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SUPERCONVERGENCE OF FINITE ELEMENT METHOD FOR PARABOLIC PROBLEM

DO Y. KWAK, SUNGYUN LEE, and QIAN LI

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ABSTRACT. We study superconvergence of a semi-discrete finite element scheme for parabolic problem. Our new scheme is based on introducing different approximation of initial condition. First, we give a superconvergence of $u_h - R_h u$, then use a postprocessing to improve the accuracy to higher order.

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1. Introduction. We consider the following parabolic problem:

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in } \Omega, \text{ for } t > 0, \\ u &= 0 \quad \text{on } \partial\Omega, \text{ for } t \geq 0, \\ u(\cdot, 0) &= v \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a domain with smooth boundary. Suppose we are given a family $\mathfrak{T}_h$ of quasi-uniform triangulation of Ω, whose maximum diameter is denoted by h. Let $S_h \subset H_0^1(\Omega)$ be a standard finite element space consisting of continuous, piecewise polynomial of degree k. Define an elliptic projection $R_h : H_0^1(\Omega) \to S_h$ by

$$(\nabla(R_h w - w), \nabla \chi) = 0 \quad \forall \chi \in S_h. \tag{1.2}$$

We consider the following map $u_h(t) : [0, T] \to S_h$ defined by

$$(u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) = (f, \chi), \qquad u_h(0) = v_h, \tag{1.3}$$

where v_h is determined by

$$(\nabla v_h, \nabla \chi) = (f(0), \chi) - (R_h u_t(0), \chi) \quad \forall \chi \in S_h, \tag{1.4}$$

and $u_t(0)$ is determined by (1.1). Superconvergence of finite element for parabolic problem has been studied by many authors. For example, Thomeé [8], Chen and Huang [1] studied superconvergence of the gradient in L^2 norm while Thomeé et al. [9] studied maximum norm superconvergence of gradient for linear finite element. Superconvergence of the lumped finite element method for linear and nonlinear parabolic problems were studied in [2] and [6], respectively. In this paper, we introduce a different way of approximating the initial condition, namely (1.4) and investigate the superconvergence of finite element for parabolic problem using any order element.

To do so, we decompose the error as $u_h - u = u_h - R_h u + R_h u - u = \theta + \rho$ and estimate θ in a superconvergent order. Next, a postprocessing technique used in [4, 5] is employed to obtain higher order convergence. The rest of the paper is organized as follows. In Section 2, we show θ in L^2 and H^1 norm when $k > 1$. For θ, the superconvergence in L^∞ and $W^{1,\infty}$ norm are also considered. In Section 3, the case $k = 1$ is considered. The superconvergence of θ_t in H^1 and θ in $W^{1,\infty}$ norm are shown. In Section 4, $W^{l,p}$, $l = 0, 1$, $(2 < p < \infty)$ norm estimates are shown. Finally, in Section 5, we give some applications of the results obtained in Sections 2, 3 and 4. For example, a postprocessing technique is employed to obtain second-order superconvergence for gradient and first-order for the solution when $k > 1$. First-order superconvergence is shown when $k = 1$.

2. Superconvergence in L^2,H^1,L^∞, and $W^{1,\infty}$ norm. We recall $\rho = R_h u - u$ and $\theta = u_h - R_h u$.

Lemma 2.1. *Let $1 < p < \infty$, $(1/p) + (1/p') = 1$. Then for any $g \in W^{1,p'}(\Omega)$, we have, for $k > 1$,*

$$\left| (D_t^s \rho, g) \right| \le C h^{k+2} \| D_t^s u \|_{k+1,p} \| g \|_{1,p'}, \tag{2.1}$$

where $D_t^s = \partial^s / \partial t^s$.

Proof. It suffices to prove the case for $s = 0$. From standard finite element theory,

$$\| \rho \|_{1,p} \le C h^k \| u \|_{k+1,p}. \tag{2.2}$$

Consider the dual problem: given $g \in L^p(\Omega)$, find $w \in W^{3,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$ satisfying

$$(\nabla v, \nabla w) = (g, v), \quad \forall v \in H_0^1(\Omega), \tag{2.3}$$

$$\| w \|_{3,p'} \le C \| g \|_{1,p'}. \tag{2.4}$$

Let $\prod_h$ denote the S_h interpolation operator. Then by (2.3), (1.2), (2.2), (2.4), and the property of interpolation, we have

$$\begin{aligned}(g, \rho) = (\nabla \rho, \nabla w) &= \left| (\nabla \rho, \nabla (w - \Pi_h w)) \right| \\ &\le \| \rho \|_{1,p} \| w - \Pi_h w \|_{1,p'} \le C h^k \| u \|_{k+1,p} h^2 \| w \|_{3,p'} \\ &\le C h^{k+2} \| u \|_{k+1,p} \| g \|_{1,p'}.\end{aligned} \tag{2.5}$$

□

Lemma 2.2. *We have*
(i) $\theta_t(0) = 0$, *i.e.,* $u_{h,t}(0) = R_h u_t(0)$.
(ii) $\| \theta(0) \|_1 \le C h^{k+2} \| u_t(0) \|_{k+1}$.

Proof. From (1.4) and (1.3),

$$(R_h u_t(0), \chi) = (f(0), \chi) - (\nabla v_h, \nabla \chi) = (u_{h,t}(0), \chi), \quad \chi \in S_h. \tag{2.6}$$

Hence $R_h u_t(0) = u_{h,t}(0)$. For (ii), we see from (1.1),

$$(u_t, v) + (\nabla u, \nabla v) = (f, v). \tag{2.7}$$

Subtraction of (1.3) from (2.7), and noting (1.2), give

$$(\theta_t,\chi)+(\nabla\theta,\nabla\chi)=-(\rho_t,\chi),\quad \chi\in S_h. \tag{2.8}$$

Set $t=0$ and noting that $\theta_t(0)=0$, we have

$$(\nabla\theta(0),\nabla\chi)=-(\rho_t(0),\chi). \tag{2.9}$$

Take $\chi=\theta(0)$ in (2.9). Then we see from Lemma 2.1,

$$\|\nabla\theta(0)\|^2=|(\rho_t(0),\theta(0))|\le Ch^{k+2}||u_t(0)||_{k+1}||\theta(0)||_1. \tag{2.10}$$

Since $|\cdot|$ and $\|\cdot\|$ are equivalent in $H_0^1(\Omega)$,

$$||\theta(0)||_1\le C\|\nabla\theta(0)\|\le Ch^{k+2}||u_t||_{k+1}. \tag{2.11}$$

□

Theorem 2.3. *We have first-order superconvergence for $\|\theta_t\|$ and second-order superconvergence for $\|\nabla\theta_t\|$. In other words,*

$$||\theta_t(t)||+\left(\int_0^t||\nabla\theta_t||^2\,d\tau\right)^{1/2}\le Ch^{k+2}\left(\int_0^t||u_{tt}||_{k+1}^2\,d\tau\right)^{1/2} \tag{2.12}$$

holds.

Proof. Differentiating error equation (2.8),

$$(\theta_{tt},\chi)+(\nabla\theta_t,\nabla\chi)=-(\rho_{tt},\chi),\quad \chi\in S_h. \tag{2.13}$$

Take $\chi=\theta_t$. Then by Lemma 2.1, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\theta_t||^2+||\nabla\theta_t||^2&=|(\rho_{tt},\theta_t)|\le Ch^{k+2}||u_{tt}||_{k+1}||\theta_t||_1\\&\le Ch^{2(k+2)}||u_{tt}||_{k+1}^2+\frac{1}{2}||\nabla\theta_t||^2,\end{aligned} \tag{2.14}$$

where arithmetic-geometric inequality was used in the last line. Elimination of $(1/2)\|\nabla\theta_t\|^2$ and integration, give, by Lemma 2.2(i),

$$\begin{aligned}||\theta_t(t)||^2+\int_0^t||\nabla\theta_t(\tau)||^2\,d\tau&\le||\theta_t(0)||^2+Ch^{2(k+2)}\int_0^t||u_{tt}(\tau)||_{k+1}^2\,d\tau\\&\le Ch^{2(k+2)}\int_0^t||u_{tt}(\tau)||_{k+1}^2\,d\tau.\end{aligned} \tag{2.15}$$

□

Theorem 2.4. *We have second-order superconvergence for $\|\theta_t\|_1$ and first-order for $\|\theta_{tt}\|$,*

$$\left(\int_0^t||\theta_{tt}||^2d\tau\right)^{1/2}+||\theta_t(t)||_1\le Ch^{k+2}\left[||u_{tt}(t)||_{k+1}+\left(\int_0^t||u_{ttt}||_{k+1}^2d\tau\right)^{1/2}\right]. \tag{2.16}$$

PROOF. From (2.13) with $\chi = \theta_{tt}$,

$$||\theta_{tt}||^2 + \frac{1}{2}\frac{d}{dt}||\nabla\theta_t||^2 = -(\rho_{tt},\theta_{tt}). \tag{2.17}$$

Integration, and noting that $\theta_t(0) = 0$, gives

$$\begin{aligned}\int_0^t ||\theta_{tt}||^2\,d\tau + \tfrac{1}{2}||\nabla\theta_t||^2 &= -\int_0^t (\rho_{tt},\theta_{tt})\,d\tau \\ &= -(\rho_{tt},\theta_t)|_0^t + \int_0^t (\rho_{ttt},\theta_t)\,d\tau \\ &= -(\rho_{tt},\theta_t) + \int_0^t (\rho_{ttt},\theta_t)\,d\tau\end{aligned} \tag{2.18}$$

Using Lemma 2.1, left-hand side of (2.18) is

$$\begin{aligned}&\le Ch^{k+2}||u_{tt}||_{k+1}||\theta_t||_1 + Ch^{k+2}\int_0^t ||u_{ttt}||_{k+1}||\theta_t||_1\,d\tau \\ &\le Ch^{2(k+2)}||u_{tt}||_{k+1}^2 + \frac{1}{4}||\theta_t||_1^2 + Ch^{2(k+2)}\int_0^t ||u_{ttt}||_{k+1}^2\,d\tau + C\int_0^t ||\theta_t||_1^2\,d\tau.\end{aligned} \tag{2.19}$$

Elimination of $(1/4)\|\theta_t\|_1^2$ and usage of Gronwall inequality give (2.16). □

THEOREM 2.5. *We have second-order superconvergence for* $\|\theta\|_1$.

$$||\theta(t)||_1 \le Ch^{k+2}\left[||u_t(0)||_{k+1} + \left(\int_0^t ||u_{tt}||_{k+1}^2\,d\tau\right)^{1/2}\right]. \tag{2.20}$$

PROOF. By Lemma 2.2 and Theorem 2.3, we have

$$\begin{aligned}||\theta(t)||_1 &\le ||\theta(0)||_1 + \int_0^t ||\theta_t||_1\,d\tau \\ &\le ||\theta(0)||_1 + C\left(\int_0^t ||\theta_t||_1^2\,d\tau\right)^{1/2} \\ &\le Ch^{k+2}||u_t(0)||_{k+1} + Ch^{k+2}\left(\int_0^t ||u_{tt}||_{k+1}^2\,d\tau\right)^{1/2}.\end{aligned} \tag{2.21}$$

□

THEOREM 2.6. *We have first-order superconvergence for* $\|\theta\|$.

$$||\theta(t)|| \le Ch^{k+2}\left[||u_t(0)||_{k+1} + \left(\int_0^t ||u_t||_{k+1}^2\,d\tau\right)^{1/2}\right]. \tag{2.22}$$

PROOF. Recall that error equation (2.8)

$$(\theta_t,\chi) + (\nabla\theta,\nabla\chi) = -(\rho_t,\chi). \tag{2.23}$$

Take $\chi = \theta$ in (2.8). Then we see from Lemma 2.1,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\theta(t)||^2 + \|\nabla\theta\|^2 = -(\rho_t,\theta) &\le Ch^{k+2}||u_t||_{k+1}\|\theta\|_1 \\ &\le Ch^{2(k+2)}||u_t||_{k+1}^2 + \|\nabla\theta\|^2.\end{aligned} \tag{2.24}$$

Elimination of $\|\nabla\theta\|^2$ and integration, give, by Lemma 2.2,

$$\begin{aligned}\|\theta(t)\|^2 &\le \|\theta(0)\|^2 + ch^{2(k+2)}\int_0^t \|u_t\|_{k+1}^2\,d\tau \\ &\le Ch^{2(k+2)}\|u_t(0)\|_{k+1}^2 + ch^{2(k+2)}\int_0^t \|u_t\|_{k+1}^2\,d\tau.\end{aligned} \tag{2.25}$$

□

Now we study L^∞, $W^{1,\infty}$ superconvergence. First we need Green's functions. The discrete Green's function $G_h^z \in S_h$ for $z \in \Omega$ is defined by

$$(\nabla G_h^z, \nabla\chi) = \chi(z), \quad \chi \in S_h. \tag{2.26}$$

The derivative type Green's function $g_{h,i}^z \in S_h$, $(i = 1,2)$ is defined by

$$(\nabla g_{h,i}^z, \nabla\chi) = \frac{\partial}{\partial x_i}\chi(z), \quad \chi \in S_h. \tag{2.27}$$

Green's functions posses the following properties (see [9, 10]).

Lemma 2.7. *We have*

$$\|G_h^z\| + \|G_h^z\|_{1,p'} \le C, \quad 1 \le p' < 2, \tag{2.28}$$

$$\|g_{h,i}^z\|^2 + \|g_{h,i}^z\|_{1,1} \le C\ \log\frac{1}{h}. \tag{2.29}$$

Theorem 2.8. *We have the following estimate:*

$$\|\theta(t)\|_{0,\infty} \le Ch^{k+2}\left[\|u_t(t)\|_{k+1,p} + \left(\int_0^t \|u_{tt}\|_{k+1}^2\,d\tau\right)^{1/2}\right], \quad p > 2. \tag{2.30}$$

Proof. By taking $\chi = \theta$ in the definition (2.26), we have by (2.8), Lemmas 2.1, 2.7, and Theorem 2.3,

$$\begin{aligned}|\theta(z,t)| = |(\nabla G_h^z, \nabla\theta)| &= |(\rho_t, G_h^z) + (\theta_t, G_h^z)| \\ &\le Ch^{k+2}\|u_t\|_{k+1,p}\|G_h^z\|_{1,p'} + \|\theta_t\|\|G_h^z\| \\ &\le Ch^{k+2}\|u_t\|_{k+1,p} + Ch^{k+2}\left(\int_0^t \|u_{tt}\|_{k+1}^2\,d\tau\right)^{1/2}.\end{aligned} \tag{2.31}$$

Now take supremum over all $z \in \Omega$. □

Theorem 2.9. *We have the following estimate:*

$$\|\theta(t)\|_{1,\infty} \le Ch^{k+2-\epsilon}\left[\|u_t\|_{k+1,p} + \left(\int_0^t \|u_{tt}\|_{k+1}^2\ d\tau\right)^{1/2}\right], \tag{2.32}$$

for any $\epsilon > 2/p$, $p < \infty$ large enough.

Proof. For $z \in \Omega$, we see from (2.27), (2.8), Lemma 2.7, and Theorem 2.3,

$$\begin{aligned}
\left|\frac{\partial}{\partial x_i}\theta(z)\right| &= |(\nabla g^z_{h,i}, \nabla\theta)| = |(\rho_t, g^z_{h,i}) + (\theta_t, g^z_{h,i})| \\
&\le Ch^{k+2}\|u_t\|_{k+1,p}\|g^z_{h,i}\|_{1,p'} + \|\theta_t\|\|g^z_{h,i}\| \\
&\le Ch^{k+2-2/p}\|u_t\|_{k+1,p}\|g^z_{h,i}\|_{1,1} + Ch^{k+2}\left(\int_0^t \|u_{tt}\|^2_{k+1}d\tau\right)^{1/2}\|g^z_{h,i}\| \\
&\le Ch^{k+2-\epsilon}\left(\int_0^t \|u_{tt}\|^2_{k+1}d\tau\right)^{1/2}\|g^z_{h,i}\|,
\end{aligned} \tag{2.33}$$

where inverse estimate

$$\|g^z_{h,i}\|_{1,p'} \le Ch^{-2/p}\|g^z_{h,i}\|_{1,1}, \quad 1 \le p' < 2,\ 2 < p \le \infty \tag{2.34}$$

was used in the second inequality. □

3. The case $k = 1$. Here the corresponding finite element space S_h is a linear finite element space. We make suitable modification of Lemma 2.2 to obtain the following lemma.

Lemma 3.1.

$$\|\theta(0)\|_1 \le Ch^2\|u_t(0)\|_2. \tag{3.1}$$

Proof. We recall (2.9)

$$(\nabla\theta(0), \nabla\chi) = -(\rho_t(0), \chi), \quad \chi \in S_h \tag{3.2}$$

Take $\chi = \theta(0)$. Then, we see that

$$\|\nabla\theta(0)\|^2 = |(\rho_t(0), \chi(0))| \le \|\rho_t(0)\| \cdot \|\theta(0)\| \le Ch^2\|u_t(0)\|_2 \cdot \|\nabla\theta(0)\|. \tag{3.3}$$

□

Theorem 3.2. *We have*

$$\|\theta_t(t)\| + \left(\int_0^t \|\nabla\theta_t\|^2 d\tau\right)^{1/2} \le Ch^2\left(\int_0^t \|u_{tt}\|_2^2 d\tau\right)^{1/2}. \tag{3.4}$$

Proof. We recall (2.13)

$$(\theta_{tt}, \chi) + (\nabla\theta_t, \nabla\chi) = -(\rho_{tt}, \chi), \quad \chi \in S_h. \tag{3.5}$$

Taking $\chi = \theta_t$, we see that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta_t\|^2 + \|\nabla\theta_t\|^2 &\le C\|\rho_{tt}\| \cdot \|\theta_t\| \\
&\le Ch^2\|u_{tt}\|_2\|\nabla\theta_t\| \\
&\le Ch^4\|u_{tt}\|_2^2 + \frac{1}{2}\|\nabla\theta_t\|^2.
\end{aligned} \tag{3.6}$$

Elimination of $(1/2)\|\nabla\theta_t\|^2$ and integration, give the result. □

Corollary 3.3. *We have*

$$\|\theta(t)\|_1 \le Ch^2\left(\int_0^t \|u_{tt}\|_2^2\, d\tau\right)^{1/2}. \tag{3.7}$$

Theorem 3.4. *We have*

$$\left(\int_0^t \|\theta_{tt}\|^2 d\tau\right)^{1/2} + \|\theta_t(t)\|_1 \le Ch^2\left[\|u_{tt}(t)\|_2 + \left(\int_0^t \|u_{ttt}\|_2^2\, d\tau\right)^{1/2}\right]. \tag{3.8}$$

Proof. Taking $\chi = \theta_{tt}$ in (2.13), we see that

$$\|\theta_{tt}\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\theta_t\|^2 = -(\rho_{tt}, \theta_{tt}). \tag{3.9}$$

Integrating and noting $\theta_t(0) = 0$, we have

$$\begin{aligned}
\int_0^t \|\theta_{tt}\|^2\, d\tau + \frac{1}{2}\|\nabla\theta_t\|^2 &= -\int_0^t (\rho_{tt}, \theta_{tt})\, dt \\
&= -(\rho_{tt}, \theta_t) + \int_0^t (\rho_{ttt}, \theta_t)\, d\tau \\
&\le \|\rho_{tt}\| \cdot \|\theta_t\| + \int_0^t \|\rho_{ttt}\| \cdot \|\theta_t\|\, d\tau \\
&\le Ch^2\|u_{tt}\|_2 \cdot \|\theta_t\| + ch^2 \int_0^t \|u_{ttt}\|_2 \cdot \|\theta_t\|\, d\tau \\
&\le Ch^4\|u_{tt}\|_2^2 + \frac{1}{4}\|\nabla\theta_t\|^2 + ch^4\int_0^t \|u_{ttt}\|_2^2\, d\tau + \int_0^t \|\nabla\theta_t\|^2\, d\tau.
\end{aligned} \tag{3.10}$$

Now Gronwall inequality gives the result. □

Lemma 3.5. *For $1 < p < 2$, we have the following estimate:*

$$\|\nabla g_{h,i}^z\|_{0,p} \le C \quad \textit{for}\ \ i = 1, 2. \tag{3.11}$$

Proof. Let $(1/p) + (1/p') = 1$. For any $\phi \in L^{p'}(\Omega)$, let Ψ be the solution of

$$-\Delta\Psi = \phi \quad \text{in } \Omega, \qquad \Psi = 0 \quad \text{on } \partial\Omega. \tag{3.12}$$

Then we have

$$\|\Psi\|_{2,p'} \le C\|\phi\|_{0,p'}. \tag{3.13}$$

Setting $g_h = g_{h,i}^z$, we have, by (3.12), (1.2), and (2.27),

$$(g_h, \phi) = (\nabla g_h, \nabla\Psi) = (\nabla g_h, \nabla R_h\Psi) = \frac{\partial}{\partial x_i} R_h\Psi(z). \tag{3.14}$$

Thus, we see from $W^{1,\infty}$ stability of R_h, imbedding theorem and (3.13) that

$$(g_h, \phi) \le \|R_h\Psi\|_{1,\infty} \le C\|\Psi\|_{1,\infty} \le C\|\Psi\|_{2,p'} \le C\|\phi\|_{0,p'}, \tag{3.15}$$

we have

$$\|g_h\|_{0,p} = \sup_{\phi\in L^{p'}(\Omega)} \frac{(g_h, \phi)}{\|\phi\|_{0,p'}} \le C. \tag{3.16}$$

□

Theorem 3.6. *We have*

$$\|\theta(t)\|_{1,\infty} \le Ch^2\left[\|u_t(t)\|_{2,p} + \|u_{tt}(t)\|_2 + \left(\int_0^t \|u_{ttt}\|_2^2\, d\tau\right)^{1/2}\right], \quad p > 2. \tag{3.17}$$

Proof. Setting $\chi = g_{h,i}^z$ in (2.8), we obtain by (2.27), (3.11) and imbedding theorem, we have

$$\begin{aligned} \frac{\partial}{\partial x_i}\theta(z,t) &\le (u_t - u_{h,t}, g_{h,i}^z) \\ &\le (\|\rho_t\|_{0,p} + \|\theta_t\|_{0,p})\|g_{h,i}^z\|_{0,p'}, \ (1/p) + (1/p') = 1 \\ &\le C\big(\|\rho_t\|_{0,p} + \|\theta_t\|_1\big). \end{aligned} \tag{3.18}$$

By standard estimate, we have

$$\|\rho_t\|_{0,p} \le Ch^2\|u_t\|_{2,p}. \tag{3.19}$$

Combining (3.8), (3.19) with (3.18), we obtain the desired result. □

Corollary 3.7. *We have*

$$\|\theta(t)\|_{0,\infty} \le Ch^2\left[\|u_t(t)\|_{2,p} + \|u_{tt}(t)\|_2 + \left(\int_0^t \|u_{ttt}\|_2^2\, d\tau\right)^{1/2}\right], \quad p > 2. \tag{3.20}$$

4. Superconvergence in L^p and $W^{1,p}$, $(2 < p < \infty)$

Theorem 4.1. *We have*

$$\|\theta\|_{0,p} \le Ch^{k+2}\left[\|u_t(0)\|_{k+1} + \left(\int_0^t \|u_{tt}\|_{k+1}^2\, d\tau\right)^{1/2}\right], \quad k > 1. \tag{4.1}$$

Proof. From Sobolev inequality, we have, for $2 < p < \infty$,

$$\|\chi\|_{0,p} \le C\|\chi\|_1, \quad \chi \in S_h. \tag{4.2}$$

The conclusion directly follows from Theorem 2.5. □

Theorem 4.2. *We have*

$$\|\theta(t)\|_{1,p} \le Ch^{k+2}\left[\|u_t(t)\|_{k+1,p} + \left(\int_0^t \|u_{tt}\|_{k+1}^2\, d\tau\right)^{1/2}\right], \quad k > 1, \tag{4.3}$$

$$\|\theta(t)\|_{1,p} \le Ch^2\left[\|u_t(t)\|_{2,p} + \|u_{tt}(t)\|_2 + \left(\int_0^t \|u_{ttt}\|_2^2\, d\tau\right)^{1/2}\right], \quad k = 1. \tag{4.4}$$

Proof. Let $p(2 < p < \infty)$ and p' be conjugate indices, and let $\phi \in L^{p'}(\Omega)$ with $\|\phi\|_{0,p'} = 1$ and ϕ_x be any component of $\nabla\phi$. If ψ is the solution of

$$(\nabla v, \nabla\psi) = -(\phi_x, v), \quad \forall v \in H_0^1(\Omega) \tag{4.5}$$

with the regularity property [7]

$$\|\psi\|_{1,p'} \le C_p \|\phi\|_{0,p'} = C_p. \tag{4.6}$$

Then by Green's formula, equations (4.5), (1.2), (2.8), Lemma 2.1, Theorem 2.3, Sobolev lemma, and (4.6), we have

$$\begin{aligned}
(\theta_x,\phi) &= -(\phi_x,\theta) = (\nabla\theta,\nabla\psi) = (\nabla\theta,\nabla R_h\psi) = -(\rho_t,R_h\psi)-(\theta_t,R_h\psi)\\
&\le Ch^{k+2}\|u_t(t)\|_{k+1,p}\|R_h\psi\|_{1,p'}+\|\theta_t(t)\|\|R_h\psi\|\\
&\le Ch^{k+2}\left[\|u_t(t)\|_{k+1,p}+\left(\int_0^t\|u_{tt}\|_{k+1}^2\,d\tau\right)^{1/2}\right]\|R_h\psi\|_{1,p'}\\
&\le Ch^{k+2}\left[\|u_t(t)\|_{k+1,p}+\left(\int_0^t\|u_{tt}\|_{k+1}^2\,d\tau\right)^{1/2}\right].
\end{aligned}\tag{4.7}$$

Now noting that

$$\|\theta_x\|_{0,p} = \sup_{\psi\in L^{p'}(\Omega)}(\theta_x,\phi), \quad \|\phi\|_{0,p'} = 1, \tag{4.8}$$

the conclusion (4.3) is obtained. To prove (4.4), we note that

$$\|\theta\|_{1,p} \le C\|\theta\|_{1,\infty}. \tag{4.9}$$

This, together with (3.17), proves the theorem. □

5. Application. We now give an application of the results derived in Sections 2 and 3.

As an example, let T_h be a quasi-uniform rectangular partition of $\Omega\subset\mathbb{R}^2$ and let S_h be the space of continuous piecewise polynomials

$$S_h = \{v\in H_0^1(\Omega),\ v\in Q^k(\tau),\ \tau\in T_h\}, \tag{5.1}$$

where

$$Q^k = \operatorname{span}\{x_1^i x_2^j,\ 0\le i,\ j\le k\}. \tag{5.2}$$

Introduce two kinds of operators (see [3, 4]), the vertices-edges-element interpolation i_h^k and the high-interpolation operator $I_{2h}^{k+l}(l=1,2)$. They satisfy the following properties:

$$\|u-I_{2h}^{k+l}u\|_{m,p}\le Ch^{k+l+1-m}\|u\|_{k+l+1,p}, \quad 1\le k,\ m=0,1,\ (2\le p\le\infty),\ l=1,2, \tag{5.3}$$

$$I_{2h}^{k+l}i_h^k = I_{2h}^{k+l}, \quad k\ge1,\ l=1,2, \tag{5.4}$$

$$\|I_{2h}^{k+l}\chi\|_{m,p}\le C\|\chi\|_{m,p}, \quad \forall\chi\in S_k,\ 1\le k,\ m=0,1,\ (2\le p\le\infty),\ l=1,2. \tag{5.5}$$

Using these properties we can improve global convergence from k-to $k+2$-order for gradient, and from $k+1$-to $k+2$-order for solution when $k\ge2$. When $k=1$, we get one order gain for the gradient.

THEOREM 5.1. *For $k \geq 2$, we have the following results:*

$$\|u - I_{2h}^{k+1} u_h\| \leq Ch^{k+2}\left[\|u_t(0)\|_{k+1} + \left(\int_0^t \|u_t\|_{k+1}^2\, d\tau\right)^{1/2} + \|u(t)\|_{k+3}\right], \tag{5.6}$$

$$\begin{aligned}&\|u - I_{2h}^{k+1} u_h\|_{0,p}\\ &\quad \leq Ch^{k+2}\left[\|u_t(0)\|_{k+1} + \left(\int_0^t \|u_{tt}\|_{k+1}^2\, d\tau\right)^{1/2} + \|u(t)\|_{k+3,p}\right], \quad p > 2,\end{aligned} \tag{5.7}$$

$$\|u_t - I_{2h}^{k+1} u_{h,t}\| \leq Ch^{k+2}\left[\left(\int_0^t \|u_{tt}\|_{k+1}^2\, d\tau\right)^{1/2} + \|u_t(t)\|_{k+3}\right], \tag{5.8}$$

$$\begin{aligned}&\|u - I_{2h}^{k+1} u_h\|_{0,\infty}\\ &\quad \leq Ch^{k+2}\left[\|u_t(t)\|_{k+1,p} + \left(\int_0^t \|u_{tt}\|_{k+1}^2\, d\tau\right)^{1/2} + \|u(t)\|_{k+3,\infty}\right], \quad p > 2,\end{aligned} \tag{5.9}$$

$$\|u - I_{2h}^{k+2} u_h\|_1 \leq Ch^{k+2}\left[\|u_t(0)\|_{k+1} + \left(\int_0^t \|u_{tt}\|_{k+1}^2\, d\tau\right)^{1/2} + \|u(t)\|_{k+3}\right], \tag{5.10}$$

$$\begin{aligned}&\|u - I_{2h}^{k+2} u_h\|_{1,p}\\ &\quad \leq Ch^{k+2}\left[\|u_t(t)\|_{k+1,p} + \left(\int_0^t \|u_{tt}\|_{k+1}^2\, d\tau\right)^{1/2} + \|u(t)\|_{k+3,p}\right], \quad p > 2,\end{aligned} \tag{5.11}$$

$$\begin{aligned}&\|u - I_{2h}^{k+2} u_h\|_{1,\infty}\\ &\quad \leq Ch^{k+2-\epsilon}\left[\|u_t(t)\|_{k+1,p} + \left(\int_0^t \|u_{tt}\|_{k+1}^2 d\tau\right)^{1/2} + \|u(t)\|_{k+3,\infty}\right]\end{aligned} \tag{5.12}$$

for any $\epsilon > 2/p, p$ large enough,

$$\|u_t - I_{2h}^{k+2} u_{h,t}\|_1 \leq Ch^{k+2}\left[\|u_{tt}(t)\|_{k+1} + \left(\int_0^t \|u_{ttt}\|_{k+1}^2 d\tau\right)^{1/2} + \|u(t)\|_{k+1}\right]. \tag{5.13}$$

PROOF. Obviously, by (5.4) and (5.5), we have

$$u - I_{2h}^{k+l} u_h = u - I_{2h}^{k+l} u + I_{2h}^{k+l}(i_h^k u - R_h u) + I_{2h}^{k+1}(R_h u - u_h), \tag{5.14}$$

$$\|u - I_{2h}^{k+l} u_h\|_{m,p} \leq \|u - I_{2h}^{k+l} u\|_{m,p} + C\|i_h^k u - R_h u\|_{m,p} + C\|R_h u - u_h\|_{m,p}, \tag{5.15}$$

for $l = 1, 2$. The estimates of first and third terms are shown in (5.3) and Theorems 2.6, 4.1, 2.3, 2.8, 2.5, 4.2, 2.9 and 2.4 (in this order). It remains estimate the second term. By [5, Corollary to Theorem 3.4.2],

$$\|i_h^k u - R_h u\|_{m,p} \leq Ch^{k+2}\|u\|_{k+3,p}, \quad 2 \leq p \leq \infty,\ m = 0, 1, \tag{5.16}$$

so that

$$\|i_h^k u_t - R_h u_t\|_{m,p} \leq Ch^{k+2}\|u_t\|_{k+3,p}. \tag{5.17}$$

Thus, the proof is complete. □

Theorem 5.2. *For $k = 1$, we have*

$$\|u - I_{2h}^2 u_h\|_1 \le Ch^2 \left[\left(\int_0^t \|u_{tt}\|_2^2 \, d\tau \right)^{1/2} + \|u(t)\|_3 \right], \tag{5.18}$$

$$\begin{aligned} &\|u - I_{2h}^2 u_h\|_{1,p} \\ &\quad \le Ch^2 \left[\|u_t(t)\|_{2,p} + \|u_{tt}\|_2 + \left(\int_0^t \|u_{ttt}\|_2^2 \, d\tau \right)^{1/2} + \|u(t)\|_{3,p} \right], \quad 2 < p \le \infty, \end{aligned} \tag{5.19}$$

$$\|u_t - I_{2h}^2 u_{h,t}\|_1 \le Ch^2 \left[\|u_{tt}(t)\|_2 + \left(\int_0^t \|u_{ttt}\|_2^2 \, d\tau \right)^{1/2} + \|u_t(t)\|_3 \right]. \tag{5.20}$$

Proof. When $k = 1$ and $m = 1$ in (5.15)

$$\|u - I_{2h}^2 u_h\|_{1,p} \le \|u - I_{2h}^2 u_h\|_{1,p} + C\|i_h^2 u - R_h u\|_{1,p} + C\|R_h u - u_h\|_{1,p}. \tag{5.21}$$

It suffices to estimate the second term. By [3], for any $\chi \in S_h$

$$(\nabla(i_h^2 u - R_h u), \nabla \chi) = (\nabla(i_h^2 u - u), \nabla \chi) \tag{5.22}$$

$$= O(h^2) \|u\|_{3,p} \|\chi\|_{1,p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \ p \ge 2. \tag{5.23}$$

Using the same method as in [4] we have

$$\begin{aligned} \|i_h^2 u - R_h u\|_{1,p} &\le Ch^2 \|u\|_{3,p}, \\ \|i_h^2 u_t - R_h u_t\|_{1,p} &\le Ch^2 \|u_t\|_{3,p}. \end{aligned} \tag{5.24}$$

These together with (3.7), (3.17), and (3.8) completes the proof. □

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References

[1] C. M. Chen and Y. Q. Huang, *High Accuracy Theory of Finite Element Methods*, Hunan Science and Technology Press, China, 1995.

[2] C. M. Chen and V. Thomée, *The lumped mass finite element method for a parabolic problem*, J. Austral. Math. Soc. Ser. B **26** (1985), no. 3, 329–354. MR 86m:65117. Zbl 576.65110.

[3] Q. Lin, *A rectangle test for finite element analysis*, Proc. System Science and System Eng. (Hong Kong), Great Wall Culture Publ. Co., 1991, pp. 213–216.

[4] Q. Lin, N. N. Yang, and A. H. Zhou, *A rectangle test for interpolated finite elements*, Proc. System Science and System Eng. (Hong Kong), Great Wall Culture Publ. Co., 1991, pp. 217–229.

[5] Q. Lin and Q. D. Zhu, *The Preprocessing and Postprocessing for the Finite Element Method*, Shanghai Scientific and Technical Publishers, China, 1994.

[6] Y. Y. Nie and V. Thomée, *A lumped mass finite-element method with quadrature for a nonlinear parabolic problem*, IMA J. Numer. Anal. **5** (1985), no. 4, 371–396. MR 87b:65163. Zbl 591.65079.

[7] M. Schechter, *On L^p estimates and regularity. I*, Amer. J. Math. **85** (1963), 1–13. MR 32#6051. Zbl 113.30603.

[8] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Lecture Notes in Mathematics 1054, Springer-Verlag, Berlin, New York, 1984. MR 86k:65006. Zbl 528.65052.

[9] V. Thomée, J. C. Xu, and N. Y. Zhang, *Superconvergence of the gradient in piecewise linear finite-element approximation to a parabolic problem*, SIAM J. Numer. Anal. **26** (1989), no. 3, 553–573. MR 90e:65165. Zbl 678.65079.

[10] C. D. Zhu and Q. Lin, *Youxianyuan Chaoshoulian Lilun [The hyerconvergence theory of finite elements]*, Hunan Science and Technology Publishing House, Changsha, 1989. MR 93j:65191.

Kwak: Department of Mathematics, KAIST, Taejon, 305-701, Korea
E-mail address: dykwak@math.kaist.ac.kr

Lee: Department of Mathematics, KAIST, Taejon, 305-701, Korea
E-mail address: sylee@mathx.kaist.ac.kr

Li: Department of Mathematics, Shandong Normal University, Jinan, Shandong, 250014, China