1}^2.$

Proof. (i) Using (1.3) and (1.4) we can easily prove that

$$x_{s+2k} \ge \mu_1, \quad y_{s+2k} \ge \mu_2, \quad x_{s+2k+1} < \mu_1, \quad y_{s+2k+1} < \mu_2, \quad k = 0,1,\dots \tag{2.1}$$

from which the solution (x_n,y_n) oscillates about (μ_1,μ_2).

(ii) The proof is similar to the proof of (i).

(iii) It holds, $y_s > -Ax_{s+1} + (1/(\mu_1 - A))x_{s+1}^2 \ge (\mu_2 - A)x_{s+1}$. Then, from (1.3) and (1.4), $x_{s+2} > \mu_1$, $y_{s+2} > \mu_2$. Moreover, since $y_s > -Ax_{s+1} + (1/(\mu_1 - A))x_{s+1}^2$, from (1.3) and (1.4) we have, $x_{s+3} < \mu_1$, $y_{s+3} < \mu_2$. Therefore, from (i) the solution (x_n, y_n) oscillates about (μ_1, μ_2).

(iv) From (1.3) and (1.4) and since $y_{s+1} > (1/(\mu_1 - A))x_s$ we have $x_{s+2} < \mu_1$, $y_{s+2} < \mu_2$. Then, from (i) the solution (x_n, y_n) oscillates about (μ_1, μ_2).

The proofs of (v), (vii), (viii) are similar to (iii), the proofs of (vi), (ix), (x) are similar to (iv).

(xi) From (1.3) and (1.4) and since $y_{s+1} > (1/(\mu_1 - A))x_s$, $y_s \le -Ax_{s+1} + (1/(\mu_1 - A))x_{s+1}^2 < (\mu_2 - A)x_{s+1}$, we can prove that $x_{s+2} < \mu_1$, $y_{s+2} < \mu_2$. Then from (x) the solution (x_n, y_n) oscillates about (μ_1, μ_2).

The proofs of (xii), (xiii), and (xiv) are similar to (xi). □

In the following proposition we study the existence of period 2 solutions of (1.3).

Proposition 2.2. (i) *Suppose that system (1.3) has a nontrivial solution (x_n, y_n) of period 2. Then $A = 1$.*

(ii) *Let $A = 1$. Then the solution (x_n, y_n) of system (1.3) is periodic of period 2 if and only if*

$$x_{-1} \neq 1, \qquad y_{-1} \neq 1, \qquad x_0 = \frac{y_{-1}}{y_{-1} - 1}, \qquad y_0 = \frac{x_{-1}}{x_{-1} - 1}. \tag{2.2}$$

Proof. (i) Let (x_n, y_n) be a nontrivial solution of system (1.3) such that $x_{n+2} = x_n$ and $y_{n+2} = y_n$, $n \in \{0, 1, \dots\}$. Then, from (1.3), we have for $n \in \{0, 1, \dots\}$

$$\begin{aligned} x_{n-1}(y_n - 1) &= Ay_n, \quad \text{so } x_{n-1}\left(A + \frac{y_n}{x_{n-1}} - 1\right) = Ay_n, \\ y_{n-1}(x_n - 1) &= Ax_n, \quad \text{so } y_{n-1}\left(A + \frac{x_n}{y_{n-1}} - 1\right) = Ax_n \end{aligned} \tag{2.3}$$

from which it follows that

$$(A - 1)(x_{n-1} - y_n) = 0, \quad (A - 1)(y_{n-1} - x_n) = 0, \quad n = 0, 1, \dots. \tag{2.4}$$

Since (x_n, y_n) is a nontrivial solution of (1.3) there exists an $n \in \{0, 1, \dots\}$ such that $x_{n-1} \neq y_n$ or $y_{n-1} \neq x_n$. Then, from (2.4), $A = 1$ and the proof of (i) is completed.

(ii) If x_n, y_n, $n = -1, 0, \dots$ are periodic functions of period 2, then $x_{-1} = x_1$, $y_{-1} = y_1$. Hence, from (1.3) for $A = 1$, it is obvious that (2.2) are satisfied. Conversely, if (2.2) holds, we can easily prove by induction that $x_{n+2} = x_n$, $y_{n+2} = y_n$, $n = -1, 0, \dots$. This completes the proof of the proposition. □

In the following proposition we find positive solutions of system (1.3) which tend to (μ_1, μ_2) as $n \to \infty$.

Proposition 2.3. *Let (x_n, y_n) be a positive solution of system (1.3). Then, if there exists an $s \in \{-1, 0, \dots\}$ such that for $n \ge s$, $x_n \ge \mu_1$, $y_n \ge \mu_2$ (resp., $x_n < \mu_1$, $y_n < \mu_2$), the solution (x_n, y_n) tends to the positive equilibrium (μ_1, μ_2) of system (1.3) as $n \to \infty$.*

Proof. Let (x_n, y_n) be a positive solution of system (1.3) such that

$$x_n \ge \mu_1, \quad y_n \ge \mu_2, \quad n \ge s, \tag{2.5}$$

where $s \in \{-1,0,\dots\}$. Then from (1.3) and (2.5) we get

$$x_{n+1} = A + \frac{x_{n-1}}{y_n} \le A + \frac{x_{n-1}}{\mu_2}, \quad n > s. \tag{2.6}$$

We can prove that the solution u_n of the difference equation

$$u_{n+1} = A + \frac{u_{n-1}}{\mu_2}, \quad n > s \tag{2.7}$$

such that

$$u_s = x_s, \qquad u_{s+1} = x_{s+1} \tag{2.8}$$

is the following:

$$u_n = \sigma_n + \frac{A\mu_2}{\mu_2 - 1} = \sigma_n + \mu_1, \quad \sigma_n = c_1\left(\frac{1}{\mu_2}\right)^{n/2} + c_2\left(-\frac{1}{\mu_2}\right)^{n/2}, \tag{2.9}$$

where c_1, c_2 depend on x_s, x_{s+1}. Moreover, relations (2.6) and (2.7) imply that

$$x_{n+1} - u_{n+1} \le \frac{x_{n-1} - u_{n-1}}{\mu_2}, \quad n > s. \tag{2.10}$$

Then, using (2.8) and (2.10) and working inductively, it follows that

$$x_n \le u_n, \quad n \ge s. \tag{2.11}$$

Therefore, from (2.5), (2.9), and (2.11) and since $\sigma_n \to 0$ as $n \to \infty$, it is obvious that

$$\lim_{n\to\infty} x_n = \mu_1. \tag{2.12}$$

Similarly we can prove that

$$\lim_{n\to\infty} y_n = \mu_2. \tag{2.13}$$

So, from (2.12) and (2.13), the solution (x_n, y_n) tends to (μ_1, μ_2) as $n \to \infty$.

Arguing as above we can easily prove that if $x_n < \mu_1$, $y_n < \mu_2$ for $n \ge s$, then (x_n, y_n) tends to (μ_1, μ_2) as $n \to \infty$. This completes the proof of the proposition. □

In the following proposition we study the stability of the positive equilibrium (μ_1, μ_2) of system (1.3).

Proposition 2.4. *Consider system (1.3). Then the following statements are true:*

(i) *If $0 \le A < 1$, the unique positive equilibrium (c, c) of (1.3) is not stable.*

(ii) *If $A > 1$, the unique positive equilibrium (c, c) of (1.3) is locally asymptotically stable.*

(iii) *If $A = 1$, then for every $\mu \in (1, \infty)$ there exist positive solutions (x_n, y_n) of (1.3) which tend to the positive equilibrium $(\mu, \mu/(\mu - 1))$.*

Proof. (i) The linearized system of (1.3) about (c, c) is the following

$$v_{n+1} = Bv_n, \quad B = \begin{pmatrix} 0 & c^{-1} & -c^{-1} & 0 \\ 1 & 0 & 0 & 0 \\ -c^{-1} & 0 & 0 & c^{-1} \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.14}$$

The characteristic equation of the matrix B is

$$\lambda^4 - \lambda^2 \frac{2A+3}{(A+1)^2} + \frac{1}{(A+1)^2} = 0. \tag{2.15}$$

Since $0 \le A < 1$, then from [2, Theorem 1.3.4, page 11] there exists a root of (2.15) of modulus greater than 1. So (c,c) is not stable. This completes the proof of part (i).

(ii) Since $A > 1$, by applying again [2, Theorem 1.3.4, page 11] all the roots of (2.15) are of modulus less than 1. Therefore (c,c) is locally asymptotically stable from which the proof of part (ii) is completed.

(iii) Consider a $\mu \in (1,\infty)$. The linearized system of (1.3) about the positive equilibrium $(\mu, \mu/(\mu-1))$ of (1.3) is

$$v_{n+1} = Bv_n, \quad B = \begin{pmatrix} 0 & \dfrac{\mu-1}{\mu} & -\dfrac{(\mu-1)^2}{\mu} & 0 \\ 1 & 0 & 0 & 0 \\ -\dfrac{1}{\mu(\mu-1)} & 0 & 0 & \mu^{-1} \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.16}$$

The characteristic equation of the matrix B is

$$\lambda^4 - \lambda^2\left(\frac{\mu^2+\mu-1}{\mu^2}\right) + \frac{\mu-1}{\mu^2} = 0. \tag{2.17}$$

We can easily prove that equation (2.17) has the following roots: $1, -1, \sqrt{\mu-1}/\mu$, $-(\sqrt{\mu-1}/\mu)$. Then since two roots of (2.17) are of modulus less than 1 there exist positive solutions of (1.3) which tend to the positive equilibrium $(\mu, \mu/(\mu-1))$ of (1.3). This completes the proof of the proposition. □

In what follows we study the asymptotic behavior of the positive solutions of system (1.3) when $0 \le A < 1$, $A = 1$, $A > 1$.

Case 2.5 ($0 \le A < 1$). In this case, we find positive solutions (x_n, y_n) of system (1.3) which are not bounded.

Proposition 2.6. *Consider system (1.3) where $0 \le A < 1$. Let (x_n, y_n) be a positive solution of system (1.3) such that*

$$y_0 < 1, \qquad x_0 < 1, \qquad x_{-1} > \frac{1}{1-A}, \qquad y_{-1} > \frac{1}{1-A}. \tag{2.18}$$

Then,

$$\lim_{n\to\infty} x_{2n+1} = \infty, \qquad \lim_{n\to\infty} x_{2n} = A, \qquad \lim_{n\to\infty} y_{2n+1} = \infty, \qquad \lim_{n\to\infty} y_{2n} = A. \tag{2.19}$$

Proof. We prove by induction that

$$x_{2n+1} > A + x_{2n-1}, \quad x_{2n} < 1, \quad y_{2n+1} > A + y_{2n-1}, \quad y_{2n} < 1, \quad n = 0,1,\ldots. \tag{2.20}$$

From (1.3) and (2.18) it is obvious that (2.20) are satisfied for $n = 0$. Suppose that (2.20) hold for $n \in \{0, 1, \dots, s\}$. Then, from (1.3) and (2.18), we have

$$x_{2s+2} = A + \frac{x_{2s}}{y_{2s+1}} < A + \frac{1}{y_{-1}} < 1, \qquad y_{2s+2} < 1. \tag{2.21}$$

Therefore, relations (1.3) and (2.21) imply that

$$x_{2s+3} = A + \frac{x_{2s+1}}{y_{2s+2}} > A + x_{2s+1}, \qquad y_{2s+3} > A + y_{2s+1}. \tag{2.22}$$

Hence (2.20) are satisfied.

If $A \neq 0$, then, from (2.20), it is obvious that

$$\lim_{n\to\infty} x_{2n+1} = \infty, \qquad \lim_{n\to\infty} y_{2n+1} = \infty. \tag{2.23}$$

From (1.3) we have

$$x_{2n} = A + \frac{x_{2n-2}}{y_{2n-1}}, \qquad y_{2n} = A + \frac{y_{2n-2}}{x_{2n-1}}. \tag{2.24}$$

Relations (2.20), (2.23), and (2.24) imply that

$$\lim_{n\to\infty} x_{2n} = A, \qquad \lim_{n\to\infty} y_{2n} = A. \tag{2.25}$$

Using (2.23) and (2.25) the proof is completed if $A \neq 0$.

Now let $A = 0$. Using (1.3), it holds

$$x_{2n+1} = \frac{x_{2n-1}}{y_{2n}}, \qquad x_{2n+2} = \frac{x_{2n}}{y_{2n+1}}, \qquad y_{2n+1} = \frac{y_{2n-1}}{x_{2n}}, \qquad y_{2n+2} = \frac{y_{2n}}{x_{2n+1}}. \tag{2.26}$$

From (2.18) and (2.20) there exist

$$\lim_{n\to\infty} x_{2n+1} = L_1, \quad \lim_{n\to\infty} y_{2n+1} = L_2, \quad L_1, L_2 \in (1, \infty]. \tag{2.27}$$

If $L_1 < \infty$ (resp., $L_2 < \infty$), then from (2.26) we have,

$$\lim_{n\to\infty} y_{2n} = 1, \qquad L_1 = 1, \quad (\text{resp., } \lim_{n\to\infty} x_{2n} = 1,\ L_2 = 1) \tag{2.28}$$

which contradicts to the fact that $L_1 > 1$ (resp., $L_2 > 1$). Hence

$$L_1 = \infty, \qquad L_2 = \infty. \tag{2.29}$$

Then, from (2.20), (2.26), and (2.29), it follows that

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = 0. \tag{2.30}$$

Therefore from (2.27), (2.29), and (2.30) the proof of the proposition is completed. □

Case 2.7 ($A = 1$)**.** In this case, we prove that every positive solution of (1.3) tends to a period 2 solution.

We need the following lemma.

Lemma 2.8. *Let $A = 1$ and (x_n, y_n) be a positive solution of system (1.3). Then, the following statements are true:*

(i) *If $x_{-1}y_0 \le y_0 + x_{-1}$ (resp., $x_{-1}y_0 \ge y_0 + x_{-1}$), then the sequences x_{2n+1}, y_{2n} are nondecreasing (resp., nonincreasing).*

(ii) *If $y_{-1}x_0 \le x_0 + y_{-1}$ (resp., $y_{-1}x_0 \ge x_0 + y_{-1}$), then the sequences x_{2n}, y_{2n+1} are nondecreasing (resp., nonincreasing).*

(iii) *There exist*

$$\lim_{n\to\infty} x_{2n+1} = L, \qquad \lim_{n\to\infty} y_{2n} = \frac{L}{L-1},$$
$$\lim_{n\to\infty} x_{2n} = M, \qquad \lim_{n\to\infty} y_{2n+1} = \frac{M}{M-1}, \quad L, M \in (1,\infty). \tag{2.31}$$

Proof. (i) If $x_{-1}y_0 \le y_0 + x_{-1}$, then from (1.3) we have

$$x_{-1} \le x_1. \tag{2.32}$$

Now, suppose that for a $k \in \{0, 1, \dots\}$

$$x_{2k-1} \le x_{2k+1}. \tag{2.33}$$

Using (1.3), it follows that

$$x_{2k+3} - x_{2k+1} = \frac{y_{2k+2} + x_{2k+1}(1 - y_{2k+2})}{y_{2k+2}} = \frac{y_{2k+2} - y_{2k}}{y_{2k+2}}. \tag{2.34}$$

Similarly, we take

$$y_{2k+2} - y_{2k} = \frac{x_{2k+1} - x_{2k-1}}{x_{2k+1}}. \tag{2.35}$$

Therefore relations (2.33), (2.34), and (2.35) imply that

$$x_{2k+1} \le x_{2k+3}. \tag{2.36}$$

Since (2.32), (2.33), and (2.36) hold, then by induction we take that x_{2n+1} is a nondecreasing function. Then from (2.35) y_{2n} is also a nondecreasing function. Similarly, if $x_{-1}y_0 \ge y_0 + x_{-1}$, we can prove that x_{2n+1}, y_{2n} are nonincreasing functions.

(ii) The proof is similar to the proof of (i).

(iii) From (i) and (ii) we have that x_{2n+1}, y_{2n} (resp., x_{2n}, y_{2n+1}) are both nondecreasing or both nonincreasing functions.

Suppose first that x_{2n+1}, y_{2n} are nondecreasing functions. Then there exist

$$\lim_{n\to\infty} x_{2n+1} = L, \quad \lim_{n\to\infty} y_{2n} = N, \quad L, N \in (1,\infty]. \tag{2.37}$$

Furthermore, from (1.3) we get

$$x_{2n+1} = 1 + \frac{x_{2n-1}}{y_{2n}}, \qquad y_{2n+2} = 1 + \frac{y_{2n}}{x_{2n+1}}. \tag{2.38}$$

From (2.37) and (2.38) we have

$$L \ne \infty \quad \text{and} \quad N \ne \infty \qquad \text{or} \qquad L = \infty \quad \text{and} \quad N = \infty. \tag{2.39}$$

Suppose that $L = \infty$ and $N = \infty$. Then relations (2.38) imply that

$$\lim_{n\to\infty} \frac{x_{2n-1}}{y_{2n}} = \lim_{n\to\infty} \frac{y_{2n}}{x_{2n+1}} = \infty \tag{2.40}$$

from which it is obvious that

$$\lim_{n\to\infty} \frac{x_{2n-1}}{x_{2n+1}} = \lim_{n\to\infty} \frac{y_{2n}}{y_{2n+2}} = \infty, \tag{2.41}$$

which contradicts to the fact that x_{2n+1}, y_{2n} are nondecreasing functions. Therefore from (2.38) and (2.39) we have

$$L \neq \infty, \quad N \neq \infty, \quad N = \frac{L}{L-1}. \tag{2.42}$$

Similarly, if x_{2n}, y_{2n+1} are nondecreasing functions, we can prove that there exist

$$\lim_{n\to\infty} x_{2n} = M, \quad \lim_{n\to\infty} y_{2n+1} = \frac{M}{M-1}, \quad M \in (1,\infty). \tag{2.43}$$

Now, suppose that x_{2n+1}, y_{2n} are nonincreasing functions. Then there exist

$$\lim_{n\to\infty} x_{2n+1} = L, \quad \lim_{n\to\infty} y_{2n} = N, \quad L,\ N \in [1,\infty). \tag{2.44}$$

Then, from (2.38), it is obvious that

$$L \neq 1, \quad N = \frac{L}{L-1}. \tag{2.45}$$

Similarly, if x_{2n}, y_{2n+1} are nonincreasing functions, we have that (2.43) are satisfied. Therefore, from (2.37), (2.42), (2.43), (2.44), and (2.45), the proof of (iii) is completed. This completes the proof of the lemma. □

Using Proposition 2.3 and Lemma 2.8 the following proposition follows immediately.

Proposition 2.9. *Consider system (1.3) where $A = 1$. Then the following statements are true:*

(i) *Every positive solution of system (1.3) tends to a period 2 solution as $n \to \infty$.*

(ii) *Moreover, if for a positive solution (x_n, y_n) of system (1.3) there exist an $s \in \{-1, 0, \dots\}$ and a $\mu \in (1,\infty)$ such that for $n \geq s$ $x_n \geq \mu$, $y_n \geq \mu/(\mu-1)$ (resp., $x_n < \mu$, $y_n < \mu/(\mu-1)$) then x_n (resp., y_n) tends to μ (resp., $\mu/(\mu-1)$) as $n \to \infty$.*

Case 2.10 ($A > 1$). In the following proposition we prove that the positive equilibrium (c, c) of system (1.3) is globally asymptotically stable.

Proposition 2.11. *Consider system (1.3) where $A > 1$. Then the positive equilibrium (c, c) of system (1.3) is globally asymptotically stable.*

Proof. We prove that every positive solution (x_n, y_n) of system (1.3) tends to the positive equilibrium (c, c) of (1.3) as $n \to \infty$. First we prove that for $n = 1, 2, \dots,$

$$\begin{aligned} A < x_n &\leq c_1\left(\frac{1}{\sqrt{A}}\right)^n + c_2\left(-\frac{1}{\sqrt{A}}\right)^n + \frac{A^2}{A-1}, \\ A < y_n &\leq c_3\left(\frac{1}{\sqrt{A}}\right)^n + c_4\left(-\frac{1}{\sqrt{A}}\right)^n + \frac{A^2}{A-1}, \end{aligned} \tag{2.46}$$

where

$$\begin{aligned} c_1 &= \frac{1}{2}\left(x_0+\sqrt{A}x_1-\frac{A^2}{A-1}\left(1+\sqrt{A}\right)\right), \\ c_2 &= \frac{1}{2}\left(x_0-\sqrt{A}x_1-\frac{A^2}{A-1}\left(1-\sqrt{A}\right)\right), \\ c_3 &= \frac{1}{2}\left(y_0+\sqrt{A}y_1-\frac{A^2}{A-1}\left(1+\sqrt{A}\right)\right), \\ c_4 &= \frac{1}{2}\left(y_0-\sqrt{A}y_1-\frac{A^2}{A-1}\left(1-\sqrt{A}\right)\right). \end{aligned} \tag{2.47}$$

From (1.3) it is obvious that

$$A \le x_n, y_n, \quad n \ge 1. \tag{2.48}$$

Then, using (1.3) and (2.48), we get

$$x_{n+1} \le A+\frac{x_{n-1}}{A}, \quad n \ge 1. \tag{2.49}$$

We can easily find that the solution u_n of the difference equation

$$u_{n+1} = A+\frac{u_{n-1}}{A}, \quad n \ge 1, \tag{2.50}$$

such that $u_0 = x_0,\ u_1 = x_1$, is the following

$$u_n = c_1\left(\frac{1}{\sqrt{A}}\right)^n + c_2\left(-\frac{1}{\sqrt{A}}\right)^n + \frac{A^2}{A-1}, \quad n \ge 0. \tag{2.51}$$

Then, from (2.48), (2.49), (2.50), and (2.51) and arguing as in Proposition 2.3, we can prove that the first inequalities of (2.46) are satisfied. Similarly we can prove the second inequalities of (2.46) are satisfied.

From (2.46), we can set

$$\limsup_{n\to\infty} x_n = L_1, \qquad \liminf_{n\to\infty} x_n = m_1, \qquad \limsup_{n\to\infty} y_n = L_2, \qquad \liminf_{n\to\infty} y_n = m_2. \tag{2.52}$$

Relations (1.3) and (2.52) imply that

$$L_1 \le A+\frac{L_1}{m_2}, \qquad m_1 \ge A+\frac{m_1}{L_2}, \qquad L_2 \le A+\frac{L_2}{m_1}, \qquad m_2 \ge A+\frac{m_2}{L_1}. \tag{2.53}$$

Relations (2.53) imply that

$$(A-1)(L_1-m_2) \le 0, \qquad (A-1)(L_2-m_1) \le 0. \tag{2.54}$$

Since $A > 1$ we have

$$L_1 \le m_2 \le L_2, \qquad L_2 \le m_1 \le L_1, \tag{2.55}$$

from which

$$L_1 = L_2 = m_1 = m_2. \tag{2.56}$$

Therefore every positive solution (x_n, y_n) of system (1.3) tends to (c,c) as $n \to \infty$. Then, since, from (ii) of Proposition 2.4 the positive equilibrium (c,c) is locally asymptotically stable, the proof of the proposition is completed. □

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