Internat. J. Math. & Math. Sci.
Vol. 23, No. 12 (2000) 855–857
S0161171200001824


# ON A CLASS OF UNIVALENT FUNCTIONS

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(Received 3 March 1998)

ABSTRACT. We consider the class of univalent functions $f(z) = z + a_3z^3 + a_4z^4 + \cdots$ analytic in the unit disc and satisfying $|(z^2 f'(z)/f^2(z)) - 1| < 1$, and show that such functions are starlike if they satisfy $|(z^2 f'(z)/f^2(z)) - 1| < (1/\sqrt{2})$.

Keywords and phrases. Analytic, univalent and starlike functions.

2000 Mathematics Subject Classification. Primary 30C45.

Let $A$ denote the class of functions which are analytic in the unit disc $U = \{z : |z| < 1\}$ and have Taylor series expansion

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots, \tag{1}$$

and let $T$ be the univalent [3] subclass of $A$ which satisfy

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \quad z \in U. \tag{2}$$

By $T_2$ we denote the subclass of $T$ for which $f''(0) = 0$. In this paper, we prove the following theorem.

**THEOREM 1.** *If $f \in T_2$, then*
(i) $\mathrm{Re}(f(z)/z) > 1/2$, $z \in U$,
(ii) *$f$ is starlike in $|z| < 1/\sqrt[4]{2} = 0.840896\ldots$,*
(iii) $\mathrm{Re}\, f'(z) > 0$ *for* $|z| < 1/\sqrt{2}$.

Items (i) and (iii) are improvements of results in [2], and (ii) is the same as in [2] but has a different proof. Furthermore, (i) and (iii) are sharp as shown by the function

$$f(z) = \frac{z}{1 - z^2}, \tag{3}$$

but the sharpness of (ii) is difficult to establish by a direct example. We also prove the following theorem which partially answers a question raised in [1].

**THEOREM 2.** *If $T_{2,\mu}$ is the subclass of $T_2$ which satisfies*

$$\left| z^2 \frac{f'(z)}{f^2(z)} - 1 \right| < \mu < 1, \tag{4}$$

*then $T_{2,\mu}$ is a subclass of starlike functions if $0 \le \mu \le 1/\sqrt{2}$.*

We define by $B$ the class of functions $\omega$ analytic in $U$ and satisfying

$$|\omega(z)| < 1, \quad z \in U, \qquad \omega(0) = \omega'(0) = 0. \tag{5}$$

From Schwarz's lemma it then follows that

$$|\omega(z)| \le |z|^2. \tag{6}$$

**Proof of Theorem 1.** If $f \in T_2$ and satisfies (2), then

$$z^2 \frac{f'(z)}{f^2(z)} - 1 = \omega(z), \quad z \in U, \ \omega \in B, \tag{7}$$

and by direct integration

$$\frac{z}{f(z)} = 1 - \int_0^1 \frac{\omega(tz)}{t^2}\, dt, \quad z \in U, \ \omega \in B. \tag{8}$$

From (8), we obtain

$$\left| \frac{z}{f(z)} - 1 \right| \le |z|^2 < 1, \tag{9}$$

and this gives

$$\left| 1 - \frac{f(z)}{z} \right| \le \left| \frac{f(z)}{z} \right|, \tag{10}$$

which is equivalent to (Re $f(z)/z) > 1/2$, This proves (i).

Furthermore, from (9), we obtain

$$\left| \arg \frac{f(z)}{z} \right| \le \sin^{-1} |z|^2. \tag{11}$$

From (7), we obtain

$$z \frac{f'(z)}{f(z)} = \frac{f(z)}{z} (1 + \omega(z)) \tag{12}$$

and, therefore,

$$\left| \arg \frac{zf'(z)}{f(z)} \right| = \left| \arg \frac{f(z)}{z} + \arg (1 + \omega(z)) \right| \le 2 \sin^{-1} |z|^2. \tag{13}$$

This gives (ii).

In order to prove (iii), we notice that (7) yields

$$f'(z) = \left( \frac{f(z)}{z} \right)^2 (1 + \omega(z)) \tag{14}$$

and, therefore,

$$|\arg f'(z)| = \left| 2 \arg \frac{f(z)}{z} + \arg (1 + \omega(z)) \right| \le 3 \sin^{-1} |z|^2. \tag{15}$$

But this is equivalent to (iii). □

**Proof of Theorem 2.** If $f \in T_{2,\mu}$, we obtain from (4)

$$z \frac{f'(z)}{f^2(z)} - 1 = \mu \omega(z), \quad \omega \in B, \ z \in U \qquad \text{and} \qquad \frac{z}{f(z)} = 1 - \mu \int_0^1 \frac{\omega(tz)}{t^2}\, dt. \tag{16}$$

Hence

$$z\frac{f'(z)}{f(z)} = \frac{1+\mu\omega(z)}{1-\mu\int_0^1(\omega(tz)/t^2)\,dt}. \tag{17}$$

Now Re $z(f'(z)/f(z)) > 0$ is equivalent to the condition

$$z\frac{f'(z)}{f(z)} = \frac{1+\mu\omega(z)}{1-\mu\int_0^1(\omega(tz)/t^2)dt} \neq -iT, \quad T \in \text{Re}. \tag{18}$$

Relation (18) is equivalent to

$$\frac{\mu}{2}\left[\left(\omega(z)+\int_0^1\frac{\omega(tz)}{t^2}\,dt\right)+\frac{1-iT}{1+iT}\left(\omega(z)-\int_0^1\frac{\omega(tz)}{t^2}\,dt\right)\right] \neq -1. \tag{19}$$

Let

$$M = \sup_{z\in U,\omega\in B,T\in\text{Re}}\left|\left[\left(\omega(z)+\int_0^1\frac{\omega(tz)}{t^2}\,dt\right)+\frac{1-iT}{1+iT}\left(\omega(z)-\int_0^1\frac{\omega(tz)}{t^2}\,dt\right)\right]\right|, \tag{20}$$

then, in view of the rotation invariance of $B$, it follows that

$$\text{Re}\,z\frac{f'(z)}{f(z)} > 0, \quad \text{if } \mu \le \frac{2}{M}. \tag{21}$$

However, from (20), we notice that

$$\begin{aligned} M &\le \sup_{z\in U,\omega\in B}\left[\left|\omega(z)+\int_0^1\frac{\omega(tz)}{t^2}\,dt\right|+\left|\omega(z)-\int_0^1\frac{\omega(tz)}{t^2}\,dt\right|\right] \\ &\le 2\sup_{z\in U_i,\omega\in B}\left[\sqrt{\left|\omega(z)\right|^2+\left|\int_0^1\frac{\omega(tz)}{t^2}\,dt\right|^2}\right] \le 2\sqrt{2}. \end{aligned} \tag{22}$$

Inequality (22) follows from the parallelogram law and the last step from (6). And (21) shows that $\mu \le 1/\sqrt{2}$. □

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