Internat. J. Math. & Math. Sci.
Vol. 23, No. 2 (2000) 103–107
S0161171200000508

ON A FUNCTIONAL EQUATION RELATED TO A GENERALIZATION OF FLETT'S MEAN VALUE THEOREM

T. RIEDEL and MACIEJ SABLIK

(Received 24 November 1997 and in revised form 10 June 1998)

ABSTRACT. In this paper, we characterize all the functions that attain their Flett mean value at a particular point between the endpoints of the interval under consideration. These functions turn out to be cubic polynomials and thus, we also characterize these.

Keywords and phrases. Flett's mean value theorem, functional equations, generalized polynomials.

2000 Mathematics Subject Classification. Primary 39B22.

1. Introduction. In [5], Sahoo and Riedel gave a generalization of Flett's mean value theorem [2] as follows.

THEOREM 1.1. *Let f be a real valued function which is differentiable in $[a,b]$, then there is a point $c \in (a,b)$ such that*

$$f(c)-f(a) = (c-a)f'(c) - \frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(c-a)^2. \tag{1.1}$$

It is easy to see that if $f'(b) = f'(a)$, then this reduces to Flett's mean value theorem.

Aczél [1] and Haruki [3] used the Lagrange mean value theorem to ask the question of which functions attained their mean value at a prescribed point $c \in (a,b)$, in particular, at the midpoint $c = (a+b)/2$. The answer is that only quadratic polynomials have the property that the mean value on any interval is attained at the midpoint of that interval. A natural question to ask is this same question for the above mean value theorem. It turns out that quadratic polynomials satisfy (1.1) for any c, but, more interestingly, cubic polynomials satisfy it for $c = (a+3b)/4$. Thus, the main question becomes whether cubic polynomials are the only functions having this property.

Following the approach in [1], we *pexiderize* (1.1) to obtain

$$f(c)-f(a) = (c-a)h(c) - \frac{1}{2}\frac{h(b)-h(a)}{b-a}(c-a)^2, \tag{1.2}$$

and now setting $c = (a+3b)/4$ yields

$$f\left(\frac{a+3b}{4}\right) - f(a) = \frac{3}{4}(b-a)h\left(\frac{a+3b}{4}\right) - \frac{9}{32}(b-a)\big(h(b)-h(a)\big) \tag{1.3}$$

or

$$f\left(\frac{a+3b}{4}\right) - f(a) = \frac{3}{4}(b-a)\left[h\left(\frac{a+3b}{4}\right) - \frac{3}{8}\big(h(b)-h(a)\big)\right]. \tag{1.4}$$

More generally, setting $c = sa+tb$ with $s+t = 1$ and $0 < s,\ t < 1$, we obtain

$$f(sa+tb)-f(a) = (sa+tb-a)h(sa+tb)-\frac{1}{2}\frac{h(b)-h(a)}{b-a}(sa+tb-a)^2. \tag{1.5}$$

The question we answer, in this paper, is: What are the functions f, h that satisfy the functional equations (1.4) and (1.5) for all $a, b \in \mathbb{R}$? In solving this functional equation, we do not assume any regularity conditions on f or h.

2. Solution of the functional equation. The main work in solving this functional equation is to reduce (1.4) and (1.5) to a form where we can apply the following result by Székelyhidi [6, Thm. 9.5] and Wilson [7].

Theorem 2.1. *Let G, S be commutative groups, n a nonnegative integer, φ_i, ψ_i additive functions from G into G and let* $\operatorname{Ran}(\varphi_i) \subseteq \operatorname{Ran}(\psi_i)$ $(i = 1,\dots,n+1)$. *Then if $h, h_i, \varphi_i, \psi_i$ $(i = 1,\dots,n+1)$ satisfy*

$$h(x)+\sum_{i=1}^{n+1} h_i(\varphi_i(x)+\psi_i(t)) = 0, \tag{2.1}$$

then h is a generalized polynomial of degree at most n.

Thus, we are able to prove our main result (Theorem 2.2).

Theorem 2.2. *The real valued functions f and h are solutions of the functional equation (1.5) if and only if*

$$f(x) = \begin{cases} Ax^3+Bx^2+Cx+D & \text{if } s = \frac{1}{4},\ t = \frac{3}{4}, \\ Bx^2+Cx+D & \text{if } s \neq \frac{1}{4},\ t \neq \frac{3}{4}, \end{cases} \tag{2.2}$$

$$h(x) = \begin{cases} 3Ax^2+2Bx+C & \text{if } s = \frac{1}{4},\ t = \frac{3}{4}, \\ 2Bx+C & \text{if } s \neq \frac{1}{4},\ t \neq \frac{3}{4}. \end{cases} \tag{2.3}$$

Proof. It is easy to check that the functions f and h, given above, do satisfy the functional equation (1.5).

To show that these are the only solutions, we start by rewriting (1.5) using $s+t = 1$ as follows:

$$f(a+t(b-a))-f(a) = t(b-a)\Big[h(a+t(b-a))-\frac{t}{2}[h(b)-h(a)]\Big]. \tag{2.4}$$

Now, letting $u = (b-a)/3$, we obtain

$$f(a+3tu)-f(a) = 3tu\Big[h(a+3tu)-\frac{t}{2}[h(3u+a)-h(a)]\Big]. \tag{2.5}$$

Now, we replace a by $a-tu$ in (2.5) and get

$$f(a+2tu)-f(a-tu) = 3tu\Big[h(a+2tu)-\frac{t}{2}[h((3-t)u+a)-h(a-tu)]\Big]. \tag{2.6}$$

Similarly, using $a = a - 2tu$ in (2.5), we get

$$f(a+tu) - f(a-2tu) = 3tu\left[h(a+tu) - \frac{t}{2}\left[h((3-2t)u+a) - h(a-2tu)\right]\right]. \tag{2.7}$$

Interchanging u with $-u$ in (2.7) gives

$$f(a-tu) - f(a+2tu) = -3tu\left[h(a-tu) - \frac{t}{2}\left[h((-3+2t)u+a) - h(a+2tu)\right]\right]. \tag{2.8}$$

Comparing (2.8) and (2.6) gives, for $a, u \in \mathbb{R}$,

$$\begin{aligned}\left[h(a-tu) - \frac{t}{2}\left[h((-3+2t)u+a) - h(a+2tu)\right]\right]& \\ = \left[h(a+2tu) - \frac{t}{2}\left[h((3-t)u+a) - h(a-tu)\right]\right]&,\end{aligned} \tag{2.9}$$

which simplifies to

$$\begin{aligned}t[h((3-t)u+a) - h((-3+2t)u+a) - (h(a-tu) - h(a+2tu))]& \\ = -2[h(a-tu) - h(a+2tu)]&.\end{aligned} \tag{2.10}$$

Collecting the terms of h that have the same argument, we obtain

$$(2-t)h(a+2tu) - (2-t)h(a-tu) - th((3-t)u+a) + th((-3+2t)u+a) = 0. \tag{2.11}$$

Writing $x = a + 2tu$ and dividing (2.11) by $(2-t)$ yields

$$h(x) - h(x-3tu) - \frac{t}{2-t}h(x+3(1-t)u) + \frac{t}{2-t}h(x-3u) = 0. \tag{2.12}$$

Thus, since $t \neq 0$ is fixed, (2.12) is of the form of equation (2.1) and hence, $h(x)$ is a generalized polynomial of degree at most 2,

$$h(x) = \beta(x,x) + \alpha(x) + C, \tag{2.13}$$

where β is a symmetric, biadditive function and α is an additive function and C is an arbitrary real constant.

Setting $a = 0$ in (2.5), we get

$$f(x) = x\left[h(x) - \frac{t}{2}\left[h\left(\frac{x}{t}\right) - h(0)\right]\right] + D, \tag{2.14}$$

and substituting from (2.13), we obtain

$$f(x) = x\beta(x,x) + x\alpha(x) + Cx - x\frac{t}{2}\beta\left(\frac{x}{t}, \frac{x}{t}\right) - x\frac{t}{2}\alpha\left(\frac{x}{t}\right) + D. \tag{2.15}$$

To prove the continuity of f and h, let us substitute the solutions given in (2.15) into (2.5). We see that both the left- and the right-hand side of (2.5) are polynomial functions in a and u. The equality of the two sides implies, therefore, the equality of terms which are of the same degree with respect to a and u. First, comparing the terms of degree 1 with respect to each variable, we get

$$3a\left[\alpha(3tu)-\frac{t}{2}\alpha(3u)\right]+3tu\left[\alpha(a)-\frac{t}{2}\alpha\left(\frac{a}{t}\right)\right]=3tu\alpha(a), \tag{2.16}$$

whence, substituting ta instead of a and dividing by $t/2$, we get

$$tu\alpha(a)=2a\alpha(tu)-ta\alpha(u). \tag{2.17}$$

Dividing both sides by tua, we obtain

$$\frac{\alpha(a)}{a}=2\frac{\alpha(tu)}{tu}-\frac{\alpha(u)}{u}\quad\forall a\neq 0\neq u. \tag{2.18}$$

In particular, $\alpha(a)/a$ does not depend on a and, therefore, $\alpha(a)=2Ba$ for some constant B.

Now, let us compare the terms of degree 2 with respect to a and those of degree 1 with respect to u. We get

$$6a\left[\beta(a,tu)-\frac{t}{2}\beta\left(\frac{a}{t},u\right)\right]+3tu\left[\beta(a,a)-\frac{t}{2}\beta\left(\frac{a}{t},\frac{a}{t}\right)\right]=3tu\beta(a,a). \tag{2.19}$$

Rearranging and simplifying, we get

$$6a\left[\beta(a,tu)-\frac{t}{2}\beta\left(\frac{a}{t},u\right)\right]=\frac{3t^2}{2}\beta\left(\frac{a}{t},\frac{a}{t}\right), \tag{2.20}$$

or, after substituting ta instead of a and dividing by $3t/2$,

$$4a\left[\beta(ta,tu)-\frac{t}{2}\beta(a,u)\right]=tu\beta(a,a). \tag{2.21}$$

Dividing (2.21) by a^2u, we obtain

$$4\left[\frac{\beta(ta,tu)}{au}-\frac{t}{2}\frac{\beta(a,u)}{au}\right]=\frac{t\beta(a,a)}{a^2}\quad\text{for } a\neq 0\neq u. \tag{2.22}$$

Using the symmetry of β, we infer that

$$\frac{\beta(a,a)}{a^2}=\frac{\beta(u,u)}{u^2}\quad\forall u\neq 0\neq a, \tag{2.23}$$

whence, it follows that $\beta(a,a)=3Aa^2$ for some constant A. Comparing this with formulae for f and h, we see that

$$\begin{aligned} f(x)&=3A\left(1-\frac{1}{2t}\right)x^3+Bx^2+Cx+D,\\ h(x)&=3Ax^2+2Bx+C. \end{aligned} \tag{2.24}$$

Inserting (2.24) into (1.5), we get, after simplifying,

$$27t\left(1-\frac{1}{2t}\right)Aa^2u+81t^2\left(1-\frac{1}{2t}\right)Aau^2=9tAa^2u+27t^2Aau^2\quad\forall a,u\in\mathbb{R}, \tag{2.25}$$

whence, it follows that $A=0$ provided $t\neq 3/4$. Note that, for $t=3/4$, we have $3A(1-(1/2t))=A$ and the assertion follows from (2.24). □

References

[1] J. Aczél, *A mean value property of the derivative of quadratic polynomials—without mean values and derivatives*, Math. Mag. **58** (1985), no. 1, 42–45. MR 86c:39012. Zbl 571.39005.

[2] T. M. Flett, *A mean value theorem*, Math. Gaz. **42** (1958), 38–39.

[3] S. Haruki, *A property of quadratic polynomials*, Amer. Math. Monthly **86** (1979), no. 7, 577–579. MR 80g:26010. Zbl 413.39003.

[4] M. Sablik, *A remark on a mean value property*, C. R. Math. Rep. Acad. Sci. Canada **14** (1992), no. 5, 207–212. MR 94b:39039. Zbl 796.39014.

[5] P. K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific Publishing Co., NJ, 1998. CMP 1 692 936.

[6] L. Székelyhidi, *Convolution Type Functional Equations on Topological Abelian Groups*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1991. MR 92f:39017. Zbl 748.39003.

[7] W. H. Wilson, *On a certain general class of functional equations*, Amer. J. Math. **40** (1918), 263–282.

Riedel: Department of Mathematics, University of Louisville, Louisville, KY 40292, USA

E-mail address: Thomas.Riedel@louisville.edu

Sablik: Instytut Matematyki, Uniwersytet Śląski, Ul. Bankowa 14, Pl-40-007 Katowice, Poland

E-mail address: mssablik@us.edu.pl