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SEMILOCAL CONNECTEDNESS OF PRODUCT SPACES AND s-CONTINUITY OF MAPS

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ABSTRACT. We consider the problem of the transfer of semilocal connectedness from factors to the product space and vice versa. Some sufficient conditions are given under which the product of semilocally connected spaces is semilocally connected. Obtained theorems are not invertible, suitable examples are given.

Keywords and phrases. Semilocally connected space, multivalued map, s-continuity.

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1. Introduction and preliminaries. A topological space (Y,T) is called semilocally connected if it has an open base consisting of sets V such that $Y\setminus V$ has a finite number of components. In contrary to Whyburn [9, page 19], where this notion was introduced, we do not assume that (Y,T) is a connected T_1-space.

In a topological space (Y,T) we denote by T^* the topology given by the subbase $\{U\in T : Y\setminus U \text{ is connected}\}$; obviously $T^*\subset T$. Then we have that (Y,T^*) is semilocally connected [4, Theorem 3.1] and (Y,T) is semilocally connected if and only if $T=T^*$ [4, Theorem 3.3].

We consider the problem of the transfer of semilocal connectedness from (Y_1,T_1) and (Y_2,T_2) to $(Y_1\times Y_2,T_1\times T_2)$ and vice versa. In Section 2, we formulate some sufficient conditions under which the product of semilocally connected spaces is semilocally connected. Obtained theorems are not invertible, suitable examples are given in Section 3. In this part, we also show that the semilocal connectedness does not generally transfer in either direction. Furthermore, for (Y_1,T_1) and (Y_2,T_2) the topologies $T_1^*\times T_2^*$ and $(T_1\times T_2)^*$ need not be even comparable.

Now let X be a topological space and let $F:X\to Y$ be a multivalued map. For a set $W\subset Y$ we will denote $F^+(W)=\{x\in X : F(x)\subset W\}$ and $F^-(W)=\{x\in X : F(x)\cap W\neq\emptyset\}$. A multivalued map $F:X\to Y$ is said to be upper (lower) s-continuous at a point $x\in X$ if for each open set $V\subset Y$ with $Y\setminus V$ connected and $F(x)\subset V$ (respectively, $F(X)\cap V\neq\emptyset$) there exists an open set $U\subset X$ such that $x\in U\subset F^+(V)$ (respectively, $x\in U\subset F^-(V)$). A multivalued map F is called upper (lower) s-continuous if it has this property at each point [2, 7]. In the case of functions, the upper and lower s-continuity coincide and mean the s-continuity defined by Kohli in [4].

Theorem 1.1. *A function $f:X\to(Y,T)$ is s-continuous if and only if $f:X\to(Y,T^*)$ is continuous* [8, Proposition 9].

As it was shown in [2], Theorem 1.1 for multivalued maps is not true in general.

The last part of the paper is devoted to s-continuity. If $F_j : X_j \to Y_j$, $j \in J$, is a multivalued map, then $\prod_{j\in J} F_j$ will denote the product map

$$\prod_{j\in J} F_j : \prod_{j\in J} X_j \longrightarrow \prod_{j\in J} Y_j \tag{1.1}$$

defined as

$$\left(\prod_{j\in J} F_j\right)\left(\{x_j\}_{j\in J}\right) = \prod_{j\in J} F_j(x_j). \tag{1.2}$$

We will show that the upper s-continuity of a product map implies the upper s-continuity of factors. Moreover, for maps with connected values, the analogous theorem for the lower s-continuity is true. These results improve the similar theorem for functions [5, Theorem 2.2], where (Y_j, T_j) were assumed connected. Finally, for a multivalued map $F : X \to Y$ we denote by φ_F the graph map, i.e., $\varphi_F : X \to X \times Y$, $\varphi_F(x) = \{x\} \times F(x)$. We show that if X is connected, then the upper (lower) s-continuity of φ_F implies the same property of F. This is an extension of the following theorem.

Theorem 1.2. *If $f : X \to Y$ is a function from a connected space X into a space Y such that the graph function is s-continuous, then f is s-continuous* [4, Theorem 2.7].

In [4] the problem was raised whether the converse of Theorem 1.2 is true. We will show that the answer is negative.

2. The semilocal connectedness of product spaces

Theorem 2.1. *If (Y_i, T_i), $i \in \{1, 2, \ldots, n\}$ are topological spaces such that each of the sets $Y_1, Y_2, \ldots, Y_n$ has a finite number of components, then*

$$T_1^* \times T_2^* \times \cdots \times T_n^* \subset (T_1 \times T_2 \times \cdots \times T_n)^*. \tag{2.1}$$

Proof. Each of T_i^* has a base B_i^* consisting of T_i-open sets which complements have finite number of components; then

$$\{U_1 \times U_2 \times \cdots \times U_n : U_i \in B_i^*,\ i = 1, 2, \ldots, n\} \tag{2.2}$$

is a base of the topology $T_1^* \times T_2^* \times \cdots \times T_n^*$. Since for $U_i \in B_i^*$ we have

$$(Y_1 \times \cdots \times Y_n) \setminus (U_1 \times \cdots \times U_n) = \bigcup_{i=1}^{n} Y_1 \times \cdots \times Y_{i-1} \times (Y_i \setminus U_i) \times Y_{i+1} \times \cdots \times Y_n \tag{2.3}$$

and each set on the right-hand side of (2.3) has a finite number of components, we obtain $U_1 \times \cdots \times U_n \in (T_1 \times \cdots \times T_n)^*$ which completes the proof. □

Theorem 2.2. *Let $\{(Y_j, T_j) : j \in J\}$ be a family of connected topological spaces; then*

$$\prod_{j\in J} T_j^* \subset \left(\prod_{j\in J} T_j\right)^*. \tag{2.4}$$

Proof. Let $\prec$ be a well order on the set J and let β be the order type of $(J,\prec)$; then the set J can be presented as a transfinite sequence

$$j_0, j_1, \dots, j_\alpha, \dots, \quad \alpha < \beta, \quad \prod_{j\in J} T_j^* = \prod_{\alpha\prec\beta} T_{j_\alpha}^*. \tag{2.5}$$

We denote by $B_{j_\alpha}^*$ the base of the topology $T_{j_\alpha}^*$ which consists of T_{j_α}-open sets which complements have a finite number of components. Then the base of $\prod_{\alpha\prec\beta} T_{j_\alpha}^*$ is composed of the sets

$$\prod_{\alpha\prec\alpha_1} Y_{j_\alpha} \times U_{j_{\alpha_1}} \times \prod_{\alpha_1\prec\alpha\prec\alpha_2} Y_{j_\alpha} \times U_{j_{\alpha_2}} \times \cdots \times \prod_{\alpha_{n-1}\prec\alpha\prec\alpha_n} Y_{j_\alpha} \times U_{j_{\alpha_n}} \times \prod_{\alpha_n\prec\alpha\prec\beta} Y_{j_\alpha}, \tag{2.6}$$

where $n = 1,2,\dots,\ 0 \prec \alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_n \prec \beta$ and

$$U_{j_{\alpha_k}} \in B_{j_{\alpha_k}}^* \quad \text{for } k \in \{1,2,\dots,n\}. \tag{2.7}$$

But the complements of such sets have finite numbers of components, so these sets belong to $(\prod_{\alpha\prec\beta} T_{j_\alpha})^*$ and this completes the proof. $\square$

As a consequence of above theorems we have the following.

Corollary 2.3. (a) *Let (Y_i,T_i), $i \in \{1,2,\dots,n\}$, be semilocally connected spaces such that each of the sets Y_i has a finite number of components. Then the space $(Y_1\times\cdots\times Y_n, T_1\times\cdots\times T_n)$ is semilocally connected.*

(b) *Let $\{(Y_j,T_j) : j \in J\}$ be a family of connected and semilocally connected spaces. Then the space $(\prod_{j\in J} Y_j, \prod_{j\in J} T_j)$ is semilocally connected.*

3. Examples.

We establish some notions and notation that will be used to construct some examples. For a subset A of a topological space (Y,T) we denote by $\mathrm{Cl}_T A$ and $\mathrm{Int}_T A$ the closure and the interior of A, respectively.

Let P be an ideal of subsets of Y and let

$$D_P(A) = \{x \in Y : U \cap A \notin P \text{ for each neighbourhood } U \text{ of } x\}. \tag{3.1}$$

If an ideal P has the property

(1) $A \in P \Leftrightarrow A \cap D_P(A) = \varnothing \Leftrightarrow D_P(A) = \varnothing$, then the family

$$T(P) = \{U \setminus H : U \in T,\ H \in P\} \tag{3.2}$$

is a topology on Y; evidently $T \subset T(P)$ (see [3]). Then we also have:

(2) A set $M \subset Y$ is $T(P)$-closed if and only if it is of the form $M = B \cup H$, where B is T-closed and $H \in P$.

(3) The condition $T \cap P = \{\varnothing\}$ is equivalent to $D_P(Y) = Y$ (see [1]).

(4) If $T \cap P = \{\varnothing\}$, then $\mathrm{Cl}_T W = \mathrm{Cl}_{T(P)} W$ for each set $W \in T(P)$ (see [3]).

Lemma 3.1. *Let (Y,T) be a topological space and let P be an ideal of subsets of Y which satisfies (1) and $T \cap P = \{\varnothing\}$. Then*

(a) *The space (Y,T) is connected if and only if $(Y,T(P))$ is connected.*

(b) $(T(P))^* = T^*$.

Proof. Since $T \subset T(P)$ the connectedness of $(Y,T(P))$ implies that (Y,T) is connected. Conversely, suppose that $(Y,T(Y))$ is not connected. Then there exists an open-closed set A in $(Y,T(P))$ such that $\varnothing \neq A \neq Y$. It follows from (4) that $A = \mathrm{Cl}_{T(P)} A = \mathrm{Cl}_T A$, so A is T-closed. On the other hand, A is $T(P)$-closed, thus as a consequence of (4) we have $A = \mathrm{Int}_{T(P)} A = \mathrm{Int}_T A$; i.e., the set A is T-open. Hence (Y,T) is not connected and (a) is proved.

Now let us observe that any set $H \in P$ consisting of at least two points is not $T(P)$-connected. Let E be a closed connected subset of $(Y,T(P))$. Then it can be presented in the form $E = B \cup H$, where B is T-closed, $H \in P$ and $B \cap H = \varnothing$, so—by the connectedness of E—we have $H = \varnothing$. Thus E is T-closed and in the consequence it is also T-connected. So we have shown $(T(P))^* \subset T^*$.

Finally, we suppose that M is a connected closed subset of (Y,T) which is not $T(P)$-connected. Then it is of the form $\bigcup E_j$, where the E_j are pairwise disjoint $T(P)$-connected and $T(P)$-closed. But as it was shown in the previous part of this proof, E_j are T-connected and T-closed, which is impossible. Thus M is $T(P)$-connected and the proof is completed. □

Let us remark that in this lemma the assumption $T \cap P = \{\varnothing\}$ is essential. For instance, let (R,T) be the space of real numbers with the natural topology and let P consist of all subsets of R. Then (1) is satisfied and $T \cap P = T$. Furthermore, $T^* = T$, $T(P)$ is the discrete topology, $(T(P))^*$ is the cofinite topology, so $T^* \neq (T(P))^*$.

Example 3.2. Let $Y = \bigcup_{n=0}^{\infty}[2n,2n+1]$ and let T be the natural topology in Y induced from the real line. Since (Y,T) and (Y,T^*) have the same classes of connected closed sets, the set $[0,1]$ is T^*-connected and T^*-closed. Thus

$$W = \bigcup_{n,k=1}^{\infty} \big([2n,2n+1]\times[2k,2k+1]\big) \tag{3.3}$$

belongs to $T^* \times T^*$. For each set $V \in T^* \times T^*$ such that $Y \times Y \setminus V$ has a finite number of components, the condition

$$(Y \times Y \setminus V) \cup \big([2n,2n+1]\times[2k,2k+1]\big) \neq \varnothing \tag{3.4}$$

holds for a finite number of sets $[2n,2n+1]\times[2k,2k+1]$ only. Hence $V \not\subset W$ which means $W \notin (T^* \times T^*)^*$ and in the consequence $T^* \times T^* \neq (T^* \times T^*)^*$. This example shows that

- the product of semilocally connected spaces need not have this property;
- the assumptions in Theorem 2.1 and Corollary 2.3(a) are essential.

Example 3.3. Let (R,T) be the space of real numbers with the natural topology and let P be the ideal of Lebesgue measure zero sets. Then (1) is satisfied and $T \cap P = \{\varnothing\}$. The family $B = \{(a,b) \setminus H : a,b \in R,\ a < b,\ H \in P\}$ is a base of the topology $T(P)$. Let

$U, V \in B$; we fix points $x_0 \in R \setminus U$ and $y_0 \in R \setminus W$. Then we have

$$R \times R \setminus U \times W = \bigcup_{a \in R \setminus U} (\{a\} \times R) \cup \bigcup_{b \in R \setminus W} (R \times \{b\}),$$
$$(\{x_0\} \times R) \cap (R \times \{b\}) \neq \varnothing, \qquad (R \times \{y_0\}) \cap (\{a\} \times R) \neq \varnothing, \tag{3.5}$$

for each $a, b \in R$ and all sets $\{a\} \times R$, $R \times \{b\}$ are connected in $(R \times R, T(P) \times T(P))$. Thus

$$(\{x_0\} \times R) \cup \bigcup_{b \in R \setminus W} (R \times \{b\}), \qquad \bigcup_{a \in R \setminus U} (\{a\} \times R) \cup (R \times \{y_0\}), \tag{3.6}$$

are $T(P) \times T(P)$-connected sets containing $(\{x_0\} \times R) \cup (R \times \{y_0\})$, so their union is $T(P) \times T(P)$-connected. This gives that $U \times V \in (T(P) \times T(P))^*$ and $(T(P) \times T(P))^* = T(P) \times T(P)$. But according to Lemma 3.1 $(T(P))^* = T^* = T \neq T(P)$. So we have shown that the semilocal connectedness of the product does not imply this property of factors even then if all spaces are connected; thus Corollary 2.3 is not invertible.

Example 3.4. Let (Y_1, T) be the set of real numbers with the natural topology and P the ideal of Lebesgue measure zero sets. We put

$$Y_2 = \bigcup_{n=0}^{\infty} [2n, 2n+1] \tag{3.7}$$

and we denote by T_2 the natural topology on Y_2 induced from the real line. Then following Lemma 3.1, we have $(T(P))^* = T$. The family

$$\{Y_2 \setminus \{p\} : p \in Y_2\} \cup \{Y_2 \setminus [a,b] : 2n \le a < b \le 2n+1,\ n = 0, 1, 2, \dots\} \tag{3.8}$$

is a subbase for T_2^*. Let $\{w_n : n = 1, 2, \dots\}$ be the set of all rational numbers from the interval $[0,1]$ and let

$$B = \left(\bigcup_{n=1}^{\infty} \{w_n\} \times [0,1] \right) \cup ([0,1] \times \{1\}). \tag{3.9}$$

The set B is connected and closed in $(Y_1 \times Y_2, T(P) \times T_2)$, so

$$U = Y_1 \times Y_2 \setminus B \in \left(T(P) \times T_2\right)^*. \tag{3.10}$$

We fix a point $p \in U$ with coordinates $x, y \in (0,1)$. The neighbourhood base of p in $(Y_1 \times Y_2, (T(P))^* \times T_2^*)$ consists of sets

$$(a,b) \times \left(Y_2 \setminus \left(\bigcup_{j=1}^{k} [c_j, d_j] \cup \{x_1, x_2, \dots, x_m\} \right) \right), \tag{3.11}$$

where $a < x < b$, $k, m \in \{1, 2, \dots\}$, $2n_j \le c_j < d_j \le 2n_j + 1$ for $j \in \{1, 2, \dots, k\}$; $x_1, x_2, \dots, x_m \in Y_2$ and

$$y \notin \bigcup_{j=1}^{k} [c_j, d_j] \cup \{x_1, x_2, \dots, x_m\}, \tag{3.12}$$

but none of these neighbourhoods is contained in U. Hence we obtain $U \notin (T(P))^* \times T_2^*$ and in the consequence

$$(T(P)\times T_2)^* \not\subset (T(P))^* \times T_2^*. \tag{3.13}$$

Now we put

$$V = Y_1 \times Y_2 \setminus \bigcup_{n=0}^{\infty} [2n,2n+1]\times[2n,2n+1]. \tag{3.14}$$

For a fixed $r \in (0,1/2)$ the set V can be presented in the form

$$\begin{aligned} V = (-\infty,0)\times Y_2 \cup \bigcup_{n=0}^{\infty}(2n+1,2n+2)\times Y_2 \\ \cup \bigcup_{n=0}^{\infty}\left((2n-r,2n+1+r)\times \bigcup_{k=0,k\neq n}^{\infty}[2k,2k+1]\right), \end{aligned} \tag{3.15}$$

so $V \in (T(P))^* \times T_2^*$. Let $p \in V$ and let W be a $(T(P)\times T_2)^*$-neighbourhood of p. Then at most a finite number of sets $Y_1 \times [2n,2n+1]$ are not contained in W, while $(Y_1 \times Y_2 \setminus V) \cap (Y_1 \times [2n,2n+1]) \neq \emptyset$ for each $n \in \{0,1,2,\dots\}$. Thus $W \not\subset V$ and $V \notin (T(P)\times T_2)^*$. So we have shown $(T(P))^* \times T_2^* \not\subset (T(P)\times T_2)^*$.

4. s-continuity of maps on product spaces

Theorem 4.1. *Let $\{X_j : j \in J\}$ and $\{Y_j : j \in J\}$ be two families of topological spaces and let $F_j : X_j \to Y_j$ be a multivalued map, $j \in J$. If $\prod_{j\in J} F_j$ is upper s-continuous, then each of F_j is upper s-continuous.*

Proof. We fix $i \in J$ and let M_i be a connected closed subset of Y_i with $F_i^-(M_i) \neq \emptyset$. For each $j \in J$, $j \neq i$, we choose a component M_j of Y_j such that $F_j^-(M_j) \neq \emptyset$. Then $\prod_{j\in J} M_j$ is a connected closed subset of $\prod_{j\in J} Y_j$, so $(\prod_{j\in J} F_j)^-(\prod_{j\in J} M_j)$ is closed. Since we have

$$\left(\prod_{j\in J} F_j\right)^-\left(\prod_{j\in J} M_j\right) = \prod_{j\in J} F_j^-(M_j) = \mathrm{Cl}\left(\prod_{j\in J} F_j^-(M_j)\right) = \prod_{j\in J}\mathrm{Cl}\,(F_j^-(M_j)), \tag{4.1}$$

and all factors are nonempty this implies that $F_i^-(M_i)$ is closed and the proof is completed. □

Theorem 4.2. *Let $\{X_j : j \in J\}$ and $\{Y_j : j \in J\}$ be two families of topological spaces and let $F_j : X_j \to Y_j$ be a multivalued map with connected values for $j \in J$. If $\prod_{j\in J} F_j$ is lower s-continuous, then each F_j is lower s-continuous.*

Proof. Let $i \in J$ be fixed and let M_i be a connected closed set with $F_i^+(M_i) \neq \emptyset$. For each $j \in J$, $j \neq i$, we choose a component M_j of Y_j such that $F_j^+(M_j) \neq \emptyset$. Now using arguments analogous to those in the proof of Theorem 4.1 we obtain $F_i^+(M_i)$ is closed which completes the proof. □

Each of the above theorems implies the following.

COROLLARY 4.3. *Let $\{X_j : j \in J\}$, $\{Y_j : j \in J\}$ be families of topological spaces and $f_j : X_j \to Y_j$ be a function for $j \in J$. If the function $\prod_{j\in J} f_j$ is s-continuous, then each f_j is s-continuous.*

Under additional assumptions that all Y_j are connected this corollary makes [5, Theorem 2.2].

Let us remark that the above corollary and theorems are not invertible.

EXAMPLE 4.4. Let (R,T) and P be such as in Example 3.3 and let $f : (R,T) \to (R,T(P))$ be the function given by $f(x) = x$ for $x \in R$. According to Lemma 3.1 the function f is s-continuous. Let us put $W = (a,b) \setminus H$, where $a,b \in R$, $a < b$ and $H \in P$. Following Example 3.3 the topology $T(P)\times T(P)$ is semilocally connected so $W\times W \in (T(P)\times T(P))^*$. But $(f\times f)^{-1}(W\times W) \notin T\times T$, hence $f\times f$ is not s-continuous.

THEOREM 4.5. *Assume that X has a finite number of components and let $F : X \to Y$ be a multivalued map with values in a topological space Y.*

(a) *If the graph map φ_F is upper s-continuous, then F is upper s-continuous.*

(b) *If φ_F is lower s-continuous and F has connected values, then F is lower s-continuous.*

PROOF. Let $M \subset Y$ be a connected closed set with $F^-(M) \neq \emptyset$. Under assumptions we have

$$X\times M = \bigcup_{k=1}^{n} (M_k\times M), \tag{4.2}$$

where M_k are components of X. Then we have

$$F^-(M) = \varphi_F^-(X\times M) = \bigcup_{k=1}^{n} \varphi_F^-(M_k\times M). \tag{4.3}$$

Since $M_k\times M$ are connected closed subsets of $X\times Y$, the set $F^-(M)$ is closed; hence F is upper s-continuous.

For the lower s-continuity the proof is analogous. □

If X is connected Theorem 4.5 gives [4, Theorem 2.7], for usual functions. In [4] problem of validity of the inverse to this theorem is stated. The answer is negative.

For instance, under notation of Example 4.4, the function f is s-continuous but φ_f^{-1} $(W\times W) = W \notin T$; thus φ_f is not s-continuous.

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