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ON CLOSED-FORM SOLUTIONS OF SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is devoted to closed-form solutions of the partial differential equation: $\theta_{xx} + \theta_{yy} + \delta \exp(\theta) = 0$, which arises in the steady state thermal explosion theory. We find simple exact solutions of the form $\theta(x,y) = \Phi(F(x)+G(y))$, and $\theta(x,y) = \Phi(f(x+y)+g(x-y))$. Also, we study the corresponding nonlinear wave equation.

Keywords and phrases. Thermal explosion theory, exothermic reaction, endothermic reaction, large activation energy.

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1. Introduction. In this paper, we study the following standard nonlinear partial differential equation, which occurs in combustion theory.

$$\theta_{xx} + \theta_{yy} + \delta \exp(\theta) = 0, \tag{1.1}$$

where as usual x and y denote cartesian coordinates, $\theta(x,y)$ is the dimensionless temperature, and δ is a strictly positive constant which is sometimes referred to as the Frank-Kamenetskii parameter. In particular, (1.1) arises in the thermal explosion theory of an exothermic reaction of a gas, taking place in a closed vessel, or of a solid in the important limiting case of large activation energy [3, 4].

In a recent related paper, Rubel [7] found simple exact solutions of (1.1) with $\delta = 0$ via quasisolutions of the form $\Phi(F(x)+G(y))$. Rubel pointed out that the organized method of differential-stäcked matrices in [5] has some advantages over the *ad hoc* method of [7], but that the calculations are still lengthy. It is worth pointing out that simple exact solutions for many nonlinear partial differential equations are always important, because closed-form (or explicit) solutions are so hard to come by that any examples are valuable in themselves.

The present paper considers (1.1), which was examined in [1] for the case of a slab with spatially periodic surface temperature, but we found simple exact solutions of a kind of separation of variables of the form

$$\theta(x,y) = \Phi(F(x)+G(y)). \tag{1.2}$$

We also study the corresponding nonlinear wave equation. However, it is worth pointing out that Stuart [9] investigated a class of solutions of (1.1) which arises in the nonlinear inviscid incompressible motion of laminar fluid.

The germ of our procedure is contained in the paper by Rubel [7] and the solutions are better left in the more general form. All functions appearing in this paper are supposed to be real analytic on a domain in the appropriate Euclidean space.

2. Laplace's equation with nonlinear source term

Theorem 2.1. *Suppose that $\theta = \Phi(F(x)+G(y))$ is a nonconstant real solution of (1.1). Then for $\delta > 0$, there exists some real constants A, B, C, and λ such that θ has at least the following six forms:*

$$\theta = -2\ln\left[A+B(x^2+y^2)\right] \quad \textit{provided } 8AB=\delta, \tag{2.1}$$

$$\begin{aligned} &\theta = -2\ln\left[A+Bx+C(x^2+y^2)\right] \quad \textit{provided } 8AC = 2B^2+\delta \\ &(\textit{also, } \theta = -2\ln\left[A+By+C(x^2+y^2)\right] \quad \textit{provided } 8AC = 2B^2+\delta, \\ &\textit{and } \theta = -2\ln\left[A+B(x+y)+C(x^2+y^2)\right] \quad \textit{provided } 8AC = 4B^2+\delta), \end{aligned} \tag{2.2}$$

$$\theta = -2\ln\left[A\cosh\lambda x\right] \quad \textit{provided } 2\lambda^2A^2=\delta, \tag{2.3}$$

$$\theta = -2\ln\left[A\cosh\lambda y\right] \quad \textit{provided } 2\lambda^2A^2=\delta, \tag{2.4}$$

$$\theta = -2\ln\left[A\cosh\lambda x - B\cos\lambda y\right] \quad \textit{provided } 2\lambda^2(A^2-B^2)=\delta, \tag{2.5}$$

$$\theta = -2\ln\left[A\cosh\lambda y - B\cos\lambda x\right] \quad \textit{provided } 2\lambda^2(A^2-B^2)=\delta. \tag{2.6}$$

Furthermore, each of these cases gives a solution of the form $\Phi(F(x)+G(y))$.

Proof. Let $z = x+iy$. Then $\bar{z} = x-iy$. Therefore,

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z\,\partial\bar{z}} + \frac{\partial^2}{\partial \bar{z}^2} \tag{2.7}$$

and

$$\frac{\partial^2}{\partial y^2} = -\left(\frac{\partial^2}{\partial z^2} - 2\frac{\partial^2}{\partial z\,\partial\bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}\right). \tag{2.8}$$

Substituting (2.7) and (2.8) into (1.1), we obtain

$$\frac{\partial^2\theta}{\partial z\,\partial\bar{z}} + \frac{\delta}{4}\exp(\theta) = 0. \tag{2.9}$$

By differentiating (2.9) partially with respect to z and then eliminating the exponential term between the resulting expression and (2.9), we find that

$$\frac{\partial^3\theta}{\partial^2 z\,\partial\bar{z}} - \frac{\partial\theta}{\partial z}\frac{\partial^2\theta}{\partial z\,\partial\bar{z}} = 0. \tag{2.10}$$

Equation (2.10) can be expressed as

$$\frac{\partial}{\partial\bar{z}}\left(\frac{\partial^2\theta}{\partial z^2} - \frac{1}{2}\left(\frac{\partial\theta}{\partial z}\right)^2\right) = 0. \tag{2.11}$$

Integrating (2.11) with respect to $\overline{z}$, we get

$$\frac{\partial^2\theta}{\partial z^2}-\frac{1}{2}\left(\frac{\partial\theta}{\partial z}\right)^2=-2P(z), \tag{2.12}$$

where $P(z)$ is an arbitrary function and can be chosen to generate all the required explicit solutions. Define $\alpha(z,\overline{z})$ by

$$\frac{\partial\theta}{\partial z}=-\frac{2}{\alpha}\frac{\partial\alpha}{\partial z}. \tag{2.13}$$

Then by substituting in (2.12), we get

$$\frac{\partial^2\alpha}{\partial z^2}=P(z)\alpha. \tag{2.14}$$

The required solution can be written in the form

$$\alpha(z,\overline{z})=\alpha_1(z)\overline{\alpha}_1(\overline{z})+\alpha_2(z)\overline{\alpha}_2(\overline{z}), \tag{2.15}$$

where $\alpha_1(z)$ and $\alpha_2(z)$ are analytic functions of z and independent solutions of (2.14). The definition (2.13) gives

$$\alpha=\exp\left(-\frac{\theta}{2}\right). \tag{2.16}$$

Differentiating (2.13) partially with respect to $\overline{z}$ gives

$$\frac{\partial^2\theta}{\partial z\,\partial\overline{z}}=-2\left(\frac{1}{\alpha}\frac{\partial^2\alpha}{\partial z\,\partial\overline{z}}-\frac{1}{\alpha^2}\frac{\partial\alpha}{\partial\overline{z}}\frac{\partial\alpha}{\partial z}\right). \tag{2.17}$$

In (2.17), substituting for α from (2.15) results in

$$\frac{\partial^2\theta}{\partial z\,\partial\overline{z}}=-\frac{2}{\alpha^2}|\mu|^2, \tag{2.18}$$

where

$$\mu(z)=\alpha_1\frac{d\alpha_2}{dz}-\alpha_2\frac{d\alpha_1}{dz}. \tag{2.19}$$

Substitution of (2.16) and (2.18) in (2.9) gives

$$|\mu|^2=\frac{\delta}{8}. \tag{2.20}$$

To obtain the low-degree polynomials in (x,y), we chose $P(z)=0$.

Case 1 of Theorem 2.1. Let $\alpha_1=A^{1/2}$ and $\alpha_2=B^{1/2}z$, where A and B are real constants. Substituting in (2.15) then gives $\alpha=\exp(-\theta/2)=A+B(x^2+y^2)$. Thus, we have $8AB=\delta$.

Case 2 of Theorem 2.1. Let $\alpha_1=B_0^{1/2}+C^{1/2}z$ and $\alpha_2=A_0^{1/2}$, where A_0, B_0, and C are real constants. Substituting in (2.15) then gives $\alpha=\exp(-\theta/2)=A+Bx+C(x^2+y^2)$, where $A=A_0+B_0$ and $B=2(B_0C)^{1/2}$. Thus, we obtain $8AC-2B^2=\delta$.

Case 3 of Theorem 2.1. Let $P(z) = \lambda^2/4$, $\alpha_1 = (A/2)^{1/2}\exp(\lambda z/2)$, and $\alpha_2 = (A/2)^{1/2}\exp(-\lambda z/2)$, where A and λ are real constants. Thus, $\alpha = \exp(-\theta/2) = A\cosh\lambda x$ with $2\lambda^2A^2 = \delta$.

Case 4 of Theorem 2.1. Let $P(z) = -\lambda^2/4$, $\alpha_1 = (A/2)^{1/2}\exp(i\lambda z/2)$, and $\alpha_2 = (A/2)^{1/2}\exp(i\lambda z/2)$, where A and λ are real constants. Hence, $\alpha = \exp(-\theta/2) = A\cosh\lambda y$ with $2\lambda^2A^2 = \delta$.

Case 5 of Theorem 2.1. Let $P(z) = \lambda^2/4$, $\alpha_1 = (A+B)^{1/2}\sinh(\lambda z/2)$, and $\alpha_2 = (A-B)^{1/2}\cosh(\lambda z/2)$, where A, B, and λ are real constants. Therefore, $\alpha = \exp(-\theta/2) = A\cosh\lambda x - B\cos\lambda y$ with $2\lambda^2(A^2-B^2) = \delta$.

Case 6 of Theorem 2.1. Let $P(z) = -\lambda^2/4$, $\alpha_1 = (A+B)^{1/2}\sin(\lambda z/2)$, and $\alpha_2 = (A-B)^{1/2}\cos(\lambda z/2)$, where A, B, and λ are real constants. Therefore, $\alpha = \exp(-\theta/2) = A\cosh\lambda y - B\cos\lambda x$ with $2\lambda^2(A^2-B^2) = \delta$.

Finally, the other choices of α_1 and α_2 for some other possibilities of $P(z)$ that we try in Remark 2 do not satisfy the conditions of Theorem 2.1. Thus, the first part of the Theorem is proved. The 'furthermore' part is easily verified by direct computations. This completes the proof of the theorem. □

Theorem 2.2. *Suppose that $\theta = \Phi(f(x+y)+g(x-y))$ is a nonconstant real solution of (1.1). Then for $\delta > 0$, there exists some real constants A, B, C, and λ such that θ has at least the following six forms:*

$$\theta = -2\ln\left[A+B(x+y)^2+B(x-y)^2\right] \quad \textit{provided}\ \ 16AB = \delta, \tag{2.21}$$

$$\begin{aligned}
&\theta = -2\ln\left[A+B(x+y)+C(x+y)^2+C(x-y)^2\right] \quad \textit{provided}\ \ 16AC = 4B^2+\delta \\
\textit{(also, }&\theta = -2\ln\left[A+B(x-y)+C(x-y)^2+C(x+y)^2\right] \quad \textit{provided}\ \ 16AC = 4B^2+\delta, \\
\textit{and}\quad &\theta = -2\ln\left[A+B(x+y)+C(x+y)^2+B(x-y)+C(x-y)^2\right] \\
&\qquad\qquad \textit{provided}\ \ 16AC = 8B^2+\delta\textit{)},
\end{aligned} \tag{2.22}$$

$$\theta = -2\ln\left[A\cosh\lambda(x+y)\right] \quad \textit{provided}\ \ 4\lambda^2A^2 = \delta, \tag{2.23}$$

$$\theta = -2\ln\left[A\cosh\lambda(x-y)\right] \quad \textit{provided}\ \ 4\lambda^2A^2 = \delta, \tag{2.24}$$

$$\theta = -2\ln\left[A\cosh\lambda(x+y)-B\cos\lambda(x-y)\right] \quad \textit{provided}\ \ 4\lambda^2(A^2-B^2) = \delta, \tag{2.25}$$

$$\theta = -2\ln\left[A\cosh\lambda(x-y)-B\cos\lambda(x+y)\right] \quad \textit{provided}\ \ 4\lambda^2(A^2-B^2) = \delta. \tag{2.26}$$

Moreover, each of these cases gives a solution of the form $\Phi(f(x+y)+g(x-y))$.

Proof. Let $\xi = x+y$ and $\eta = x-y$. Then (1.1) reduces to

$$\frac{\partial^2\theta}{\partial\xi^2}+\frac{\partial^2\theta}{\partial\eta^2}+\frac{\delta}{2}\exp(\theta) = 0. \tag{2.27}$$

The general solution of (2.27) follows easily by exploiting Theorem 2.1 and this implies the conclusion of the theorem. □

3. Nonlinear wave equation in two variables. It is easy to show that $\theta(x,y)$ is a solution of the wave equation in two variables if and only if $\theta(x,iy)$ is a solution of the Laplace equation in two variables. Hence, since Theorems 2.1 and 2.2 are purely formal, we have the following results, with no need for further proofs.

Theorem 3.1. *Suppose that $\theta(x,y)=\Phi(F(x)+G(y))$ is a nonconstant solution of*

$$\theta_{xx}-\theta_{yy}+\delta\exp(\theta)=0. \tag{3.1}$$

Then for $\delta>0$, there exists some real constants A, B, C, D, and λ such that θ has at least the following six forms:

$$\theta=-2\ln[A+B(x^2-y^2)] \quad \textit{provided } 8AB=\delta, \tag{3.2}$$

$$\begin{gathered}\theta=-2\ln[A+Bx+C(x^2-y^2)] \quad \textit{provided } 8AC=2B^2+\delta \\ (\textit{also, } \theta=-2\ln[A+By+C(x^2-y^2)] \quad \textit{provided } 8AC+2B^2=\delta, \\ \textit{and } \theta=-2\ln[A+Bx+Cy+D(x^2-y^2)] \quad \textit{provided } 8AD+2C^2=2B^2+\delta),\end{gathered} \tag{3.3}$$

$$\theta=-2\ln[A\cosh\lambda x] \quad \textit{provided } 2\lambda^2A^2=\delta, \tag{3.4}$$

$$\theta=-2\ln[A\cos\lambda y] \quad \textit{provided } 2\lambda^2A^2=\delta, \tag{3.5}$$

$$\theta=-2\ln[A\cosh\lambda x-B\cosh\lambda y] \quad \textit{provided } 2\lambda^2(A^2-B^2)=\delta, \tag{3.6}$$

$$\theta=-2\ln[A\cosh\lambda y-B\cos\lambda x] \quad \textit{provided } 2\lambda^2(A^2-B^2)=\delta. \tag{3.7}$$

In addition, each of these cases gives a solution of the form $\Phi(F(x)+G(y))$.

Theorem 3.2. *Suppose that $\theta(x,y)=\Phi(f(x+iy)+g(x-iy))$ is a nonconstant real solution of*

$$\theta_{xx}-\theta_{yy}+\delta\exp(\theta)=0. \tag{3.8}$$

Then, for $\delta>0$, there exists real constants A, B, C, and λ such that θ has at least the following six forms.

$$\theta=-2\ln[A+B(x+iy)^2+B(x-iy)^2] \quad \textit{provided } 16AB=\delta, \tag{3.9}$$

$$\begin{gathered}\theta=-2\ln[A+B(x+iy)+C(x+iy)^2+B(x-iy)+C(x-iy)^2] \\ \textit{provided } 16AC=8B^2+\delta,\end{gathered} \tag{3.10}$$

$$\theta=-2\ln[A\cosh\lambda(x+iy)] \quad \textit{provided } 4\lambda^2A^2=\delta, \tag{3.11}$$

$$\theta=-2\ln[A\cosh\lambda(x-iy)] \quad \textit{provided } 4\lambda^2A^2=\delta, \tag{3.12}$$

$$\theta=-2\ln[A\cosh\lambda(x+iy)-B\cos\lambda(x-iy)] \quad \textit{provided } 4\lambda^2(A^2-B^2)=\delta, \tag{3.13}$$

$$\theta=-2\ln[A\cosh\lambda(x-iy)-B\cos\lambda(x+iy)] \quad \textit{provided } 4\lambda^2(A^2-B^2)=\delta. \tag{3.14}$$

Moreover, each of (3.9), (3.10), (3.11), (3.12), (3.13), and (3.14) is a solution of the form $\Phi(f(x+iy)+g(x-iy))$.

4. Remarks

Remark 1. In the case of $\delta < 0$, which corresponds to the situation of endothermic reaction, Theorems 2.1 and 2.2 can be suitably modified.

Remark 2. Finally, it remains to verify that the structures of α_1 and α_2 for other choices of $P(z)$ are not in accordance with the statements of the theorems:

(a) It is well known from the theory of ordinary differential equations that for the case where $P(z)$ is a complex number (say $P(z) = 2\lambda^2 i$),

$$\alpha_1 = A\exp\left(\lambda(1+i)z\right) \qquad \text{and} \qquad \alpha_2 = B\exp\left(-\lambda(1+i)z\right). \tag{4.1}$$

It is evident that $\alpha \neq F(x)+G(y)$.

(b) The nature of the solution for the case where $P(z)$ is periodic (the general Mathieu's equation, Schrödinger equation, or any linear differential equation with periodic coefficient which are one-valued functions of z like the Hill's types) is established in the results known as Floquet's and Bloch's theorems (see [6] for details). However the simple case $\alpha'' - (2\sec^2 z)\alpha = 0$ leads to $\alpha_1 = A(1+z\tan z)$ and $\alpha_2 = B\tan z$ which invariably does not conform to the statements of Theorem 2.1.

(c) Application of Fuch's theorem to the case of $P(z) = K(z-z_0)^{-n}$ $(n \geq 3)$, where K and z_0 are constants, clearly shows that a solution of the form $\alpha = \Sigma a_n(z-z_0)^{q+n}$ cannot exist at singular points z_0. Nevertheless, this theorem guarantees the convergence of solutions for the case $n = 2$ (i.e., Euler's equation) and $n = 1$ but definitely not of the required form. Observe that for the case $n = 2$, if

(i) $K = 1/4$, then $\alpha_1 = Az^{1/2}$ and $\alpha_2 = B(\ln z)z^{1/2}$.

(ii) $K > 1/4$, then $\alpha_1 = Az^{\lambda}\cos(\mu\ln z)$ and $\alpha_2 = Bz^{\lambda}\sin(\mu\ln z)$, where λ and μ are real constants.

(iii) $K < 1/4$, then $\alpha_1 = Az^r$ and $\alpha_2 = Bz^p$ with r and p being real numbers.

(d) It is often possible to transform a differential equation with variable coefficients into a Bessel equation of a certain order by a suitable change of variables. For example, it is easy to show that a solution of

$$z^2\alpha'' + \left(p^2\beta^2 z^{2\beta} + \tfrac{1}{4} - \nu^2\beta^2\right)\alpha = 0, \quad z > 0 \tag{4.2}$$

is given by $\alpha = z^{1/2}f(pz^{\beta})$, where $f(\eta)$ is a solution of the Bessel equation of order ν. Using this result, we can show that the general solution of the Airy's equation $\alpha'' - z\alpha = 0$, $z > 0$, is $\alpha_1 = Az^{1/2}f_1((2/3)iz^{3/2})$ and $\alpha_2 = Bz^{1/2}f_2((2/3)iz^{3/2})$, where $f_1(\eta)$ and $f_2(\eta)$ are linearly independent solutions of the Bessel equation of order one-third. Note also that $U(z)$ satisfies the Hermite differential equation if and only if $\alpha = \exp(-z^2/2)U$ satisfies $\alpha'' + (\lambda+1-z^2)\alpha = 0$. Hence, for $P(z)$ is a polynomial in z, the pair of functions α_1 and α_2 clearly will not lead to the required separation of variables needed in Theorem 2.1.

(e) It suffices to say that the case of $P(z)$ is a continuous function leads to two linearly independent convergent power series solutions in powers of z which are not of closed-form in nature (see [2]).

(f) Another important possibility of $P(z)$ is obtained by using the notation of elliptic functions which gives rise to Lamé's equation when

$$P(z) = n(n+1)\mathcal{P}(z) + B \tag{4.3}$$

with

$$\mathcal{P}(z) = \frac{1}{sn^2(z:k)} - \frac{(1+k^2)}{3} \quad k \neq 0 \text{ and } k \neq 1. \tag{4.4}$$

If $z = 2V$ and $\alpha = [\mathcal{P}'(V)]^{-n}L$, then Lamé's equation is transformed into

$$\frac{d^2L}{dV^2} - 2n\frac{\mathcal{P}''(V)}{\mathcal{P}'(V)}\frac{dL}{dV} + 4\{n(2n-1)\mathcal{P}'(V) - B\}L = 0. \tag{4.5}$$

It is easy to show that if $n = 1/2$ and $B = 0$, we obtain

$$\alpha_1 = A_0\left\{\mathcal{P}'\left(\tfrac{1}{2}z\right)\right\}^{-1/2}\mathcal{P}\left(\tfrac{1}{2}z\right) \qquad \text{and} \qquad \alpha_2 = B_0\left\{\mathcal{P}'\left(\tfrac{1}{2}z\right)\right\}^{-1/2}, \tag{4.6}$$

where A_0 and B_0 are arbitrary constants. However, if $B \neq 0$, then the constant B is determined by the condition that Lamé's equation should have a solution in the form of a polynomial in $\mathcal{P}(z)$ or in the form of a product of this polynomial and a factor of the form $\sqrt{\mathcal{P}(z)-e_1}$, $\sqrt{\mathcal{P}(z)-e_2}$, $\sqrt{\mathcal{P}(z)-e_3}$, where $e_1+e_2+e_3 = 0$ (see [8]). Further investigation reveals that if

$$P(z) = \frac{1}{4}\left(\frac{2}{sn^2(z/2:k)} - 1 - k^2\right), \tag{4.7}$$

then

$$\alpha_1 = \frac{(A_0+B_0)^{1/2}}{\sqrt{k}sn(z/2:k)} \tag{4.8}$$

and

$$\alpha_2 = \frac{(A_0-B_0)^{1/2}}{k^{3/2}sn(z/2:k)}\left[\frac{z}{2} - E\left(\frac{z}{2}:k\right)\right] \tag{4.9}$$

with $2(A_0^2-B_0^2) = k^4\delta$, where $E(u:k)$ is the fundamental elliptic integral of the second kind. It is well known that $E(u:k)$ is not doubly periodic and it has been established that no algebraic relation can exist connecting them with $sn(u:k)$, $cn(u:k)$, and $dn(u:k)$ [10]. Nevertheless, it is observed that if $A_0 = B_0 = 1/2$, then we obtain the limiting case $\delta = 0$ and

$$\begin{aligned}
\alpha = \frac{1}{ksn(z/2:k)sn(\bar{z}/2:k)} &= \coth\left(\frac{1}{2}\ln\left\{\frac{1+ksn(z/2:k)sn(\bar{z}/2:k)}{1-ksn(z/2:k)sn(\bar{z}/2:k)}\right\}\right) \\
&= \coth\left(\frac{1}{2}\ln\left\{\frac{dn((z+\bar{z})/2:k)-kcn((z+\bar{z})/2:k)}{dn((z-\bar{z})/2:k)-kcn((z-\bar{z})/2:k)}\right\}\right) \\
&= \coth\left(\frac{(A(x)+B(y))}{2}\right),
\end{aligned} \tag{4.10}$$

where $A(x) = \ln[dn(x:k)-kcn(x:k)]$ and $B(y) = -\ln[dn(iy:k)-kcn(iy:k)]$. Therefore, $\theta = -2\ln[\coth((A(x)+B(y))/2)]$ and [7, Thm. 1, eq. (1.5)] follows immediately. It is clear that the solutions related to elliptic functions do not conform to the statement of Theorem 2.1.

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