VACCINATION IN A MODEL OF AN EPIDEMIC

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ABSTRACT. Vaccination has been included in a model which describes an epidemic. A traveling wave solution together with an equilibrium and stability analysis have been done to the model.

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1. Introduction. When a population is infected by a disease it is partitioned into several distinct classes such as the infectives (denoted by I), the susceptibles (denoted by S) and others. Kermack and McKendrick [6] have originally introduced the epidemic models. In this paper, we consider only two classes of the population, namely the infectives and the susceptibles. The infectives is the class of the population who already have caught the disease and can transmit it, and the susceptibles is the class of the population who can catch the disease. In our model we consider vaccination which in turns keep the number of infectives and susceptibles unchanged and specific in the long run. Anderson [2] considered this subject into a population of laboratory mice. Also, Anderson and May [3, 4] talked about that idea further. Greenhalgh (1988) discussed vaccination in age-dependent epidemic models and in later paper Greenhalgh [5] talked about vaccination in density dependent epidemic models. In our model, we included logistic growth of the susceptibles, which is more realistic.

2. The model. Assume that the susceptible individuals are vaccinated at a constant rate v, which in turn implies that the total rate of vaccination of individuals in the population is vS. After non-dimensionalization we consider the following epidemic model with logistic growth of the susceptibles and vaccination:

$$\frac{dS}{dt} = (a-v)S - SI - aS^2, \qquad \frac{dI}{dt} = -bI + SI, \tag{2.1}$$

where $S = S(t)$, and $I = I(t)$. $a, b,$ and v are positive constants with $b < 1$. Here v is the vaccination rate. Abual-Rub [1] gave an explanation to such terms as follows:

The terms aS and bI denote the growth or death of the susceptibles and infectives, respectively. The term SI represents the binary interaction between the susceptibles and infectives.

Finally, the term S^2 represents the interaction between the same kind of species in the susceptible population. In the next two sections we investigate the equilibrium and stability analysis of the model (2.1)

3. Equilibrium analysis. In order to find the equilibrium or steady states of our model, let us set the right-hand side of (2.1) equal zero and suppose that the steady states of susceptibles and infectives are, respectively, $\hat{S}$ and $\hat{I}$. Solving the system of equations we obtain the following possibilities for the steady states:

    (i) $\hat{S} = \hat{I} = 0$.

This means that the population has died out which is always possible.

    (ii) $\hat{S} = 1 - v/a$, $\hat{I} = 0$.

This means that due to vaccination the disease has died out and therefore the population maintains itself at a steady level and of course this case is possible in reality.

    (iii) $\hat{S} = b$, $\hat{I} = a - ab - v$.

This means that the disease is possible and due to vaccination the population remains at the above two steady levels of susceptibles and infectives. Of course, this case is possible.

4. Stability analysis. We now consider small perturbations of the stability for the solutions, which are possible, namely:

    (i) $\hat{S} = \hat{I} = 0$.

Let $S = s$, $I = i$ and substitute into (2.1).

    To the first order, we get

$$\frac{ds}{dt} = (a - v)s, \qquad \frac{di}{dt} = -bi. \tag{4.1}$$

By calculating the characteristic equation we get two roots, namely $-b$ and $a - v$. In order to get local stability to small perturbations in this state, the two roots above must be negative, and therefore we require that

$$v > a. \tag{4.2}$$

This means that the effect of vaccination is so high on the susceptibles. Of course, local unstability occurs if and only if $v < a$.

    (ii) $\hat{S} = 1 - v/a$, $\hat{I} = 0$.

Let $S = 1 - v/a + s$, $I = i$. By the same way of (i) we conclude that this state is locally stable if and only if

$$1 - b < v < 1. \tag{4.3}$$

    (iii) $\hat{S} = b$, $\hat{I} = a - ab - v$.

Let $S = b + s$, $I = a - ab - v + i$. After very long calculations we obtain that this state is a locally stable node if

$$v \geq a\left(1 - b - \frac{ab}{4}\right), \tag{4.4}$$

a locally stable spiral if

$$v < a\left(1 - b - \frac{ab}{4}\right), \tag{4.5}$$

and unstable otherwise.

In the next two sections, we include diffusion to our model and then analyze the traveling wave solution.

5. Spatial spread model. Now we consider the spatial spread of the infectives and susceptibles. By remodeling our basic model (2.1) by simple diffusion and letting d_1 and d_2 to be the diffusion coefficients of susceptibles and infectives, respectively, we obtain the following model:

$$\frac{\partial S}{\partial t} = (a - v)S - SI - aS^2 + d_1 \Delta S, \qquad \frac{\partial I}{\partial t} = -bI + SI + d_2 \Delta I, \tag{5.1}$$

where $S = S(x,t)$ and $I = I(x,t)$. ΔS and ΔI represent the diffusion of the susceptibles and infective densities respectively.

6. Traveling wave solution. Now, as done in Abual-Rub [1], we seek a constant shape traveling wave solution of (5.1) by setting

$$S(x,t) = \tilde{S}(z), \qquad I(x,t) = \tilde{I}(z), \qquad z = x - ct, \tag{6.1}$$

where c is the wave speed, which has to be determined. Substitute (6.1) into (5.1) we get

$$\begin{aligned} c\tilde{S}' &= (v - a)\tilde{S} + \tilde{S}\tilde{I} + a\tilde{S}^2 - d_1\tilde{S}'', \\ c\tilde{I}' &= b\tilde{I} - \tilde{S}\tilde{I} - d_2\tilde{I}'', \end{aligned} \tag{6.2}$$

where $'$ denotes the differentiation with respect to z.

Before analyzing (6.2) we assume that d_1 is much smaller than d_2. This assumption is a legitimate one because the infective population is very active in infecting other individuals in the total population and it is capable of moving more but the susceptibles is not so.

Therefore we assume that d_1 is negligible compared to d_2.

Hence, with $d_1 = 0$, we can rewrite (6.2) as three ordinary differential equations,

$$\tilde{S}' = \frac{a\tilde{S}^2 + \tilde{S}\tilde{I} + (v - a)\tilde{S}}{c}, \qquad \tilde{I}' = \tilde{T}, \qquad \tilde{T}' = \frac{-c\tilde{T} + b\tilde{I} - \tilde{S}\tilde{I}}{d_2} \tag{6.3}$$

In the $(\tilde{S}, \tilde{I}, \tilde{T})$ phase space there are three steady states, namely

$$(0,0,0), \qquad \left(1 - \frac{v}{a}, 0, 0\right), \qquad (b, a - ab - v, 0). \tag{6.4}$$

From the analysis we have done in [1], we expect the traveling wavefront solution to be from $(0,0,0)$ to $(b, a - ab - v, 0)$ and from $(1 - v/a, 0, 0)$ to $(b, a - ab - v, 0)$. Therefore we have to seek solutions $(\tilde{S}(z), \tilde{I}(z))$ of (6.3) with the following boundary conditions:

$$\tilde{S}(-\infty) = 0, \qquad \tilde{I}(-\infty) = 0, \qquad \tilde{S}(\infty) = b, \qquad \tilde{I}(\infty) = a - ab - v, \tag{6.5}$$

$$\tilde{S}(-\infty) = 1 - \frac{v}{a}, \qquad \tilde{I}(-\infty) = 0, \qquad \tilde{S}(\infty) = b, \qquad \tilde{I}(\infty) = a - ab - v. \tag{6.6}$$

Let us consider only (6.3) with (6.6) and the analysis of (6.3) with (6.5) is analogous. Now, we linearize (6.3) about the point $(1 - v/a, 0, 0)$, i.e., $\tilde{S} = 1 - v/a$ and $\tilde{I} = 0$ then determine the eigenvalues λ, which are the roots of

$$\begin{vmatrix} \dfrac{a-v}{c} - \lambda & \dfrac{a-v}{ac} & 0 \\ 0 & -\lambda & 1 \\ 0 & \dfrac{ab-a+v}{ad_2} & \dfrac{-c}{d_2} - \lambda \end{vmatrix}, \tag{6.7}$$

we obtain

$$\lambda_1 = \frac{a-v}{c}; \qquad \lambda_2, \lambda_3 = \frac{-c \pm \left[c^2 + 4\left(bd_2 - d_2 + \frac{vd_2}{a} \right) \right]^{1/2}}{2d_2}. \tag{6.8}$$

Using (6.8) we can see that the only possibility for the existence of a traveling wavefront solution which tends to $\tilde{S} = 1 - v/a$ and $\tilde{I} = 0$ as $z \to -\infty$ if

$$c \geq 2\left(d_2 - bd_2 - \frac{vd_2}{a} \right)^{1/2}; \text{ provided that } v < a(1-b). \tag{6.9}$$

Now, let us consider the steady state $(b, a - ab - v)$ and linearize (6.3) about the point $(b, a - ab - v, 0)$ then determine the roots of

$$\begin{vmatrix} \dfrac{ab}{c} - \lambda & \dfrac{b}{c} & 0 \\ 0 & -\lambda & 1 \\ \dfrac{v+ab-a}{d_2} & 0 & \dfrac{-c}{d_2} - \lambda \end{vmatrix}, \tag{6.10}$$

which are the roots of the characteristic polynomial

$$P(\lambda) = \lambda^3 + \left(\frac{c}{d_2} - \frac{ab}{c} \right)\lambda^2 - \frac{ab}{d_2}\lambda + \frac{ab - bv - ab^2}{cd_2}. \tag{6.11}$$

Using the theory of cubic polynomials we can easily see that its impossible to have three real roots of $P(\lambda)$. Therefore, we have one real root and two complex roots.

To get stability to small linear perturbations we use the Routh-Hurwitz conditions for the roots $P(\lambda)$ to have negative real parts, i.e., we require $\operatorname{Re}\lambda < 0$. This holds if

$$\frac{c}{d_2} - \frac{ab}{c} > 0, \qquad \frac{ab-bv-ab^2}{cd_2} > 0, \qquad \frac{ab}{d_2}\left(\frac{c}{d_2} - \frac{ab}{c} \right) + \frac{ab-bv-ab^2}{cd_2} < 0. \tag{6.12}$$

It is obvious that if the first and the second inequalities of (6.12) hold, the third of (6.12) does not hold unless $d_2 < 0$. Therefore in order to have a traveling wavefront solution which approach the steady state $(b, a - ab - v, 0)$ in an oscillatory manner as $z \to +\infty$, we require that $d_2 < 0$ and (6.12) must hold. Otherwise we have instability.

7. Interpretations and conclusions. We talk about the biological interpretation of some of the results. For example, since we assumed that the death rate b is constant we may say, as in (Anderson and May [4]), that the proportion of susceptibles in the population is the same no matter what the strength of the vaccination campaign will be.

The result (4.2) is very reasonable in the steady state where the population has died out because (4.2) means that the birth rate of susceptibles, a, is very small and much smaller than the vaccination rate. On the other hand, the result (4.3) means that vaccination has a very strong effect in the population and therefore the disease has died out which might always happen.

We expect that the most realistic result can be found from (4.4) and (4.5), namely $v = a(1 - b - ab/4)$ and this gives us an estimate on the vaccination rate we should use in order to overcome the disease and to keep the population at a steady state of susceptibles and infectives and thus we may control the proportion of infectives in the population.

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