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SUPERCONVERGENCE OF A FINITE ELEMENT METHOD FOR LINEAR INTEGRO-DIFFERENTIAL PROBLEMS

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ABSTRACT. We introduce a new way of approximating initial condition to the semidiscrete finite element method for integro-differential equations using any degree of elements. We obtain several superconvergence results for the error between the approximate solution and the Ritz-Volterra projection of the exact solution. For $k > 1$, we obtain first order gain in $L_p(2 \le p \le \infty)$ norm, second order in $W^{1,p}(2 \le p \le \infty)$ norm and almost second order in $W^{1,\infty}$ norm. For $k = 1$, we obtain first order gain in $W^{1,p}(2 \le p \le \infty)$ norms. Further, applying interpolated postprocessing technique to the approximate solution, we get one order global superconvergence between the exact solution and the interpolation of the approximate solution in the L_p and $W^{1,p}(2 \le p \le \infty)$.

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1. Introduction. Consider the following problem with memory:

$$u_t = \nabla \cdot \left\{ a(x,t)\nabla u + \int_0^t b(x,t,\tau)\nabla u(x,\tau)d\tau \right\} + f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{1.1a}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.1b}$$

$$u(x,t) = 0, \quad x \in \partial\Omega \times [0,T], \tag{1.1c}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and a, b, f, u_0 are bounded together with their derivatives and

$$0 < a_0 \le a(x,t), \quad (x,t) \in \Omega \times [0,T]. \tag{1.2}$$

For the existence, uniqueness and stability of the above integro-differential equations, we refer to [2, 10, 12]. The weak form of (1.1) is to find a map $u(t) : [0,T] \to H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$, the following holds:

$$(u_t, v) + (a(t)\nabla u, \nabla v) + \left(\int_0^t b(t,\tau)\nabla u(\tau)d\tau, \nabla v \right) = (f(t), v), \quad u(0) = u_0. \tag{1.3}$$

For the semidiscrete finite element methods of such problems, various error estimates are known, for example, maximum norm error estimate for $k = 1$ was shown by Lin [8], Zhang and Lin [14], optimal L_p error estimates are by Lin, Thomée and Wahlbin [9], maximum norm estimates are shown by [4, 13], optimal L_2 error estimates for nonlinear equations are shown by Cannon and Lin [1], while optimal L_2 estimates for

Crank-Nicolson scheme are shown in [7]. In this paper, we introduce a semidiscrete finite element formulation for solving (1.3) using a new initial approximation U_0 (see (2.5)). This initial condition enables us to obtain a higher order accuracy of our finite element solution U to the Ritz-Volterra projection $V_h u$ of the exact solution u for all $k \geq 1$. Furthermore, our estimates can be used to improve the approximation accuracy of U to u by certain postprocessing as in [4, 5, 6]. For this purpose, we proceed as follows. First, we adopt the rectangular partition τ_h on the domain Ω and then introduce two interpolation operators, the "vertices-edges-element" interpolation operator i_h^k and high interpolation operator I_{2h}^{k+1}. Applying the properties of these operators and superconvergence of $U - V_h u$, we can easily establish superconvergence results of $u - I_{2h}^{k+1} U$ to gain one order comparing with the standard finite element methods of degree k in both L_p and $W^{1,p}$, $(2 \leq p \leq \infty)$ norms by one order.

The rest of this paper is organized as follows. In Section 2, we give the semidiscrete Galerkin approximation scheme of the problem and define the Ritz-Volterra projection. Section 3 is devoted to the superconvergence of $U - V_h u$ for $k > 1$. The superconvergence estimate of $U - V_h u$ for $k = 1$ are derived in Section 4. Finally, the interpolated postprocessing technique is discussed in Section 5. The global superconvergence results of $u - I_{2h}^{k+1} U$ are demonstrated in Theorem 5.5.

2. The approximation scheme and Ritz-Volterra projection. First, let us describe some of the notation used throughout this paper. Let $L_2(\Omega)$, $L_p(\Omega)$ and $W^{m,p}(\Omega)$, $H^m(\Omega) = W^{m,2}(\Omega)$ for any integer $m \geq 0$ and $1 \leq p \leq \infty$, denote the usual Lebesgue and Sobolev spaces on Ω, respectively. The L_2 and L_p norms are denoted by $\|\cdot\|$ and $\|\cdot\|_{0,p}$, the Sobolev norms by $\|\cdot\|_m$ and $\|\cdot\|_{m,p}$. For any $t \in [0,T]$ define

$$\|u(t)\|_{s,m,p} = \sum_{j=0}^{s} \left[\left\| D_t^j u(t) \right\|_{m,p} + \int_0^t \left\| D_t^j u(\tau) \right\|_{m,p} d\tau \right], \tag{2.1}$$

where $D_t^j = \partial^j / \partial t^j$. Let $(\cdot,\cdot)$ denote the inner product in $L_2(\Omega)$ or $L_2(\Omega)^2$. In this paper, C denote a generic positive constant independent of u and h, not necessarily the same at each occurrence. Moreover, we also use the notation p' to denote the conjugate index of $p, 2 \leq p \leq \infty$ with $1/p + 1/p' = 1$.

Assume that $S_h \subset H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$ is a finite element space which satisfies the following approximation properties:

$$\begin{gathered} \inf_{\chi \in S_h} \left\{ \|v - \chi\|_{0,p} + h \|v - \chi\|_{1,p} \right\} \leq C h^{k+1} \|v\|_{k+1,p}, \\ v \in W^{k+1,p}(\Omega) \cap H_0^1(\Omega), \quad k \geq 1,\ 2 \leq p \leq \infty. \end{gathered} \tag{2.2}$$

We also suppose that the standard inverse properties hold on S_h. Define the Ritz projection operator $R_h = R_h(t) : H_0^1(\Omega) \to S_h$ for $0 \leq t \leq T$ by

$$(a(t) \nabla (R_h w - w), \nabla \chi) = 0, \quad \chi \in S_h, \tag{2.3}$$

where $a(t) = a(x,t)$. Then the semidiscrete finite element approximation to (1.1) is to find a map $U(t) : (0,T] \to S_h$ such that

$$(U_t,\chi)+(a(t)\nabla U,\nabla\chi)+\Big(\int_0^t b(t,\tau)\nabla U(\tau)d\tau.\nabla\chi\Big)=(f(t),\chi),\quad \chi\in S_h,\ 0<t\le T, \tag{2.4a}$$

$$U(0)=U_0,\quad x\in\Omega, \tag{2.4b}$$

where $U_0\in S_h$ is determined by

$$(a(0)\nabla U_0,\nabla\chi)=(f(0),\chi)-(R_hu_t(0),\chi),\quad \chi\in S_h \tag{2.5}$$

with

$$u_t(0)=\nabla\cdot(a(0)\nabla u_0)+f(0). \tag{2.6}$$

Now we define the Ritz-Volterra projection operator. For any given function $w\in H_0^1(\Omega)$ define a function $V_hw\in S_h$ such that

$$(a(t)\nabla(V_hw-w),\nabla\chi)+\Big(\int_0^t b(t,\tau)\nabla(V_hw(\tau)-w(\tau))\,d\tau,\nabla\chi\Big)=0,\quad \chi\in S_h. \tag{2.7}$$

Obviously, when $t=0, V_h$ is the same as Ritz projection operator R_h. Let $\eta=V_hu-u$. The next lemmas concern the estimates of η which come from [9] and [11], respectively.

Lemma 2.1 (see [9]). *For $k\ge 1$, we have*

$$||D_t^s\eta(t)||_{0,p}+h||D_t^s\eta(t)||_{1,p}\le Ch^{k+1}||u(t)||_{s,k+1,p},\quad 2\le p<\infty. \tag{2.8}$$

Lemma 2.2. *For $k>1$, we have*

$$|(D_t^s\eta,\phi)|\le Ch^{k+2}\|u\|_{s,k+1,p}\|\phi\|_{1,p'},\quad \phi\in W^{1,p'}(\Omega),\ 1<p<\infty. \tag{2.9}$$

3. Superconvergence of $U-V_hu$ when $k>1$. Let u and U be the solutions of the problem (1.1) and (2.4), respectively, and let $\xi=U-V_hu$. In this section, we study superconvergence of ξ for $k>1$. We begin with the estimates for initial value of ξ and ξ_t.

Lemma 3.1. *We have, for $k>1$*

$$\xi_t(0)=0,\quad \textit{i.e.},\ U_t(0)=R_hu_t(0), \tag{3.1}$$

$$||\xi(0)||_1\le Ch^{k+2}\{\|u_0\|_{k+1}+\|u_t(0)\|_{k+1}\}. \tag{3.2}$$

Proof. From (2.5) and (2.4)

$$(R_hu_t(0),\chi)=(f(0),\chi)-(a(0)U_0,\nabla\chi)=(U_t(0),\chi),\quad \chi\in S_h. \tag{3.3}$$

Hence $R_hu_t(0)=U_t(0)$. For (3.2), we see from (1.3), (2.4a), and (2.7) that

$$(\xi_t,\chi)+(a(t)\nabla\xi,\nabla\chi)+\Big(\int_0^t b(t,\tau)\nabla\xi(\tau)d\tau,\nabla\chi\Big)=-(\eta_t,\chi),\quad \chi\in S_h. \tag{3.4}$$

Setting $t = 0$ and noting that $\xi_t(0) = 0$, we have

$$(a(0)\nabla\xi(0), \nabla\chi) = -(\eta_t(0), \chi). \tag{3.5}$$

Take $\chi = \xi(0)$ in (3.5). Then it follows, from Lemma 2.2, that

$$\|\nabla\xi(0)\| \le Ch^{k+2}[\|u(0)\|_{k+1} + \|u_t(0)\|_{k+1}]. \tag{3.6}$$

Since $\|\nabla\cdot\|$ and $\|\cdot\|_1$ are equivalent in $H_0^1(\Omega)$, the proof is completed. □

We turn to the superconvergence estimates for ξ and show the following theorem.

Theorem 3.2. *We have, for $k > 1$*

$$\begin{aligned}\|\xi_t(t)\| + \|\xi(t)\|_1 + \left(\int_0^t \|\xi_t(\tau)\|_1^2\,d\tau\right)^{1/2} \\ \le Ch^{k+2}\left\{\|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + \left(\sum_{j=0}^{2}\int_0^t \|D_\tau^j u\|_{k+1}\,d\tau\right)^{1/2}\right\}.\end{aligned} \tag{3.7}$$

Proof. Setting $\chi = \xi_t$ in (3.4), we obtain by Lemma 2.2

$$\begin{aligned}&\|\xi_t\|^2 + \frac{1}{2}\frac{d}{dt}(a(t)\nabla\xi, \nabla\xi) \\ &\quad= \frac{1}{2}(a(t)\nabla\xi, \nabla\xi) - \left(\int_0^t b(t,\tau)\nabla\xi(\tau)\,d\tau, \nabla\xi_t\right) - (\eta_t, \xi_t) \\ &\quad\le C\left[\|\nabla\xi\|^2 + \int_0^t \|\nabla\xi\|\,d\tau\,\|\nabla\xi_t\| + h^{k+2}\|u\|_{1,k+1,2}\|\xi_t\|_1\right] \\ &\quad\le C\left[h^{2(k+2)}\|u\|_{1,k+1,2}^2 + \|\nabla\xi\|^2 + \int_0^t \|\nabla\xi\|^2\,d\tau\right] + \frac{a_0}{4}\|\nabla\xi_t\|^2.\end{aligned} \tag{3.8}$$

Differentiating (3.4) with respect to t, we see that

$$\begin{aligned}&(\xi_{tt}, \chi) + (a(t)\nabla\xi_t, \nabla\chi) \\ &\quad= -(\eta_{tt}, \chi) - (\alpha(t)\nabla\xi, \nabla\chi) - \left(\int_0^t b_t(t,\tau)\nabla\xi(\tau)\,d\tau, \nabla\chi\right).\end{aligned} \tag{3.9}$$

Setting $\chi = \xi_t$ in (3.9) and using the similar technique to deriving (3.8), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\xi_t\|^2 + (a(t)\nabla\xi_t, \nabla\xi_t) \\ &\quad\le C\left[h^{2(k+2)}\|u\|_{2,k+1,2}^2 + \|\nabla\xi\|^2 + \int_0^t \|\nabla\xi\|^2\,d\tau\right] + \frac{a_0}{4}\|\nabla\xi_t\|^2.\end{aligned} \tag{3.10}$$

Adding (3.10) onto (3.8) and using (1.2), we have

$$\begin{aligned}&\frac{d}{dt}[\|\xi_t\|^2 + (a(t)\nabla\xi, \nabla\xi)] + \|\nabla\xi_t\|^2 \\ &\quad\le C\left[h^{2(k+2)}\|u\|_{2,k+1,2}^2 + \|\nabla\xi\|^2 + \int_0^t \|\nabla\xi\|^2\,d\tau\right].\end{aligned} \tag{3.11}$$

Integrating (3.11) with respect to t, we get, by Lemma 3.1

$$\begin{aligned}
&\|\xi_t\|^2 + a_0\|\nabla\xi\|^2 + \int_0^t \|\nabla\xi_t\|^2\,d\tau \\
&\quad\le \|\xi_t(0)\|^2 + \big(a(0)\nabla\xi(0),\nabla\xi(0)\big) + Ch^{2(k+2)}\int_0^t \|u\|_{2,k+1,2}^2\,d\tau + C\int_0^t \|\nabla\xi\|^2 d\tau \\
&\quad\le Ch^{2(k+2)}\left[\|u_0\|_{k+1}^2 + \|u_t(0)\|_{k+1}^2 + \int_0^t \|u\|_{2,k+1,2}^2\,d\tau\right] + C\int_0^t \|\nabla\xi\|^2\,d\tau.
\end{aligned} \tag{3.12}$$

Finally, applying Gronwall's inequality, we obtain the result. □

To derive maximum norm superconvergence estimates, we introduce Green's functions. Let the discrete Green's functions $G_h^z \in S_h$ and $g_{h,i}^z \in S_h,\ i = 1,2$ for $z \in \Omega$ be defined by

$$\big(a(t)\nabla G_h^z, \nabla\chi\big) = \chi(z), \quad \chi \in S_h, \tag{3.13}$$

$$\big(a(t)\nabla g_{h,i}^z, \nabla\chi\big) = \frac{\partial}{\partial x_i}\chi(z), \quad \chi \in S_h, \tag{3.14}$$

respectively. Let the pre-Green's functions G_*^z and $g_{*,i}^z (i = 1,2)$ be defined by

$$\big(a(t)\nabla G_*^z, \nabla v\big) = P_h v(z), \quad v \in H_0^1(\Omega), \tag{3.15}$$

$$\big(a(t)\nabla g_{*,i}^z, \nabla v\big) = P_h\frac{\partial}{\partial x_i}v(z), \quad v \in H_0^1(\Omega), \tag{3.16}$$

respectively, where $P_h : L_2(\Omega) \to S_h$ is the L_2 projection operator.

Lemma 3.3 [15]. *We have*

$$||G_h^z|| + ||G_h^z||_{1,p'} + ||G_*^z||_{1,1} + ||g_{h,i}^z - g_{*,i}^z||_{1,1} + h||g_{*,i}^z||_{2,1} \le C \tag{3.17}$$

for $1 \le p' < 2$ *and*

$$||G_h^z||_{2,1}' + ||g_{h,i}^z||^2 + ||g_{*,i}^z||_{1,1} \le C\log\frac{1}{h}, \tag{3.18}$$

$$||G_h^z - G_*^z||_{1,1} \le C\begin{cases} h, & \text{if } k > 1, \\ h\log\dfrac{1}{h}, & \text{if } k = 1, \end{cases} \tag{3.19}$$

where

$$||G_h^z||_{2,1}' = \sum_{\tau\in\tau_h} ||G_h^z||_{2,1,\tau}. \tag{3.20}$$

Thus we can prove the superconvergence results for ξ in L_∞ and $W^{1,\infty}$.

Theorem 3.4. *We have, for* $k > 1$

$$\begin{aligned}
||\xi(t)||_{0,\infty} \le Ch^{k+2}\Bigg\{ &||u_0||_{k+1} + ||u_t(0)||_{k+1} + ||u(t)||_{1,k+1,p} \\
&+ \left(\sum_{j=0}^{2}\int_0^t ||D_t^j u||_{k+1}^2\,d\tau\right)^{1/2}\Bigg\}, \quad p > 2.
\end{aligned} \tag{3.21}$$

Proof. By (3.4) and (3.13), we have

$$\begin{aligned}\xi(z) &= (a(t)\nabla\xi,\nabla G_h^z)\\ &= -\left(\xi_t+\eta_t,G_h^z\right)-\left(\int_0^t b(t,\tau)\nabla\xi(\tau)d\tau,\nabla(G_h^z-G_*^z)\right)\\ &\quad -\int_0^t (b(t,\tau)\nabla\xi(\tau),\nabla G_*^z)\,d\tau \equiv I_1+I_2+I_3.\end{aligned} \tag{3.22}$$

By Lemma 2.2, Theorem 3.2, and Lemma 3.3, we get

$$\begin{aligned}|I_1| &\le \left|(\eta_t,G_h^z)\right|+\left|(\xi_t,G_h^z)\right| \le Ch^{k+2}\|u\|_{1,k+1,p}\|G_h^z\|_{1,p'}+\|\xi_t\|\|G_h^z\|\\ &\le Ch^{k+2}\left\{\|u\|_{1,k+1,p}+\|u_0\|_{k+1}+\|u_t(0)\|_{k+1}+\left(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2\,d\tau\right)^{1/2}\right\},\end{aligned} \tag{3.23}$$

$$|I_2| \le C\int_0^t \|\xi\|_{1,\infty}\,d\tau\|G_h^z-G_*^Z\|_{1,1} \le Ch\int_0^t \|\xi\|_{1,\infty}\,d\tau \le C\int_0^t \|\xi\|_{0,\infty}\,d\tau. \tag{3.24}$$

By $\nabla(fg)=f\nabla g+g\nabla f$ and (3.15), we have

$$\begin{aligned}|I_3| &= \left|\int_0^t \left(a\nabla\left(\frac{b}{a}\xi(\tau)\right),\nabla G_*^z\right)d\tau-\int_0^t\left(a\xi(\tau)\nabla\left(\frac{b}{a}\right),\nabla G_*^z\right)d\tau\right|\\ &\le \left|\int_0^t P_h\left(\frac{b(z,t,\tau)}{a(z,\tau)}\xi(z,\tau)\right)d\tau\right|+C\int_0^t\|\xi\|_{0,\infty}d\tau\|G_*^z\|_{1,1}\le C\int_0^t\|\xi\|_{0,\infty}d\tau.\end{aligned} \tag{3.25}$$

Substituting estimates on I_1, I_2, and I_3 into (3.22), we obtain

$$\begin{aligned}|\xi(Z)| &\le Ch^{k+2}\Bigg\{\|u\|_{1,k+1,p}+\|u_0\|_{k+1}+\|u_t(0)\|_{k+1}\\ &\quad +\left(\sum_{j=0}^{2}\int_0^t\|D_t^j u\|_{k+1}^2\,d\tau\right)^{1/2}\Bigg\}+C\int_0^t\|\xi\|_{0,\infty}\,d\tau.\end{aligned} \tag{3.26}$$

Applying Gronwall's lemma, we complete the proof. □

Theorem 3.5. *We have, for $k>1$*

$$\begin{aligned}\|\xi(t)\|_{1,\infty} &\le Ch^{k+2-\epsilon}\Bigg\{\|u_0\|_{k+1}+\|u_t(0)\|_{k+1}+\|u(t)\|_{1,k+1,p}\\ &\quad +\left(\sum_{j=0}^{2}\int_0^t\|D_t^j u\|_{k+1}^2\right)^{1/2}\Bigg\},\end{aligned} \tag{3.27}$$

with $\epsilon>2/p$, p large enough.

Proof. Writing $g_h=g_{h,i}^z$ and $g_*=g_{*,i}^z$, we have, by definition (3.14) of g_h,

$$\begin{aligned}\frac{\partial}{\partial x_i}\xi(z) &= (a(t)\nabla\xi,\nabla g_h)\\ &= -(\xi_t+\eta_t,g_h)-\left(\int_0^t b(t,\tau)\nabla\xi(\tau)\,d\tau,\nabla(g_h-g_*)\right)\\ &\quad -\int_0^t(b(t,\tau)\nabla\xi(\tau),\nabla g_*)\,d\tau \equiv J_1+J_2+J_3.\end{aligned} \tag{3.28}$$

Using similar argument as in Theorem 3.4, we have by inverse properties and Theorems 3.2 and 3.4,

$$
\begin{aligned}
|J_1| &\le \big|(\eta_t, g_h)\big| + \big|(\xi_t, g_h)\big| \le Ch^{k+2}\|u\|_{1,k+1,p}\|g_h\|_{1,p'} + \|\xi_t\|\|g_h\| \\
&\le Ch^{k+2}\bigg\{\|u\|_{1,k+1,p}h^{-2/p}\|g_h\|_{1,1} \\
&\qquad + \bigg[\|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + \bigg(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2\,d\tau\bigg)^{1/2}\bigg]\|g_h\|\bigg\} \\
&\le Ch^{k+2}\bigg\{\|u\|_{1,k+1,p}h^{-2/p} + \|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + \bigg(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2\bigg)^{1/2}\bigg\}\log\frac{1}{h},
\end{aligned}
\tag{3.29}
$$

$$
|J_2| \le C\int_0^t \|\xi\|_{1,\infty}d\tau\|g_h - g_*\|_{1,1} \le C\int_0^t \|\xi\|_{1,\infty}d\tau. \tag{3.30}
$$

and

$$
\begin{aligned}
|J_3| &\le \bigg|\int_0^t \Big(a\nabla\Big(\frac{b}{a}\xi(\tau)\Big), \nabla g_*\Big)d\tau - \int_0^t \Big(a\xi(\tau)\nabla\Big(\frac{b}{a}\Big), \nabla g_*\Big)d\tau\bigg| \\
&\le \int_0^t \bigg|P_h\frac{\partial}{\partial x_i}\Big(\frac{b(z,t,\tau)}{a(z,\tau)}\xi(z,\tau)\Big)d\tau\bigg| + C\int_0^t \|\xi\|_{0,\infty}d\tau\|g_*\|_{1,1} \\
&\le C\int_0^t \|\xi\|_{0,\infty}\,d\tau + Ch^{k+2}\bigg\{\|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + \|u\|_{1,k+1,p} \\
&\qquad + \bigg(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2 d\tau\bigg)^{1/2}\bigg\}\log\frac{1}{h}.
\end{aligned}
\tag{3.31}
$$

Combining (3.29), (3.30), and (3.31) with (3.28) and applying Gronwall's lemma we derive the conclusion. □

Now let us turn to superconvergence of ξ in L_p and $W^{1,p}$ for $2 < p < \infty$.

Theorem 3.6. *We have, for $k > 1$*

$$
\|\xi(t)\|_{0,p} \le Ch^{k+2}\bigg\{\|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + \bigg(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2\,d\tau\bigg)^{1/2}\bigg\}, \quad 2 < p < \infty. \tag{3.32}
$$

Proof. By Sobolev inequality, we have

$$
\|\chi\|_{0,p} \le C\|\chi\|_1, \quad \chi \in S_h. \tag{3.33}
$$

The conclusion follows from Theorem 3.2. □

Theorem 3.7. *We have, for $k > 1$*

$$
\begin{aligned}
\|\xi(t)\|_{1,p} &\le Ch^{k+2}\bigg\{\|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + \|u_t(t)\|_{k+1,p} \\
&\qquad + \bigg(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2\,d\tau\bigg)^{1/2}\bigg\}, \quad 2 < p < \infty.
\end{aligned}
\tag{3.34}
$$

Proof. For any $\phi \in W^{1,p'}(\Omega)$ with $\|\phi\|_{0,p'} = 1$, let Φ be the solution of

$$(a(t)\nabla v, \nabla\Phi) = -(\phi_x, v), \quad v \in H_0^1(\Omega), \tag{3.35}$$

where ϕ_x is any component of $\nabla\phi$. Then

$$\|\Phi\|_{1,p'} \le C_p\|\phi\|_{0,p'} \le C_p. \tag{3.36}$$

Now by Green's formula, (3.4), (3.35), (3.36), and Theorem 3.2, we have

$$\begin{aligned}
(\xi_x, \phi) &= -(\phi_x, \xi) = (a(t)\nabla\xi, \nabla\Phi) = (a(t)\nabla\xi, \nabla R_h\Phi) \\
&= -(\eta_t, R_h\Phi) - (\xi_t, R_h\Phi) - \left(\int_0^t b(t,\tau)\nabla\xi(\tau)\,d\tau, \nabla R_h\Phi\right) \\
&\le Ch^{k+2}\|u_t\|_{k+1,p}\|R_h\Phi\|_{1,p'} + \|\xi_t(t)\|\|R_h\Phi\| \\
&\quad + C\int_0^t \|\nabla\xi\|_{0,p}\,d\tau\|R_h\Phi\|_{1,p'} \\
&\le Ch^{k+2}\Bigg\{\|u_t\|_{k+1,p} + \|u_0\|_{k+1} + \|u_t(0)\|_{k+1} \\
&\qquad + \left(\sum_{j=0}^{2}\int_0^t \|D_t^j\|_{k+2}^2\,d\tau\right)^{1/2}\Bigg\} + C\int_0^t \|\nabla\xi\|_{0,p}\,d\tau.
\end{aligned} \tag{3.37}$$

It follows from (3.37) that

$$\begin{aligned}
\|\xi_x\|_{0,p} &= \sup_{\phi\in L^{p'}(\Omega)} (\xi_x, \phi) \\
&\le Ch^{k+2}\Bigg\{\|u_t\|_{k+1,p} + \|u_0\|_{k+1} + \|u_t(0)\|_{k+1} \\
&\qquad + \left(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2\,d\tau\right)^{1/2}\Bigg\} + C\int_0^t \|\nabla\xi\|_{0,p}\,d\tau.
\end{aligned} \tag{3.38}$$

Summing both components of $\nabla\xi$ and using Gronwall's lemma we get (3.34). □

4. Superconvergence of $U - V_h u$ when $k = 1$. In this section, we consider superconvergence when $k = 1$. Throughout this section $k = 1$. If Lemma 2.2 is replaced by Lemma 2.1 in the proof of Lemma 3.1 and Theorem 3.2, we obtain the following two results, respectively.

Lemma 4.1. *We have*

$$\xi_t(0) = 0, \quad \textit{i.e.,}\ U_t(0) = R_h u_t(0), \tag{4.1}$$

$$\|\xi(0)\|_1 \le Ch^2[\|u_0\|_2 + \|u_t(0)\|_2]. \tag{4.2}$$

Theorem 4.2. *We have*

$$\begin{aligned}
&\|\xi_t(t)\| + \|\xi(t)\|_1 + \left(\int_0^t \|\xi_t\|_1^2\,d\tau\right)^{1/2} \\
&\quad \le Ch^2\left\{\|u_0\|_2 + \|u_t(0)\|_2 + \left(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_2^2\,d\tau\right)^{1/2}\right\}.
\end{aligned} \tag{4.3}$$

Theorem 4.3. *We have*

$$\|\xi_t(t)\|_1 \le Ch^2\left\{\|u_0\|_2 + \|u_t(0)\|_2 + \left(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_2^2 d\tau\right)^{1/2}\right\}. \tag{4.4}$$

Proof. Taking $\chi = \xi_{tt}$ in (3.9), we have

$$\begin{aligned}\|\xi_{tt}\|^2 &+ \frac{1}{2}\left[\frac{d}{dt}(a(t)\nabla\xi_t, \nabla\xi_t) - (a_t(t)\nabla\xi_t, \nabla\xi_t)\right] \\ &= -(\eta_{tt}, \xi_{tt}) - (\alpha(t)\nabla\xi, \nabla\xi_{tt}) - \left(\int_0^t b_t(t,\tau)\nabla\xi(\tau)d\tau, \nabla\xi_{tt}\right).\end{aligned} \tag{4.5}$$

Integrating this, we obtain, by $\xi_t(0) = 0$,

$$\begin{aligned}\int_0^t \|\xi_{tt}\|^2 d\tau + \frac{a_0}{2}\|\nabla\xi_t\|^2 &\le \frac{1}{2}\int_0^t (a_t\nabla\xi_t, \nabla\xi_t)\,d\tau \\ &\quad - \int_0^t (\eta_{tt}, \xi_{tt})\,d\tau - \int_0^t (\alpha\nabla\xi, \nabla\xi_{tt})\,d\tau \\ &\quad - \int_0^t \left(\int_0^s (b_t(s,\tau)\nabla\xi(\tau)d\tau, \nabla\xi_{tt}(s))\right) ds \\ &\equiv K_1 + K_2 + K_3 + K_4.\end{aligned} \tag{4.6}$$

Obviously,

$$|K_1| \le C\int_0^t \|\nabla\xi_t\|^2 d\tau, \qquad |K_2| \le C\int_0^t \|\eta_{tt}\|^2 d\tau + \int_0^t \|\xi_{tt}\|^2 d\tau. \tag{4.7}$$

By integration by parts, we have

$$K_3 = (\alpha(t)\nabla\xi, \nabla\xi_t) - \int_0^t [(\alpha_t\nabla\xi, \nabla\xi_t) + (\alpha\nabla\xi_t, \nabla\xi_t)]\,d\tau. \tag{4.8}$$

Noting (4.1) and

$$\|\nabla\xi(t)\|^2 \le \|\nabla\xi(0)\|^2 + \int_0^t \|\nabla\xi_t\|^2 d\tau, \tag{4.9}$$

we see by arithmetic-geometric inequality

$$|K_3| \le C\left[\|\nabla\xi(0)\|^2 + \int_0^t \|\nabla\xi_t\|^2 d\tau\right] + \frac{a_0}{8}\|\nabla\xi_t\|^2. \tag{4.10}$$

By integrating by part and changing order of integration, we have

$$\begin{aligned}K_4 &= -\int_0^t d\tau \int_\tau^t (b_t(s,\tau)\nabla\xi(\tau), \nabla\xi_{tt}(s))\,ds \\ &= \int_0^t \{(b_t(\tau,\tau)\nabla\xi(\tau), \nabla\xi_t(\tau)) - (b_t(t,\tau)\nabla\xi(\tau), \nabla\xi_t(t))\}\,d\tau \\ &\quad + \int_0^t d\tau \int_\tau^t (b_{tt}(s,\tau)\nabla\xi(\tau), \nabla\xi_t(s))\,ds,\end{aligned} \tag{4.11}$$

and hence by using similar argument as before, we have

$$|K_4| \le C\Big[||\nabla\xi(0)||^2 + \int_0^t \|\nabla\xi_t\|^2\,d\tau\Big] + \frac{a_0}{8}\|\nabla\xi_t\|. \tag{4.12}$$

Substituting estimates of K_1–K_4 into (4.6), we get

$$||\nabla\xi_t||^2 \le C\Big[||\nabla\xi(0)||^2 + \int_0^t ||\eta_{tt}||^2\,d\tau + \int_0^t \|\nabla\xi_t\|^2 d\tau\Big]. \tag{4.13}$$

Now Gronwall's inequality gives

$$||\nabla\xi_t||^2 \le C\Big[||\nabla\xi(0)||^2 + \int_0^t ||\eta_{tt}||^2\,d\tau\Big]. \tag{4.14}$$

This together with (2.8) and (4.2) completes the proof. □

To derive superconvergence in $W^{1,\infty}$ we first bound $\|g^z_{h,i}\|_{0,p}$.

Lemma 4.4. *For* $1 < p < 2$*, we have*

$$||g^z_{h,i}||_{0,p} \le C \quad \textit{for } i = 1,2. \tag{4.15}$$

Proof. We introduce an auxiliary problem: for any given $\psi \in L_{p'}(\Omega)$, find $\Psi \in H^1_0(\Omega)$ such that

$$(a(t)\nabla v, \nabla\Psi) = (v,\psi), \quad v \in H^1_0(\Omega). \tag{4.16}$$

Then Ψ satisfies the elliptic regularity

$$\|\Psi\|_{2,p'} \le C\|\psi\|_{0,p'}. \tag{4.17}$$

Writing $g_h = g^z_{h,i}$, we have, by (3.14) and (4.16),

$$(g_h,\psi) = (a(t)\nabla g_h, \nabla\Psi) = (a(t)\nabla g_h, \nabla R_h\Psi) = \frac{\partial}{\partial x_i}R_h\Psi(z). \tag{4.18}$$

It follows from $W^{1,\infty}$ stability of the Ritz projection operator R_h, imbedding theorem and (4.17) that

$$(g_h,\psi) \le \|R_h\Psi\|_{1,\infty} \le C\|\Psi\|_{1,\infty} \le C\|\Psi\|_{2,p'} \le C\|\psi\|_{0,p'} \quad \forall\psi \in L_{p'}(\Omega). \tag{4.19}$$

Thus the proof is completed. □

Now we show the following superconvergence estimates for ξ in $W^{1,\infty}$.

Theorem 4.5. *We have*

$$\begin{aligned}\|\xi(t)\|_{1,\infty} \le Ch^2\Big\{&\|u_0\|_2 + \|u_t(0)\|_2 + \|u(t)\|_{1,2,p}\\ &+ \Big(\sum_{j=0}^{2}\int_0^t ||D^j_t u||_2^2\,d\tau\Big)^{1/2}\Big\}, \quad p > 2.\end{aligned} \tag{4.20}$$

Proof. Writing $g_h = g^z_{h,i}$ and $g_* = g^z_{*,i}$, we have, by (3.4) and (3.14),

$$\begin{aligned}\left|\frac{\partial}{\partial x_i}\xi(z)\right| &= |(a(t)\nabla\xi,\nabla g_h)| \\ &= -(\xi_t+\eta_t,g_h)-\int_0^t (b(t,\tau)\nabla\xi(\tau),\nabla(g_h-g_*))\,d\tau \\ &\quad -\int_0^t (b(t,\tau)\nabla\xi(\tau),\nabla g_*)\,d\tau \\ &\equiv R_1+R_2+R_3.\end{aligned} \tag{4.21}$$

Applying similar argument as before, we see, by Lemmas 3.3 and 4.4, that

$$\begin{aligned}|R_1| &\le (\|\xi_t\|_{0,p}+\|\eta_t\|_{0,p})\|g_h\|_{0,p'} \le C(\|\xi_t\|_1+\|\eta_t\|_{0,p}), \\ |R_2| &\le C\int_0^t \|\nabla\xi\|_{0,\infty}\,d\tau\,\|g_h-g_*\|_{1,1} \le C\int_0^t \|\nabla\xi\|_{0,\infty}\,d\tau, \\ |R_3| &= \left|\int_0^t \left(a\nabla\left(\frac{b}{a}\xi(\tau)\right),\nabla g_*\right)d\tau - \int_0^t \left(a\xi(\tau)\nabla\left(\frac{b}{a}\right),\nabla g_*\right)d\tau\right| \\ &\le \left|\int_0^t P_h\frac{\partial}{\partial x_i}\left(\frac{b(z,t,\tau)}{a(z,\tau)}\xi(z,\tau)\right)d\tau\right| + \left|\int_0^t \left(\nabla\cdot\left(a\xi(\tau)\nabla\left(\frac{b}{a}\right)\right),g_*\right)d\tau\right| \\ &\le C\int_0^t \|\xi\|_{1,\infty}\,d\tau\,(1+\|g_*\|_{0,1}),\end{aligned} \tag{4.22}$$

and

$$\|g_*\|_{0,1} \le \|g_*-g_h\|_{1,1}+\|g_h\|_{0,p'} \le C. \tag{4.23}$$

Combining above estimate with (4.21), we get

$$\|\xi(t)\|_{1,\infty} \le C\left[\|\xi_t\|_1+\|\eta_t\|_{0,p}+\int_0^t \|\xi\|_{1,\infty}\,d\tau\right]. \tag{4.24}$$

This together with Lemma 2.1, Theorem 4.3, and Gronwall's inequality completes the proof. □

Next we derive the superconvergence results of ξ in $W^{1,p}$.

Theorem 4.6. *We have for* $2 < p < \infty$

$$\|\xi(t)\|_{1,p} \le Ch^2\left\{\|u_0\|_2+\|u_t(0)\|_2+\|u_t(t)\|_{2,p}+\left(\sum_{j=0}^2\int_0^t \|D_t^j u\|_2^2\,d\tau\right)^{1/2}\right\}. \tag{4.25}$$

Proof. By Lemma 2.1 and Theorem 4.2, we see that

$$\|\eta_t\|_{0,p} \le Ch^2\|u\|_{1,2,p} \tag{4.26}$$

and

$$\|\xi(t)\| \le Ch^2\left\{\|u_0\|_2+\|u_t(0)\|_2+\left(\sum_{j=0}^2\int_0^t \|D_t^j u\|_2^2\,d\tau\right)^{1/2}\right\}. \tag{4.27}$$

Applying the same argument as in the proof of Theorem 3.7, we get the desired results at once. □

5. The interpolated postprocessing technique. In this section, we apply the interpolated postprocessing technique to enhance the accuracy of the approximate solution U. The global one order superconvergence result in L_p and $W^{1,p}$, $2 \le p \le \infty$, are established. Let T_h be a quasi uniform rectangular partition of $\Omega \subset \mathbb{R}^2$ and let S_h be the space of continuous piecewise polynomials

$$S_h = \{v \in H_0^1(\Omega),\ v \in Q^k(\tau),\ \tau \in T_h\}, \tag{5.1}$$

where

$$Q^k = \operatorname{span}\{x_1^i x_2^j,\ 0 \le i,\ j \le k\}. \tag{5.2}$$

Introduce two kinds of operators (see [4, 5, 6]), the vertices-edges-element interpolation operator i_h^k and the high interpolation operator I_{2h}^{k+1}. They satisfy the following properties:

$$||u - I_{2h}^{k+1}u||_{m,p} \le Ch^{k+2-m}\|u\|_{k+2,p}, \quad k \ge 1,\ m = 0,1,\ 2 \le p \le \infty. \tag{5.3}$$

$$I_{2h}^{k+1} i_h^k = I_{2h}^{k+1}, \quad k \ge 1. \tag{5.4}$$

$$\|I_{2h}^{k+1}\chi\|_{m,p} \le C\|\chi\|_{m,p} \quad \forall \chi \in S_k,\ k \ge 1,\ m = 0,1,\ 2 \le p \le \infty. \tag{5.5}$$

Using these properties, we can improve the global convergence for the solution and its gradient. Let $\theta = V_h u - i_h^k u$. We begin by demonstrating Lemma 5.1.

Lemma 5.1. *We have*

$$(a(t)\nabla\theta, \nabla\chi) + \left(\int_0^t b(t,\tau)\nabla\theta(\tau)\,d\tau, \nabla\chi\right) = O(h^{k+m})\|u(t)\|_{0,r,p}\|\chi\|_{m,p'}, \quad \chi \in S_h, \tag{5.6}$$

where

$$2 \le p \le \infty, \qquad r = \begin{cases} 3, & \text{if } k = 1, \\ k+3, & \text{if } k > 1, \end{cases} \qquad m = \begin{cases} 1, & \text{if } k \ge 1, \\ 2, & \text{if } k \ge 3. \end{cases} \tag{5.7}$$

Proof. By [4, 5], we see that for any $\alpha(x)$

$$(\alpha(x)\nabla(u - i_h^k u), \nabla\chi) = O(h^{k+m})\|u\|_{r,p}\|\chi\|_{m,p'}, \tag{5.8}$$

and so that by (2.7) and (5.8)

$$\begin{aligned} &(a(t)\nabla\theta,\nabla\chi) + \int_0^t (b(t,\tau)\nabla\theta(\tau)\,d\tau, \nabla\chi)\,d\tau \\ &\quad = (a(t)\nabla(u - i_h^k u), \nabla\chi) + \int_0^t (b(t,\tau)\nabla(u(\tau) - i_h^k u(\tau)), \nabla\chi)\,d\tau \\ &\quad = O(h^{k+m})\left[\|u\|_{r,p} + \int_0^t \|u\|_{r,p}\,d\tau\right]\|\chi\|_{m,p'}. \end{aligned} \tag{5.9}$$

The proof is completed. □

Now we aim for superconvergence of θ in L_p and $W^{1,p}$, $2 \le p \le \infty$, which are given in the following theorems.

Theorem 5.2. *For $k \ge 3$, we have*

$$\|\theta(t)\|_{0,p} \le Ch^{k+2}\|u(t)\|_{0,k+3,p}, \quad 2 \le p < \infty. \tag{5.10}$$

Proof. Again using (4.20) and (5.6), we have

$$\begin{aligned}
(\theta,\psi) &= (a(t)\nabla\theta,\nabla\Psi) = (a(t)\nabla\theta,\nabla R_h\Psi) \\
&= O(h^{k+2})\|u(t)\|_{0,r,p}\|R_h\Psi\|_{2,p'} \\
&\quad + \int_0^t (b(t,\tau)\nabla\theta(\tau),\nabla(\Psi - R_h\Psi))\,d\tau - \int_0^t (b(t,\tau)\nabla\theta(\tau),\nabla\Psi)\,d\tau \\
&\le C\Big[h^{k+2}\|u\|_{0,r,p}\|R_h\Psi\|_{2,p'} + \int_0^t \|\theta\|_{1,p}\,d\tau\|\Psi - R_h\Psi\|_{1,p'}\Big] \\
&\quad + \int_0^t (\theta(\tau)\nabla\cdot(b(t,\tau)\nabla\Psi))\,d\tau \\
&\le C\Big[h^{k+2}\|u\|_{0,r,p} + \int_0^t \|\theta\|_{0,p}\,d\tau\Big]\|\Psi\|_{2,p'}.
\end{aligned} \tag{5.11}$$

Hence, by (4.21),

$$\|\theta\|_{0,p} \le \sup_{\psi\in L^{p'}(\Omega)} \frac{(\theta,\psi)}{\|\psi\|_{0,p'}} \le C\Big[h^{k+2}\|u\|_{0,r,p} + \int_0^t \|\theta\|_{0,p}\,d\tau\Big]. \tag{5.12}$$

This together with Gronwall's inequality completes the proof. □

Theorem 5.3. *We have, for $k \ge 1$,*

$$\|\theta(t)\|_{1,p} \le Ch^{k+1}\|u(t)\|_{0,r,p}, \quad 2 \le p < \infty, \tag{5.13}$$

where $r = 3$ if $k = 1$, and $r = k+3$ if $k \ge 2$.

Proof. We have, by (3.35) and (3.36)

$$\begin{aligned}
(\theta_x,\phi) &= -(\phi_x,\theta) = (a(t)\nabla\theta,\nabla\Phi) = (a(t)\nabla\theta,\nabla R_h\Phi) \\
&\le Ch^{k+1}\|u\|_{0,r,p}\|R_h\Phi\|_{1,p'} - \int_0^t (b(t,\tau)\nabla\theta(\tau),\nabla R_h\Phi)\,d\tau \\
&\le C\Big[h^{k+1}\|u\|_{0,r,p} + \int_0^t \|\theta\|_{1,p}\,d\tau\Big]\|R_h\Phi\|_{1,p'} \\
&\le C\Big[h^{k+1}\|u\|_{0,r,p} + \int_0^t \|\theta\|_{1,p}\,d\tau\Big].
\end{aligned} \tag{5.14}$$

Thus the proof is completed. □

Theorem 5.4. *We have*

$$\|\theta(t)\|_{0,\infty} \le Ch^{k+2}\log\frac{1}{h}\|u(t)\|_{0,k+3,\infty}, \quad \text{if } k \ge 3, \tag{5.15}$$

$$\|\theta(t)\|_{1,\infty} \le Ch^{k+1}\left(\log\frac{1}{h}\right)^{\bar{\alpha}}\|u(t)\|_{0,r,\infty}, \tag{5.16}$$

where $\bar{\alpha} = 1$ if $k = 1,2$, $\bar{\alpha} = 0$ if $k \ge 3$, $r = 3$ if $k = 1$, and $r = k+3$ if $k \ge 2$.

Proof. By Lemma 5.1, we have

$$\begin{aligned} |\theta(z)| &= |(a(t)\nabla\theta, \nabla G_h^z)| \\ &\le Ch^{k+2}\|u\|_{o,r,\infty}\|G_h^z\|_{2,1}' + \left|\int_0^t (b(t,\tau)\nabla\theta(z), \nabla G_h^z)\,d\tau\right|, \end{aligned} \tag{5.17}$$

where

$$\begin{aligned} &\left|\int_0^t (b(t,\tau)\nabla\theta(z), \nabla G_h^z)\,d\tau\right| \\ &\quad\le \left|\int_0^t (b(t,\tau)\nabla\theta(z), \nabla(G_h^z - G_*^z))\,d\tau + \int_0^t \left(a\nabla\left(\frac{b}{a}\theta(z)\right), \nabla G_*^z\right)d\tau \right.\\ &\qquad \left. - \int_0^t \left(a\theta(\tau)\nabla\left(\frac{b}{a}\theta(z)\right), \nabla G_*^z\right)d\tau\right| \\ &\quad\le C\int_0^t \|\theta\|_{1,\infty}\,d\tau\,\|G_h^z - G_*^z\|_{1,1} + \left|\int_0^t P_h\left(\frac{b}{a}\theta(\tau)\right)d\tau\right| \\ &\qquad + C\int_0^t \|\theta\|_{0,\infty}\,d\tau\,\|G_*^z\|_{1,1}. \end{aligned} \tag{5.18}$$

Combining (5.18) with (5.17), we have by Lemma 3.3

$$\|\theta(z)\|_{0,\infty} \le C\left[h^{k+2}\log\frac{1}{h}\|u(t)\|_{0,r,\infty} + \int_0^t \|\theta\|_{0,\infty}\,d\tau\right]. \tag{5.19}$$

The conclusion (5.15) follows from Gronwall's inequality.

Writing $g_h = g_{h,i}^z$ and $g_* = g_{*,i}^z$, we see by a similar argument as above, for $k = 1,2$

$$\begin{aligned} \left|\frac{\partial}{\partial x_i}\theta(z)\right| &= |(a(t)\nabla\theta, \nabla g_h)| \\ &\le Ch^{k+1}\|g_h\|_{1,1}\|u\|_{0,r,\infty} + \left|\int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla g_h)\,d\tau\right| \\ &\le Ch^{k+1}\log\frac{1}{h}\cdot\|u(t)\|_{0,r,\infty} + \left|\int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla(g_h - g_*))\,d\tau\right. \\ &\qquad \left. + \int_0^t \left(a\nabla\left(\frac{b}{a}\theta(\tau)\right), \nabla g_*\right)d\tau - \int_0^t \left(a\theta\nabla\left(\frac{b}{a}\right), \nabla g_*\right)d\tau\right| \\ &\le Ch^{k+1}\log\frac{1}{h}\cdot\|u\|_{0,r,\infty} + C\int_0^t \|\theta\|_{1,\infty}\,d\tau\,\|g_h - g_*\|_{1,1} \\ &\qquad + \left|\int_0^t P_h\frac{\partial}{\partial x_i}\left(\frac{b}{a}\theta(\tau)\right)d\tau + \int_0^t \left(\nabla\cdot\left(a\theta(\tau)\nabla\left(\frac{b}{a}\right)\right), g_*\right)d\tau\right| \\ &\le Ch^{k+1}\log\frac{1}{h}.\|u\|_{0,r,\infty} + C\int_0^t \|\theta\|_{1,\infty}\,d\tau\,(1 + \|g_h - g_*\|_{1,1} + \|g_*\|_{0,1}) \\ &\le C\left[h^{k+1}\log\frac{1}{h}\|u\|_{0,r,\infty} + \int_0^t \|\theta\|_{1,\infty}\,d\tau\right]. \end{aligned} \tag{5.20}$$

For $k \geq 3$, we have

$$
\begin{aligned}
\frac{\partial}{\partial x_i}\theta(z) &= (a(t)\nabla\theta, \nabla g_h) \\
&= \left[(a(t)\nabla\theta, \nabla(g_h - \Pi_h g_*)) + \int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla(g_h - \Pi_h g_*))\, d\tau \right] \\
&\quad + \left[(a(t)\nabla\theta, \nabla\Pi_h g_*) + \int_0^t (b(t,\tau), \nabla(\Pi_h g_*))\, d\tau \right] \\
&\quad - \int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla g_h)\, d\tau \equiv J_1 + J_2 + J_3,
\end{aligned}
\tag{5.21}
$$

where by Lemma 5.1, inverse property and Lemma 3.3, we have

$$
\begin{aligned}
|J_1| &\leq Ch^{k+2}\|u\|_{0,r,\infty}\|g_h - \Pi_h g_*\|'_{2,1} \\
&\leq Ch^{k+2}\|u\|_{0,r,\infty}[h^{-1}\|g_h - g_*\|'_{1,1} + \|G_*\|_{2,1}] \\
&\leq Ch^{k+1}\|u\|_{0,r,\infty}.
\end{aligned}
\tag{5.22}
$$

Similarly,

$$
\begin{aligned}
|J_2| &\leq Ch^{k+2}\|\Pi_h g_*\|'_{2,1}\|u\|_{0,r,\infty} \leq Ch^{k+2}\|g_*\|_{2,1}\|u\|_{0,r,\infty} \leq Ch^{k+1}\|u\|_{0,r,\infty}, \\
|J_3| &= \left| \int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla(g_h - g_*))\, d\tau + \int_0^t \left(a\nabla\frac{b}{a}\theta(\tau), \nabla g_* \right) d\tau \right. \\
&\quad \left. - \int_0^t \left(a\theta(\tau)\nabla\left(\frac{b}{a}\right), \nabla g_* \right) d\tau \right| \\
&\leq C\int_0^t \|\theta\|_{1,\infty}\, d\tau \|g_h - g_*\|_{1,1} + \left| \int_0^t P_h \frac{\partial}{\partial x_i}\left(\frac{b}{a}\theta(\tau)\right) d\tau \right. \\
&\quad \left. + \int_0^t \left(\nabla\cdot\left(a\theta(\tau)\nabla\left(\frac{b}{a}\right)\right), g_* \right) d\tau \right| \\
&\leq C\int_0^t \|\theta\|_{1,\infty}\, d\tau (1 + \|g_h - g_*\|_{1,1} + \|g_*\|_{0,1}) \leq C\int_0^t \|\theta\|_{1,\infty}\, d\tau.
\end{aligned}
\tag{5.23}
$$

Substituting $J_1 - J_3$ into (5.21) completes the proof. □

Finally, we give the main results of this paper.

Theorem 5.5. *We have the following superconvergence:*

$$
\begin{aligned}
&\|u(t) - I_{2h}^{k+1}U(t)\|_{0,p} \\
&\quad \leq Ch^{k+2}\left\{ \|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + \left(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2\, d\tau \right)^{1/2} + \|u(t)\|_{0,k+3,p} \right\}, \\
&\qquad\qquad k \geq 3,\ 2 \leq p < \infty,
\end{aligned}
\tag{5.24}
$$

$$
\begin{aligned}
\|u(t) - I_{2h}^{k+1}U(t)\|_{0,\infty} \leq Ch^{k+2}\Big\{ &\|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + \|u(t)\|_{1,k+1,p} \\
&+ \left(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2\, d\tau \right)^{1/2} \\
&+ \log\frac{1}{h}\|u(t)\|_{0,k+3,\infty} \Big\}, \quad k \geq 3,\ p > 2,
\end{aligned}
\tag{5.25}
$$

$$\|u(t)-I_{2h}^{k+1}U(t)\|_1 \le Ch^{k+1}\Bigg\{\|u_0\|_{k+1}+\|u_t(0)\|_{k+1} + \left(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2\,d\tau\right)^{1/2} + \|u(t)\|_{0,r,2}\Bigg\},\quad k\ge 1, \tag{5.26}$$

$$\|u(t)-I_{2h}^{k+1}U(t)\|_{1,p} \le Ch^{k+1}\Bigg\{\|u_0\|_{k+1}+\|u_t(0)\|_{k+1}+\|u(t)\|_{1,k+1,p} + \left(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2\,d\tau\right)^{1/2} + \|u(t)\|_{0,k+3,p}\Bigg\},\quad k\ge 1,\ 2<p<\infty, \tag{5.27}$$

$$\|u(t)-I_{2h}^{k+1}U(t)\|_{1,\infty} \le Ch^{k+1}\Bigg\{\|u_0\|_{k+1}+\|u_t(0)\|_{k+1}+\|u(t)\|_{1,k+1,p} + \left(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2\,d\tau\right)^{1/2} + \left(\log\frac{1}{h}\right)^{\bar\alpha}\|u(t)\|_{0,r,\infty}\Bigg\},\quad k\ge 1,\ p>2, \tag{5.28}$$

where $\bar\alpha = 1$ *if* $k=1$, $\bar\alpha = 0$ *if* $k\ge 3$, $r=3$ *if* $k=1$, *and* $r=k+3$ *if* $k\ge 2$.

Proof. By (5.4), we have

$$u-I_{2h}^{k+1}U = u-I_{2h}^{k+1}u + I_{2h}^{k+1}(i_h^k u - V_h u) + I_{2h}^{k+1}(V_h u - U). \tag{5.29}$$

Then by (5.5)

$$\|u-I_{2h}^{k+1}U\|_{m,p} = \|u-I_{2h}^{k+1}u\|_{m,p} + C\|i_h^k u - V_h u\|_{m,p} + C\|V_h u - U\|_{m,p}. \tag{5.30}$$

The estimate of first term is shown in (5.3), second term in Theorems 5.2, 5.3, and 5.4, third one in Theorems 3.2, 3.4, 3.5, 4.2, 4.5, and 4.6, Thus, we complete the proof. □

Remark 5.6. In the case Ω is a general domain, we may take the piecewise "regular partition" or "most rectangular" partition to do the postprocessing (see [6]). Then we can still get half order gain for $u-I_{2h}^{k+1}U$ in L_p and $W^{1,p}$, $2\le p\le\infty$.

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