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ON RATIONAL APPROXIMATION IN A BALL IN $\mathbb{C}^N$

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Abstract. We study rational approximations of elements of a special class of meromorphic functions which are characterized by their holomorphic behavior near the origin in balls in $\mathbb{C}^N$ by means of their rational approximants. We examine two modes of convergence for this class: almost uniform-type convergence analogous to Montessus-type convergence and weaker form of convergence using capacity based on the classical Tchebychev constant. These methods enable us to generalize and extend key results of Pommeranke and Gonchar.

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1. Introduction. This paper is an attempt to extend the theoretical basis of rational approximation by means of rational approximants in $\mathbb{C}^N$ to elements of a certain class of meromorphic functions on the ball $\mathbb{B}_\rho^N := \{z \in \mathbb{C}^N : \sum_{k=1}^N |z_k|^2 < \rho\}$, that are holomorphic at the origin. This investigation of rational approximants in several complex variables, which began in the early seventies, offers new insights into the problem of analytic continuation from the local neighborhoods of holomorphy into the open connected regions of meromorphy. The local holomorphic expansions from which one traditionally extracted rational approximants, largely involved polynomial expansions of multiple degrees and power series expansions not in terms of homogeneous polynomials. This approach, although it gave rise to some interesting results (see [8, 9, 10]) lacked the flexibility of the formulation discussed in this paper. Part of the advantage gained in the latter formulation, is that one sets up initial definitions in a relatively simple, almost Padé-like fashion using slice functions. A useful consequence of this is that any investigation of the vertical and diagonal sequences of a (μ,ν)-rational approximant table, analogous to the Padé table, is easily accessible.

There are two main types of convergence behavior of interest in this paper. The first is the almost uniform-type (see [8]) associated with vertical sequences analogous to Montessus-type convergence. The second convergence behavior is the weaker of the two, and it is given in terms of convergence in capacity. The methods of investigations of the main diagonal sequences in $\mathbb{C}^N$, chiefly use capacity based on the classical Tchebychev constant (see [1, 2, 12]). Our main result associated with the Tchebychev constant (transfinite diameter) generalizes a result of Pommeranke [11] and also extends a result of Gonchar [5].

We now give a brief description of the contents of this paper. Section 2 introduces and develops most of the required preliminaries by way of definitions, lemmas, and propositions. Section 3, considers Montessus-type convergence analogous to that given in [8] for the polydisc. In Section 4, we discuss convergence in capacity which gives a considerably sharp version of a result of Gonchar (see [5]), proved using $\mathbb{R}^{2N}$-Lebesgue measure.

From the contents of [3, 4, 6, 7], the authors take a different tack on Padé approximants and the problems of convergence of de Montessus de Ballore type and hence prove totally different theorems than our own, represented in this paper.

2. Some preliminaries.

We begin this section by developing index sets most suited for handling expansions expressed in terms of homogeneous polynomials.

Let $\mathbb{I} := \{0,1,2,\dots,\}$ and $\mathbb{I}^N := \mathbb{I}\times\cdots\times\mathbb{I}$, N copies. We introduce a partial order in $\mathbb{I}^N$ as follows: for each pair $\alpha,\beta\in\mathbb{I}^N$, $\alpha\preceq\beta \Leftrightarrow \alpha_i\le\beta_i$, $i=1,\dots,N$. If $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_N)$, then we shall write $|\alpha|=\sum_{j=1}^N\alpha_j$.

Let $E_\mu^N=\{\alpha\in\mathbb{I}^N : 0\le|\alpha|\le\mu\}$ so that it has the following properties:

(1) $E_0^N := \{(0,0,\dots,0)\}\subset E_\mu^N$, $\forall\mu\ge 1$.

(2) $E_\lambda^N\subset E_\mu^N$, $\forall 0\le\lambda\le\mu$.

(3) $|E_\mu^N| = \binom{N+\mu}{N}$, the cardinality of E_μ^N.

Here it should be noted that $\lambda,\mu\in\mathbb{I}$ but not in $\mathbb{I}^N$ whereas $\alpha,\beta\in\mathbb{I}^N$. We now introduce the notion of an index set $E_{\mu\nu}^N$ for $\mu,\nu\in\mathbb{I}$.

Definition 2.1. $E_{\mu\nu}^N\subset\mathbb{I}^N$ is called an index interpolation set if

(a) $E_\mu^N\subset E_{\mu\nu}^N$, for each $\mu,\nu\in\mathbb{I}$;

(b) $\beta\in E_{\mu\nu}^N \Rightarrow \alpha\in E_{\mu\nu}^N$, $\forall 0\preceq\alpha\preceq\beta$;

(c) $\exists\lambda_{\mu\nu}\in\mathbb{I}$ with $\mu+1\le\lambda_{\mu\nu}\le\mu+\nu$, $\nu\ge 1$ such that

$$\binom{N+\lambda_{\mu\nu}-1}{N} \le \left|E_{\mu\nu}^N\right| \le \binom{N+\lambda_{\mu\nu}}{N}, \tag{2.1}$$

where $|E_{\mu\nu}^N| \le \binom{N+\mu}{N}+\binom{N+\nu}{N}-1$.

Definition 2.2. $E_{\mu\nu}^N$ is called maximal if its cardinality satisfies

$$\left|E_{\mu\nu}^N\right| \ge \binom{N+\mu}{N}+\binom{N+\nu}{N}-1. \tag{2.2}$$

Remark 2.3. The concept of maximality, as will be determined later, becomes central in dealing with the question of normality for rational approximants.

Let $\mathbb{O}(\mathbb{B}_\rho^N)$ be the ring of holomorphic functions on $\mathbb{B}_\rho^N$ and $\mathcal{M}\mathrm{er}(\mathbb{B}_\rho^N)$ the ring of meromorphic functions on $\mathbb{B}_\rho^N$. In particular, let $\mathcal{M}\mathrm{er}^1(\mathbb{B}_\rho^N)$ be the subring of $\mathcal{M}\mathrm{er}(\mathbb{B}_\rho^N)$ characterised by the following properties:

(P.I) $\forall f\in\mathcal{M}\mathrm{er}^1(\mathbb{B}_\rho^N)$, there exists a neighborhood of the origin in $\mathbb{B}_\rho^N$, where f is holomorphic.

(P.II) For each $f \in \mathcal{M}\mathrm{er}^1(\mathbb{B}_\rho^N)$, there is a nonhomogeneous normalized polynomial $q(z)$ of minimal degree such that in $\mathbb{B}_\rho^N$, the zero set of $q(z)$ denoted by $\mathscr{Z}(q)$ coincides with the polar set $\mathscr{Z}(f^{-1})$ of f, that is

$$\mathscr{Z}(f^{-1}) \cap \mathbb{B}_\rho^N = \mathscr{Z}(q) \cap \mathbb{B}_\rho^N. \tag{2.3}$$

(P.III) For each $f \in \mathcal{M}\mathrm{er}^1(\mathbb{B}_\rho^N)$ and its corresponding minimal polynomial q as in (P.II), $fq \in \mathbb{O}(\mathbb{B}_\rho^N)$.

(P.IV) For f and q as in (P.III) $\mathscr{Z}(fq) \cap \mathscr{Z}(q) \cap \mathbb{B}_\rho^N = \varnothing$ except possibly at the points of indeterminacy of f on $\mathbb{B}_\rho^N$.

We shall now introduce the slice function on $\mathbb{B}_\rho^N$. For any $g \in \mathbb{O}(\mathbb{B}_\rho^N)$ and for each $z \in \partial\mathbb{B}_\rho^N$, let $\mathbb{L}_z$ denote the complex line through the origin zero and z. The slice function of g is determined from

$$\Delta_1 \times \partial\mathbb{B}_\rho^N \longrightarrow \mathbb{B}_\rho^N \longrightarrow \mathbb{C}, \tag{2.4}$$

so that $(t,z) \mapsto tz \mapsto g(tz)$ and we have

$$g_z(t) = g(tz), \tag{2.5}$$

here $\Delta_1 = \{w \in \mathbb{C} : |w| < 1\}$. In the rest of this paper, we shall apply the slice definitions to rational as well as polynomial functions.

Proposition 2.4. *Let U be a neighborhood of the origin $0 \in \mathbb{C}^N$, where $U \Subset \mathbb{B}_\rho^N$. Then $f \in \mathbb{O}(U) \Rightarrow f_z(t) \in \mathbb{O}(\Delta_1)$, with $\Delta_1 \subset \mathbb{L}_z \cap U$.*

For each $\mu, \nu \in \mathbb{I}$, we let $\mathscr{R}_{\mu\nu}$ be the class of rational functions of the form $P_\mu(z)/Q_\nu(z)$, where $P_\mu(z)$ and $Q_\nu(z)$ are nonhomogeneous polynomials expended in terms of homogeneous polynomials up to degrees μ and ν, respectively. Furthermore, $Q_\nu(0) \neq 0$; $P_\mu(z)$ and $Q_\nu(z)$ are relatively prime in $\mathbb{C}^N$ except at the points of indeterminacy of $P_\mu(z)/Q_\nu(z)$. That is $\mathscr{Z}(P_\mu) \cap \mathscr{Z}(Q_\nu) \cap \mathbb{B}_\rho^N = \varnothing$ except possibly at the points of indeterminacy of P_μ/Q_ν in $\mathbb{B}_\rho^N$.

Definition 2.5. A rational function $P_{\mu z}(t)/Q_{\mu z}(t) \in \mathscr{R}_{\mu\nu}$ is called a rational approximant to $f_z(t) \in \mathbb{O}(\Delta_1)$ at $0 \in \Delta_1$, if

$$\frac{d^k}{dt^k}\left(Q_{\nu z}(t) f_z(t) - P_{\mu z}(t)\right)\big|_{t=0} = 0, \quad 0 \le k \le \mu, \tag{2.6}$$

$$\frac{d^k}{dt^k}\left(Q_{\nu z}(t) f_z(t)\right)\big|_{t=0} = 0, \quad \mu + 1 \le k \le \lambda_{\mu\nu}. \tag{2.7}$$

Proposition 2.6. *Suppose $f \in \mathbb{O}(U)$, U an open neighborhood of the origin in $\mathbb{B}_\rho^N$ with $U \Subset \mathbb{B}_\rho^N$ and let $t \in \Delta_1 \subset \mathbb{L}_z \cap U$. Then*

$$\frac{d^k}{dt^k} f_z(t)\big|_{t=0} = \sum_{|\alpha|=k} \left(\partial^{|\alpha|} f(\xi)\big|_{\xi=0}\right) z^\alpha, \tag{2.8}$$

where

$$\partial^{|\alpha|} \equiv \frac{\partial^{\alpha_1 + \cdots + \alpha_N}}{\partial z_1^{\alpha_1} \cdots \partial z_N^{\alpha_N}}. \tag{2.9}$$

Proof. The result follows from comparing the coefficients of t^k in the equal but separate Taylor expansions of $f_z(t)$ in Δ_1 and $f(tz)$ in U. □

Proposition 2.7. *Let $f \in \mathbb{O}(U)$, $U \Subset \mathbb{B}_\rho^N$ and let $P_\mu(z)/Q_\nu(z) \in \mathcal{R}_{\mu\nu}$. Then*
(i) *Equation (2.6) holds if and only if*

$$\partial^{|\alpha|}\big(Q_\nu(\xi)f(\xi) - P_\mu(\xi)\big)\big|_{\xi=0} = 0, \quad \alpha \in E_\mu^N. \tag{2.10}$$

(ii) *Equation (2.7) holds if and only if*

$$\partial^{|\alpha|}\big(Q_\nu(\xi)f(\xi)\big)\big|_{\xi=0} = 0, \quad \alpha \in E_{\mu\nu}^N \backslash E_\mu^N. \tag{2.11}$$

Proof. The results follow from Proposition 2.6, using the linear independence of monomial vectors $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_N^{\alpha N}$, with $\alpha := (\alpha_1, \ldots, \alpha_N)$, in the right-hand side of the equation (2.8) that generate the homogeneous subspace $\{z^\alpha : |\alpha| = k\} \subset \mathbb{C}[z_1, \ldots, z_N]$, the latter being the algebra of polynomials in $\mathbb{C}^N$. Here the two cases are covered as follows:
(i) $|\alpha| = k,\ 0 \le k \le \mu \Leftrightarrow \alpha \in E_\mu^N$,
(ii) $|\alpha| = k,\ \mu + 1 \le k \le \lambda_{\mu\nu} \Leftrightarrow \alpha \in E_{\mu\nu}^N \backslash E_\mu^N$. □

When $E_{\mu\nu}^N$ is maximal, the system of equations produced by (2.11) with the normalization $Q_\nu(0) = 1$, gives rise to a linear system of maximal rank $\binom{N+\nu}{N} - 1$. The latter equals the number of unknown coefficients of $Q_\nu(z)$. The solution of the linear system of equations with maximal rank leads to the uniqueness of the resulting (μ,ν)-rational approximant with respect to $E_{\mu\nu}^N$ maximal. We call such (μ,ν)-rational approximants each with a normalized denominator polynomial, (μ,ν)-unisolvent rational approximants (in short URA) and denote it by $\pi_{\mu\nu}$.

The array or table of $\{\pi_{\mu\nu}\}_{\mu\nu}$ of uniquely determined entries is called *normal*. The fact that there are many $E_{\mu\nu}^N$ that are maximal, suggests the following conjecture: "There are as many maximal $E_{\mu\nu}^N$ index interpolation sets as there are normal tables (all analogs of a normal Padé-table) associated with a given function holomorphic at zero."

This situation is very different from the one variable case where there is one and only one maximal Padé index set giving rise to a single normal Padé table.

Lemma 2.8. *Let $f \in \mathcal{M}\mathrm{er}^1(\mathbb{B}_\rho^N)$ and let $q_\omega(z)$ be its corresponding nonhomogeneous polynomial of minimal fixed degree $\omega \in \mathbb{I}$. Suppose $\pi_{\mu\nu z}(t) = P_{\mu\nu z}(t)/Q_{\mu\nu z}(t)$ is a (μ,ν)-URA to $f_z(t)$ at $t = 0$ for each $z \in \partial\mathbb{B}_\rho^N$. Then for each integer k with $0 \le k \le \mu$,*

$$\frac{d^k}{dt^k}\big(Q_{\mu\nu z}(t)f_z(t)q_{\omega z}(t) - P_{\mu\nu z}(t)q_{\omega z}(t)\big)\Big|_{t=0} = 0. \tag{2.12}$$

Proof. By direct computation for $0 \le k \le \mu$,

$$\begin{aligned}
&\frac{d^k}{dt^k}\big(Q_{\mu\nu z}(t)f_z(t)q_{\omega z}(t) - P_{\mu\nu z}(t)q_{\omega z}(t)\big)\Big|_{t=0} \\
&\quad = \frac{d^k}{dt^k}\big(Q_{\mu\nu z}(t)f_z(t)q_{\omega z}(t)\big)\Big|_{t=0} - \frac{d^k}{dt^k}\big(P_{\mu\nu z}(t)q_{\omega z}(t)\big)\Big|_{t=0} \\
&\quad = \sum_{l=0}^{k}\binom{k}{l}\left[\frac{d^l}{dt^l}Q_{\mu\nu z}(t)f_z(t) - \frac{d^l}{dt^l}P_{\mu\nu}(t)\right]\frac{d^{k-l}}{dt^{k-l}}q_{\omega z}(t)\Big|_{t=0},
\end{aligned} \tag{2.13}$$

where we have used Leibniz rule. But by (2.6), for $0 \le k \le \mu$,

$$\frac{d^l}{dt^l} Q_{\mu\nu z}(t) f_z(t) \Big|_{t=0} = \frac{d^l}{dt^l} P_{\mu\nu z}(t) \Big|_{t=0}. \tag{2.14}$$

The result then follows from (2.13) and (2.14). □

Corollary 2.9. *Suppose the hypothesis of Lemma 2.8 is satisfied. Then for $\alpha \in E_\mu^N$,*

$$\partial^{|\alpha|} \left(Q_{\mu\nu}(\xi) f(\xi) q_\omega(\xi) - P_{\mu\nu}(\xi) q_\omega(\xi) \right) \big|_{\xi=0} = 0. \tag{2.15}$$

Proof. This follows from (2.10) and (2.12) of Proposition 2.7. □

3. Montessus-type convergence.

One of the key results in this section is Lemma 3.4 which establishes an inequality that is central to all the proofs of convergence.

As in the hypothesis of Lemma 2.8, we shall assume $f \in \mathcal{M}\mathrm{er}^1(\mathbb{B}_\rho^N)$ and let $q_\omega(z)$ be its corresponding nonhomogeneous normalized polynomial of minimal degree ω, for which the characterizing properties (P.I)–(P.IV) given in Section 2 hold.

Take $\pi_{\mu,\nu}(z) = P_{\mu\nu}(z)/Q_{\mu\nu}(z)$ to be a (μ,ν)-URA to f at $z = 0$ and let

$$H_{\mu\nu\omega}(z) = Q_{\mu\nu}(z) f(z) q_\omega(z) - P_{\mu\nu}(z) q_\omega(z). \tag{3.1}$$

Then $H_{\mu\nu\omega} \in \mathbb{O}(\mathbb{B}_\rho^N)$ for each μ, ν, $\omega \in \mathbb{I}$, with ω fixed.

Theorem 3.1. *$H_{\mu\nu\omega} \to 0$ compactly in $\mathbb{B}_\rho^N$ as $\mu \to \infty$.*

To prove Theorem 3.1, we first need to introduce some notation and some lemmata. Let $C(\bar{\mathbb{B}}_\rho^N)$ be the space of functions continuous on $\bar{\mathbb{B}}_\rho^N$ and set $\mathbb{O}C(\bar{\mathbb{B}}_\rho^N) \equiv \mathbb{O}(\mathbb{B}_\rho^N) \cap C(\bar{\mathbb{B}}_\rho^N)$ to be the space of those functions that are holomorphic in $\mathbb{B}_\rho^N$ and continuous on $\bar{\mathbb{B}}_\rho^N$.

Lemma 3.2. *Let $\rho' < \rho$ so that $\mathbb{B}_{\rho'}^N \Subset \mathbb{B}_\rho^N$. Then $H_{\mu\nu\omega}(z) \in \mathbb{O}C(\bar{\mathbb{B}}_{\rho'}^N)$ and has a power series expansion given in $\mathbb{B}_\rho^N$ by*

$$H_{\mu\nu\omega}(z) = \sum_{k=\mu+\omega+1}^{\infty} \frac{1}{k!} \frac{d^k t}{dt^k} \left(Q_{\mu\nu z}(t) f_z(t) q_{\omega z}(t) \right) \Big|_{t=0}. \tag{3.2}$$

Proof. Take $\rho' > \sqrt{N}$ and $0 < \epsilon < (\rho - \rho')/2$. Let $\Delta_{1+\epsilon} := \{\tau \in \mathbb{C} : |\tau| < 1+\epsilon\}$. With the above choice of ρ', choose a unit polydisc $\Delta_1^N \Subset \mathbb{B}_{\rho'}^N$ so that $(1+\epsilon)\Delta_1^N \Subset \mathbb{B}_{\rho'}^N$. Then for $z \in \Delta_1^N$ with $\tau z \in \mathbb{B}_{\rho'}^N$, where $\tau \in \Delta_1$, Cauchy's integral formula yields

$$H_{\mu\nu\omega}(z) = \frac{1}{2\pi i} \int_{\partial\Delta_{1+\epsilon}} \frac{H_{\mu\nu\omega z}(t)}{1-t} \, dt, \tag{3.3}$$

where $z \in \mathbb{B}_{\rho'}^N$. But $1/(1-t) = \sum_{k=0}^{\infty} 1/t^k$ is absolutely and uniformly convergent in $\partial\Delta_{1+\epsilon}$, so (3.3) gives

$$H_{\mu\nu\omega}(z) = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\partial\Delta_{1+\epsilon}} \frac{H_{\mu\nu\omega z}(t)}{t^{k+1}} dt = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dt^k} \left(H_{\mu\nu\omega z}(t) \right) \Big|_{t=0}. \tag{3.4}$$

This series is compactly convergent in $\mathbb{B}^N_{\rho'}$. Using the expression (3.1) for $H_{\mu\nu\omega}(z)$, it is immediate that

$$H_{\mu\nu\omega}(z) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^k}{dt^k}(Q_{\mu\nu z} f_z(t) q_{\omega z}(t))\Big|_{t=0} - \sum_{k=0}^{\mu+\omega} \frac{1}{k!}\frac{d^k}{dt^k}(P_{\mu\nu z}(t) q_{\omega z}(t))\Big|_{t=0}, \quad (3.5a)$$

where we have used Lemma 2.8 to yield

$$H_{\mu\nu\omega}(z) = \sum_{k=\mu+1}^{\infty} \frac{1}{k!}\frac{d^k}{dt^k}(Q_{\mu\nu z}(t) f_z(t) q_{\omega z}(t))\Big|_{t=0} - \sum_{k=\mu+1}^{\mu+\omega} \frac{1}{k!}\frac{d^k}{dt^k}(P_{\mu\nu z}(t) q_{\omega z}(t))\Big|_{t=0}. \quad (3.5b)$$

The proof then follows from Claim 3.3. □

Claim 3.3. For $\mu+1 \le k \le \mu+\omega$,

$$\frac{d^k}{dt^k}(P_{\mu\nu z}(t) q_{\omega z}(t))\Big|_{t=0} = \frac{d^k}{dt^k}(Q_{\mu\nu z}(t) f_z(t) q_{\omega z}(t))\Big|_{t=0}. \quad (3.6)$$

Proof. This is immediate from direct computation, using Leibniz rule as follows:

$$\begin{aligned}\frac{d^k}{dt^k}(P_{\mu\nu z}(t) q_{\omega z}(t))\Big|_{t=0} &= \sum_{l=0}^{\min(k,\mu)} \binom{k}{l} \frac{d^l}{dt^l} P_{\mu\nu z}(t) \frac{d^{k-l}}{dt^{k-l}} q_{\omega z}(t)\Big|_{t=0} \\ &= \sum_{l=0}^{\mu} \binom{k}{l} \frac{d^l}{dt^l} P_{\mu\nu z}(t) \frac{dt^{k-l}}{dt^{k-l}} q_{\omega z}(t)\Big|_{t=0}.\end{aligned} \quad (3.7)$$

From (2.6) (cf. (2.14)) we get for $0 \le l \le \mu$,

$$\frac{d^l}{dt^l} P_{\mu\nu z}(t)\Big|_{t=0} = \frac{d^l}{dt^l}(Q_{\mu\nu z}(t) f_z(t))\Big|_{t=0}. \quad (3.8)$$

This completes the proof of the claim. □

The proof of Lemma 3.2 is thus an immediate consequence of this claim together with (3.5b).

Lemma 3.4. *Given K compact in $\mathbb{B}^N_\rho$, there is a ρ' (cf. Lemma 3.2) such that $K \subset \mathbb{B}^N_{\rho'} \Subset \mathbb{B}^N_\rho$. Then with $\epsilon > 0$ as in the proof of Lemma 3.2,*

$$\|H_{\mu\nu\omega}(z)\|_K \le \frac{M}{\epsilon(1+\epsilon)^\mu}, \quad (3.9)$$

where $\|\,\|_K := \sup_K |\,|$ and

$$M := \max_{t\in\partial\Delta_{1+\epsilon}} \left\{ \max_{z\in\bar{\mathbb{B}}^N_{\rho'}} (|Q_{\mu\nu z}(t) f_z(t) q_{\omega z}(t)|) \right\}. \quad (3.10)$$

Proof. Following the set up of the preceding lemma, with $z \in \mathbb{B}^N_{\rho'}$, and using Cauchy's estimate we obtain

$$\begin{aligned}|H_{\mu\nu\omega}(z)| &\le \sum_{k=\mu+\omega+1}^{\infty} \frac{1}{2\pi} \int_{\partial\Delta_{1+\epsilon}} \frac{|Q_{\mu\nu z}(t) f_z(t) q_{\omega z}(t)|}{|t|^{k+1}} |dt| \\ &\le \sum_{k=\mu+1}^{\infty} \frac{M}{(1+\epsilon)^k} \le \frac{M}{\epsilon(1+\epsilon)^\mu},\end{aligned} \quad (3.11)$$

where M is as stated in the lemma. Hence for $z \in K$ the desired inequality follows. □

Proof of Theorem 3.1. This is obtained immediately from the inequality (3.9) on letting $\mu \to \infty$ while ϵ remains positive. □

Theorem 3.5 (Montessus-type). *Let $\nu \in \mathbb{I}$ be fixed. Suppose $f \in \mathcal{M}\mathrm{er}^1(\mathbb{B}_\rho^N)$, i.e., f is characterized by the four properties (P.I)–(P.IV). Suppose $\{\pi_{\mu\nu}(z)\}_\mu$ is a "column" sequence of a (μ,ν)-URA table to f at the origin $z = 0$, with its polar set on $\mathbb{B}_\rho^N$ determined by $\mathfrak{X}(\pi_{\mu\nu}^{-1}) \cap \mathbb{B}_\rho^N$ which is closed in $\mathbb{B}_\rho^N$. Then as $\mu \to \infty$, (modulo the sets of indeterminacy of f and $\pi_{\mu\nu}$)*

(i) $\mathfrak{X}(\pi_{\mu\nu}^{-1}) \cap \mathbb{B}_\rho^N \to \mathfrak{X}(f^{-1}) \cap \mathbb{B}_\rho^N$,

(ii) $\pi_{\mu\nu}(z) \to f(z)$ *compactly in* $\mathbb{B}_\rho^N \backslash \mathfrak{X}(f^{-1})$.

Proof. Recall that $\pi_{\mu\nu}(z) = P_{\mu\nu}(z)\backslash Q_{\mu\nu}(z)$. Without loss of generality, we shall assume that both $Q_{\mu\nu}$ and q_ν have been normalized in the same manner.

To prove (i), let K be any compact subset of $\mathbb{B}_\rho^N$ such that there is a ρ', satisfying $1 < \rho' < \rho$ with $K \subset \mathbb{B}_{\rho'}^N \Subset \mathbb{B}_\rho^N$. Now since $Q_{\mu\nu}$'s are normalized, $\{Q_{\mu\nu}\}_\mu$ is a uniformly bounded sequence in K and therefore, it contains a subsequence $\{Q_{\mu_j\nu}\}_j$ that converges uniformly to, say, S_ν, in K. From Theorem 3.1, we know that $H_{\mu\nu}(z) = Q_{\mu\nu}(z)f(z)q_\nu(z) - P_{\mu\nu}(z)q_\nu(z) \to 0$ uniformly on K, and so the uniform convergence of the subsequence $\{Q_{\mu_j\nu}(z)\}_j$ induces the uniform convergence of a similar subsequence $\{P_{\mu_j\nu}(z)\}_j$ of $\{P_{\mu\nu}(z)\}_\mu$ to a limit, say, $T(z)$ on K. Thus from the limit of $H_{\mu\nu}(z) \to 0$, we get

$$S_\nu(z)f(z)q_\nu(z) = T(z)q_\nu(z). \tag{3.12}$$

□

Claim 3.6. $\mathfrak{X}(S_\nu) \cap \mathbb{B}_\rho^N = \mathfrak{X}(q_\nu) \cap \mathbb{B}_\rho^N$ except possibly at the points of indeterminacy of f on $\mathbb{B}_\rho^N$.

Proof. Take any point $b \in \mathfrak{X}(q_\nu) \cap \mathbb{B}_\rho^N$ with b not in the set of points of indeterminacy of f. Then $q_\nu(b) = 0$ makes $T(b)q_\nu(b) = 0$ and from (3.12) we deduce that $S_\nu(b)f(b)q_\nu(b) = 0$. But from property (P.IV) of $f \in \mathcal{M}\mathrm{er}^1(\mathbb{B}_\rho^N)$, we have $\mathfrak{X}(fq_\nu) \cap \mathfrak{X}(q_\nu) \cap \mathbb{B}_\rho^N = \varnothing$, so that $f(b)q_\nu(b) \neq 0 \Rightarrow S_{\nu(b)} = 0$, i.e., $b \in \mathfrak{X}(S_\nu) \cap \mathbb{B}_\rho^N$ modulo any points of indeterminacy. Therefore, $\mathfrak{X}(q_\nu) \cap \mathbb{B}_\rho^N \subset \mathfrak{X}(S_\nu) \cap \mathbb{B}_\rho^N$. Now start with $b \in \mathfrak{X}(S_\nu) \cap \mathbb{B}_\rho^N$, with b not in the set of points of indeterminacy of f, then $S_\nu(b) = 0$, and hence $S_\nu(b)f(b)q_\nu(b) = 0$. Once again from (3.12), we deduce that $T(b)q_\nu(b) = 0$. From the relative primeness condition of $P_{\mu\nu}$ and $Q_{\mu\nu}$ for all $\mu,\nu \in \mathbb{I}$ we get $\mathfrak{X}(Q_{\mu_j\nu}) \cap \mathfrak{X}(P_{\mu_j\nu}) \cap \mathbb{B}_\rho^N = \varnothing$, except at points of indeterminacy of $\pi_{\mu_j\nu}$ for all j with $\nu \in \mathbb{I}$ fixed. Thus we must have $\mathfrak{X}(S_\nu) \cap \mathfrak{X}(T) \cap \mathbb{B}_\rho^N = \varnothing$, except at points of indeterminacy of f. This implies that $T(b) \neq 0$. Therefore, $q_\nu(b) = 0$, i.e., $b \in \mathfrak{X}(q_\nu) \cap \mathbb{B}_\rho^N$ modulo the points of indeterminacy of f and so $\mathfrak{X}(S_\nu) \cap \mathbb{B}_\rho^N \subset \mathfrak{X}(q_\nu) \cap \mathbb{B}_\rho^N$. This completes the proof of the claim. □

Proof of Theorem 3.5 continued. Now every subsequence $\{Q_{\mu_j\nu}(z)\}_j$ is constrained by (3.12) to converge to S_ν. Hence the sequence $\{Q_{\mu\nu}(z)\}_\mu$ converges uniformly in $\mathbb{B}_\rho^N$ to S_ν. This implies that

$$\mathfrak{X}(Q_{\mu\nu}) \cap \mathbb{B}_\rho^N \longrightarrow \mathfrak{X}(S_\nu) \cap \mathbb{B}_\rho^B. \tag{3.13}$$

By the claim, the desired result follows for (i).

To prove (ii), we return to Lemma 3.4, which says that in any compact set $K \subset \mathbb{B}^N_{\rho'} \Subset \mathbb{B}^N_\rho$, $0 < \rho' < \rho$ and $0 < \epsilon < (\rho - \rho')/2$,

$$\|Q_{\mu\nu}(z) f(z) q_\nu(z) - P_{\mu\nu}(z) q_\nu(z)\|_K \le \frac{M}{\epsilon(1+\epsilon)^\mu}. \tag{3.14}$$

Since $Q_{\mu\nu}(z) \to q_\nu(z)$ uniformly on K as $\mu \to \infty$, given $\eta > 0$ and an η-neighborhood $\mathcal{N}(\mathfrak{X}(q_\nu), \eta)$ of $\mathfrak{X}(q_\nu) \cap \mathbb{B}^N_\rho$, there exists μ_0 such that for $\mu > \mu_0$ implies that

$$(\mathfrak{X}(Q_{\mu\nu}) \cup \mathfrak{X}(q_\nu)) \cap \mathbb{B}^N_\rho \subset \mathcal{N}(\mathfrak{X}(q_\nu), \eta). \tag{3.15}$$

Thus $\exists \delta > 0$, $\forall z \in K \backslash \mathcal{N}(\mathfrak{X}(q_\nu), \eta)$ such that $\|Q_{\mu\nu}(z) q_\nu(z)\| > \delta$, $\forall \mu > \mu_0$.

Finally, from (3.14), we obtain

$$\|f(z) - \pi_{\mu\nu}(z)\|_{K \backslash \mathcal{N}(\mathfrak{X}(q_\nu), \eta)} \le \frac{M}{\epsilon\delta(1+\epsilon)^\mu}. \tag{3.16}$$

Then letting $\eta \to 0$ first followed by $\mu \to \infty$ leads to the desired result from (3.16). □

4. Convergence in capacity. In this final section of the paper, we consider diagonal sequences $\{\pi_{\mu\mu}\}_\mu$ of (μ, μ)-sequences of URA's to f at zero for $f \in \mathcal{M}\mathrm{er}^1(\mathbb{B}^N_\rho)$. Here convergence is considered in terms of capacity and not in terms of uniform convergence. This is done because there is a significant growth in the polar set of $\pi_{\mu\mu}$ for sufficiently large values of μ, which tend to thwart uniform convergence on compact subsets of $\mathbb{B}^N_\rho$, except in a small neighborhood of the origin zero.

For any integer $d \ge 1$, let $\mathcal{P}_d(\mathbb{C}^N)$ be the class of polynomials $P_d(z) = \sum_{|\alpha| \le d} a_\alpha z^\alpha$, with normalization $\max_{|\alpha| \le d}\{|a_\alpha|\} = 1$, so that on the unit ball $\mathbb{B}^N_\rho$, $\|\tilde{P}_d\|_{\mathbb{B}^N_\rho} \ge 1$, where $\tilde{P}_d$ is a homogeneous polynomial of degree d. Let $K \subset \mathbb{B}^N_\rho$ with $\rho > 1$, be compact. Then we can find a ρ' such that $1 < \rho' < \rho$ and $K \subset \mathbb{B}^N_{\rho'} \Subset \mathbb{B}^N_\rho$. The Tchebychev constant for a compact set K may be defined from (see [1, 2])

$$M_d(K) = \inf_{\mathcal{P}_d} \{\|p_d\|_K : p_d \in \mathcal{P}_d(\mathbb{C}^N)\}, \tag{4.1}$$

as a capacity of K by

$$T(K) := \inf_d \{M_d(K)\}^{1/d} = \lim_{d \to \infty} \{M_d(K)\}^{1/d}. \tag{4.2}$$

Lemma 4.1. *Let $0 < \delta < 1$ be given. Let $g_\sigma(z) \in \mathcal{P}_d(\mathbb{C}^N)$. Suppose the compact set K is defined by $K := \{z \in \mathbb{C}^N : |g_\sigma(z)| \le \delta^\sigma\}$. Then, there exists a $c_1 > 0$ such that*

$$T(K) < c_1 \delta. \tag{4.3}$$

Proof. From [12] we know that for each $d \ge 1$, there is a Tchebychev polynomial $p^*_d \in \mathcal{P}_d(\mathbb{C}^N)$, such that $\|p^*_d\| = M_d(K)$. With respect to $\sigma > 0$, $\sigma \in \mathbb{I}$, following [12], we can find numbers τ and r with $0 \le r < \tau$, σ satisfying $\sigma = k\tau + r$, so that $g_\sigma(z) = z^r (p^*_\tau(z))^k$. Thus

$$\|g_\sigma(z)\|_K = \sup |z^r| \, |p^*_\tau(z)|^k \le \delta^\sigma. \tag{4.4}$$

Now let $L = \sup_{(z_1,\dots,z_N)\in K} |z^r| > 0$, and take $c_1 = \max(1/L, 1)$. Then

$$\{M_\tau(K)\}^{1/\tau} = \|p_\tau^*\|_K^{1/\tau} \le c_1\delta. \tag{4.5}$$

Hence from the definition of $T(K)$, the desired result follows. □

Theorem 4.2. *Let $0 < \delta < 1$ be given. Suppose $f \in \mathcal{M}\mathrm{er}^1(\mathbb{B}_\rho^N)$, i.e., there is a normalized nonhomogeneous polynomial $q_\sigma(z)$ of degree σ, such that $\mathfrak{X}(f^{-1}) \cap \mathbb{B}_\rho^N = \mathfrak{X}(q_\sigma) \cap \mathbb{B}_\rho^N$. Suppose, $\{\pi_{\mu\mu}\}_\mu$, is a diagonal (μ,μ)-sequence of URA's to f at zero. Let*

$$K_\mu := \{z \in \mathbb{B}_\rho^N : |f(z) - \pi_{\mu\mu}(z)| \ge \delta^{-\mu}\}. \tag{4.6}$$

Then there exist μ_0 and a constant c_0 such that for $\mu > \mu_0$ we have

$$T(K_\mu) \le c_0\delta. \tag{4.7}$$

Proof. From Lemma 3.4, the inequality that was obtained in relation to (μ,ν)-URA sequences also holds for diagonal URA-sequences in the ball $\mathbb{B}_{\rho'}^N$. That is to say,

$$\|H_{\mu\mu\sigma}(z)\| \le \frac{M}{\epsilon(1+\epsilon)^\mu}, \tag{4.8}$$

where $0 < \epsilon < (\rho - \rho')/2$ as before and M is defined in Lemma 3.4. Note that in the inequality (3.9), σ replaces ω. Now from the inequality (4.8), we obtain

$$|f(z) - \pi_{\mu\mu}(z)| \le \frac{M}{\epsilon(1+\epsilon)^\mu |Q_{\mu\mu}(z) q_\sigma(z)|}. \tag{4.9}$$

Now recall that $Q_{\mu\mu}(0) \ne 0$ and from the definition of f, it remains holomorphic at zero and so $q_\sigma(0) \ne 0$. Thus in some small neighborhood $\mathbb{B}_\eta^N$ of the origin, with $0 < \eta \ll 1$, so that $\mathbb{B}_\eta^N \Subset \mathbb{B}_1^N$, the latter being the unit ball, there exist $\delta \in (0,1)$ and $\mu_1 \in \mathbb{I}$ such that $|q_\sigma(z) Q_{\mu\mu}(z)| > \delta$, for $\mu > \mu_1$ and

$$|f(z) - \pi_{\mu\mu}(z)| \le \frac{M}{\epsilon\delta(1+\epsilon)^\mu}, \quad z \in \mathbb{B}_\eta^N. \tag{4.10}$$

However, for $z \in \mathbb{B}_{\rho'}^N \Supset \mathbb{B}_1^N \Supset \mathbb{B}_\eta^N$, $|f(z) - \pi_{\mu\mu}(z)| \ge \delta^{-\mu}$, leads to

$$|Q_{\mu\mu}(z) q_\sigma(z)| \le \left(\frac{M}{\delta\epsilon}\right)\left(\frac{\delta}{1+\epsilon}\right)^\mu. \tag{4.11}$$

Thus we get

$$\limsup_{\mu\to\infty} |Q_{\mu\mu}(z)|^{1/\mu} \le \frac{\delta}{1+\epsilon} < \delta. \tag{4.12}$$

Now given $\psi > 0$, there exists $\mu_0 \in \mathbb{I}$, with $\mu_0 > \mu_1$, such that $\mu > \mu_0 \Rightarrow |Q_{\mu\mu}| < (\delta + \psi)^\mu$. Following the definition of K_μ, we find that for $\mu > \mu_0$,

$$K_\mu \subset \{z \in \mathbb{B}_\rho^N : |Q_{\mu\mu}(z)| \le (\delta + \psi)^\mu\}. \tag{4.13}$$

Then by Lemma 4.1, there exists a constant c_0 such that when $\mu > \mu_0$, we obtain

$$T(K_\mu) \le c_0(\delta + \psi), \tag{4.14}$$

since ψ is positive and arbitrary, the result follows. □

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