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SYMMETRIC DUALITY FOR A CLASS OF MULTIOBJECTIVE PROGRAMMING

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ABSTRACT. We formulate a pair of symmetric dual nondifferentiable multiobjective programming and establish appropriate duality theorems. We also show that differentiable and nondifferentiable analogues of several pairs of symmetric dual problems can be obtained as special cases of our general symmetric programs.

Keywords and phrases. Symmetric duality, multiobjective, nondifferentiable, pseudoconvexity.

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1. Introduction. The concept of symmetric dual programs, in which the dual of the dual equals the primal, was introduced and developed in, e.g., [2, 4, 5]. Recently, Chandra, Craven, and Mond [1] formulated a pair of symmetric dual programs with a square root term. Weir and Mond [7] discussed symmetric duality in multiobjective programming. Mond, Husain, and Prasad gave symmetric duality result for nondifferentiable multiobjective programs in [6]. In this paper, a pair of symmetric dual nondifferentiable multiobjective programming problems is formulated and appropriate duality theorems are established under suitable generalized invexity assumptions. These results include duality results for multiobjective programs given in [6, 7] as special cases.

2. Notation and preliminaries. The following conventions for vectors in R^n will be used:

$x > y$ if and only if $x_i > y_i$, $i = 1,2,3,\dots,n$;
$x \geqq y$ if and only if $x_i \geq y_i$, $i = 1,2,3,\dots,n$;
$x \geq y$ if and only if $x_i \geq y_i$, $i = 1,2,3,\dots,n$, but $x \neq y$;
$x \not\geq y$ is the negation of $x \geq y$.

If F is a twice differentiable function from $R^n \times R^m$ to R, then $\nabla_x F$ and $\nabla_y F$ denote gradient (column) vectors of F with respect to x and y, respectively, and $\nabla_{yy} F$ and $\nabla_{yx} F$ denote the $(m\times m)$ and $(m\times n)$ matrices of second-order partial derivatives, respectively.

If F is a twice differentiable function from $R^n \times R^m$ to R^k, then $\nabla_x F$ and $\nabla_y F$ denote, respectively, the $(n\times k)$ and $(m\times k)$ matrices of first-order partial derivatives.

Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x \mid C) = \max\{x^T y,\ y \in C\}. \tag{2.1}$$

A support function, being convex and everywhere finite, has a subdifferential in the sense of convex analysis, that is, there exists z such that $s(y \mid C) \geq s(x \mid C) + z^T(y - x)$ for all x. The subdifferential of $s(x \mid C)$ is given by

$$\partial s(x \mid C) = \{z \in C : z^T x = s(x \mid C)\}. \tag{2.2}$$

We also require the concept of a normal cone. For any set S the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y : y^T(z - x) \leq 0 \ \forall z \in S\}. \tag{2.3}$$

There is a relationship between normal cones and support functions of a compact convex set C, namely, y is in $N_C(x)$ if and only if $s(y \mid C) = x^T y$ or equivalently, x is in the subdifferential of s at y.

Consider the multiple objective programming problem:

$$\min f(x) \text{ subject to } x \in X, \tag{2.4}$$

where $f : R^n \to R^k$ and $X \subset R^n$.

A feasible point x_0 is said to be an efficient solution of (2.4) if for any feasible x,

$$f_i(x_0) \geq f_i(x) \quad \forall i = 1, 2, \ldots, k \tag{2.5}$$

implies

$$f_i(x_0) = f_i(x) \quad \forall i = 1, 2, \ldots, k. \tag{2.6}$$

A feasible point x is said to be properly efficient (see [6]) if it is efficient for (2.4) and if there exists a scalar $M > 0$ such that, for each i, $f_i(x_0) - f_i(x) \leq M(f_j(x) - f_j(x_0))$ for some j such that $f_j(x) > f_j(x_0)$ whenever x is feasible for (2.4) and $f_i(x) < f_i(x_0)$.

A feasible point x_0 is said to be a weak efficient solution of (2.4) if there exists no other feasible point x for which $f(x_0) > f(x)$. If a feasible point x_0 is efficient, then it is clear that it is also a weak efficient.

Definition 2.1. A differentiable numerical function ψ defined on a set $C \subset R^n$ is said to be η-convex at $\bar{x} \in C$ if there exists a function $\eta(x, \bar{x})$ defined on $C \times C$ such that

$$\psi(x) - \psi(\bar{x}) \geq \eta(x, \bar{x})^T \nabla \psi(\bar{x}) \quad \forall x \in C. \tag{2.7}$$

If $-\psi$ is η-convex at $\bar{x} \in C$, then ψ is said to be η-concave at $\bar{x} \in C$.

Definition 2.2. A differentiable numerical function ψ defined on a set $C \subset R^n$ is said to be η-pseudoconvex at $\bar{x} \in C$ if there exists a function $\eta(x, \bar{x})$ defined on $C \times C$ such that

$$\eta(x, \bar{x})^T \nabla \psi(\bar{x}) \geq 0 \longrightarrow \psi(x) \geq \psi(\bar{x}) \quad \forall x \in C. \tag{2.8}$$

If $-\psi$ is η-pseudoconvex at $\bar{x} \in C$, then ψ is said to be η-pseudoconcave at $\bar{x} \in C$.

3. Symmetric duality. Consider the following pair of symmetric dual nondifferentiable multiobjective programs.

Primal (VP),

$$\text{minimize } \left(f_1(x,y)+s(x \mid C_1)-y^Tz_1,\dots,f_k(x,y)+s(x \mid C_k)-y^Tz_k\right) \tag{3.1}$$

subject to

(1) $\sum_{i=1}^k \lambda_i(\nabla_y f_i(x,y)-z_i) \le 0$,
(2) $y^T\sum_{i=1}^k \lambda_i(\nabla_y f_i(x,y)-z_i) \ge 0$,
(3) $z_i \in D_i,\ 1 \le i \le k$,
(4) $\lambda > 0,\ \lambda^T e = 1,\ x \ge 0$.

Dual (VD),

$$\text{maximize } \left(f_1(u,v)-s(v \mid D_1)+u^Tw_1,\dots,f_k(u,v)-s(v \mid D_k)+u^Tw_k\right) \tag{3.2}$$

subject to

(5) $\sum_{i=1}^k \lambda_i(\nabla_u f_i(u,v)+w_i) \ge 0$,
(6) $u^T\sum_{i=1}^k \lambda_i(\nabla_u f_i(u,v)+w_i) \le 0$,
(7) $w_i \in C_i,\ 1 \le i \le k$.
(8) $\lambda > 0,\ \lambda^T e = 1,\ v \ge 0$.

Here $e(1,1,\dots,1)^T \in R^k$; $f_i,\ i=1,2,\dots,k$, are twice differentiable functions from $R^n \times R^m$ into R. $C_i,\ i=1,2,\dots,k$, are compact convex sets in R^n, and $D_i,\ i=1,2,\dots,k$, are compact convex sets in R^m.

Now we establish weak and strong duality theorems between (VP) and (VD).

Theorem 3.1 (weak duality). *Let $(x,y,\lambda,z_1,z_2,\dots,z_k)$ be feasible for (VP) and let $(u,v,\lambda,w_1,w_2,\dots,w_k)$ be feasible for (VD). Let*

$$\sum_{i=1}^k \lambda_i\left(f_i(\cdot,v)+(\cdot)^Tw_i\right) \text{ be } \eta_1\text{-pseudoconvex at } u \tag{3.3}$$

and let

$$\sum_{i=1}^k \lambda_i\left(f_i(x,\cdot)-(\cdot)^Tz_i\right) \text{ be } \eta_2\text{-pseudoconcave at } y. \tag{3.4}$$

Assume that $\eta_1(x,u)+u \ge 0,\ \eta_2(v,y)+y \ge 0$. Then the following cannot hold:

$$f_i(x,y)+s(x \mid C_i)-y^Tz_i \le f_i(u,v)-s(v \mid D_i)+u^Tw_i \quad \forall i \in \{1,2,\dots,k\}, \tag{3.5}$$

$$f_j(x,y)+s(x \mid C_j)-y^Tz_j < f_j(u,v)-s(v \mid D_j)+u^Tw_j \quad \text{for some } j. \tag{3.6}$$

Proof. From $\eta_1(x,u)+u \ge 0$, (5), and (6) we have

$$\eta_1(x,u)^T\sum_{i=1}^k \lambda_i\left(\nabla_u f_i(u,v)+w_i\right) \ge 0. \tag{3.7}$$

Since $\sum_{i=1}^k \lambda_i(f_i(\cdot,v)+(\cdot)^Tw_i)$ is η_1-pseudoconvex at u it follows that

$$\sum_{i=1}^k \lambda_i\left(f_i(x,v)+x^Tw_i\right) \ge \sum_{i=1}^k \lambda_i\left(f_i(u,v)+u^Tw_i\right). \tag{3.8}$$

Since $x^T w_i \leq s(x \mid C_i)$, $1 \leq i \leq k$, and (7), then

$$\sum_{i=1}^{k} \lambda_i f_i(x,v) \geq \sum_{i=1}^{k} \lambda_i f_i(u,v) + u^T w^i - s(x \mid C_i). \tag{3.9}$$

From $\eta_2(v,y)+y \geq 0$, (1), and (2) we have

$$\eta_2(v,y)^T \sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x,y) - z_i) \leq 0. \tag{3.10}$$

The η_2-pseudoconcavity assumption of $\sum_{i=1}^{k} \lambda_i (f_i(x,\cdot) - (\cdot)^T z_i)$ implies

$$\sum_{i=1}^{k} \lambda_i (f_i(x,v) - v^T z_i) \leq \sum_{i=1}^{k} \lambda_i (f_i(x,y) - y^T z_i). \tag{3.11}$$

Since $v^T z_i \leq s(v \mid D_i)$, $1 \leq i \leq k$, and (4), then

$$\sum_{i=1}^{k} \lambda_i [f_i(x,v)] \leq \sum_{i=1}^{k} \lambda_i [f_i(x,y) + s(v \mid D_i) - y^T z_i]. \tag{3.12}$$

Combining (8), (3.9), and (3.12) yields the conclusion that (3.5) and (3.6) do not hold. □

Theorem 3.2 (weak duality). *Let $(x,y,\lambda,z_1,z_2,\dots,z_k)$ be feasible for (VP) and $(u,v,\lambda,w_1,w_2,\dots,w_k)$ be feasible for (VD). Let for all $i \in \{1,2,\dots,k\}$, $f_i(\cdot,v)+(\cdot)^T w_i$ and $-f_i(x,\cdot)+(\cdot)^T z_i$ are η_1-convex for fixed v and η_2-convex for fixed x, respectively. Let $\eta_1(x,u)+u \geq 0$, $\eta_2(v,y)+y \geq 0$. Then the following cannot hold:*

$$\begin{gathered} f_i(x,y) + s(x \mid C_i) - y^T z_i \leq f_i(u,v) - s(v \mid D_i) + u^T w_i \quad \forall i \in \{1,2,\dots,k\}; \\ f_j(x,y) + s(x \mid C_j) - y^T z_j < f_j(u,v) - s(v \mid D_j) + u^T w_j \quad \text{for some } j. \end{gathered} \tag{3.13}$$

Proof. Since $f_i(\cdot,v)+(\cdot)^T w_i$ is η_1-convex for fixed $v\,(1 \leq i \leq k)$, we have

$$[f_i(x,v) + x^T w_i] - [f_i(u,v) + u^T w_i] \geq \eta_1(x,u)^T [\nabla_u f_i(u,v) + w_i], \quad 1 \leq i \leq k. \tag{3.14}$$

Since $\lambda > 0$, then

$$\sum_{i=1}^{k} \lambda_i [f_i(x,v) + x^T w_i] - \sum_{i=1}^{k} \lambda_i [f_i(u,v) + u^T w_i] \geq \eta_1(x,u)^T \left\{ \sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u,v) + w_i] \right\}. \tag{3.15}$$

Since $-f_i(x,\cdot)+(\cdot)^T z_i$ is η_2-convex for fixed $x\,(1 \leq i \leq k)$, we have

$$[f_i(x,v) - v^T z_i] - [f_i(x,y) - y^T z_i] \leq \eta_2(v,y)^T [\nabla_y f_i(x,y) - z_i], \quad 1 \leq i \leq k. \tag{3.16}$$

Since $\lambda > 0$ it follows that

$$\sum_{i=1}^{k} \lambda_i [f_i(x,v) + v^T z_i] - \sum_{i=1}^{k} \lambda_i [f_i(x,y) + y^T z_i] \le \eta_2(v,y)^T \left\{ \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x,y) - z_i] \right\}. \tag{3.17}$$

Now from $\eta_1(x,u) + u \ge 0$, (5), and (6), we have

$$\eta_1(x,u)^T \left\{ \sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u,v) + w_i] \right\} \ge 0. \tag{3.18}$$

From (3.15), (3.18), and $x^T w_i \le s(x \mid C_i),\ i = 1,2,\dots,k$; we obtain

$$\sum_{i=1}^{k} \lambda_i [f_i(x,v)] \ge \sum_{i=1}^{k} \lambda_i [f_i(u,v) - s(x \mid C_i) + u^T w_i]. \tag{3.19}$$

By $\eta_2(v,y) + y \ge 0$, (2), and (3), we have

$$\eta_2(v,y)^T \left\{ \sum_{i=1}^{k} [\nabla_y f_i(x,y) - z_i] \right\} \le 0. \tag{3.20}$$

From (3.17), (3.20), and $v^T z_i \le s(v \mid D_i),\ i = 1,2,\dots,k$; we obtain

$$\sum_{i=1}^{k} \lambda_i [f_i(x,v)] \le \sum_{i=1}^{k} \lambda_i [f_i(x,y) - y^T z_i + s(v \mid D_i)]. \tag{3.21}$$

The proof now follows along similar lines as in Theorem 3.1. □

Theorem 3.3 (strong duality). *Let $(x_0, y_0, \lambda^0, z_1^0, z_2^0, \dots, z_k^0)$ be a properly efficient solution for (VP) and fix $\lambda = \lambda^0$ in (VD), and let suppositions of Theorem 3.1 be fulfilled. Assume that*

(i) *the set*

$$\sum_{i=1}^{k} \lambda_i^0 [\nabla_{yy} f_i(x_0, y_0)] \tag{3.22}$$

is positive or negative definite

(ii) *and the set*

$$\{\nabla_y f_i(x_0, y_0) - z_i^0,\ i = 1,2,\dots,k\} \tag{3.23}$$

is linearly independent. Then there exist $w_i^0 \in R^n,\ i = 1,2,\dots,k$ such that $(x_0, y_0, \lambda^0, w_1^0, w_2^0, \dots, w_k^0)$ is a properly efficient solution of (VD).

Proof. Since $(x_0, y_0, \lambda^0, w_1^0, w_2^0, \dots, w_k^0)$ is a properly efficient solution of (VP), then it is a weakly efficient solution. Hence there exists $\alpha \in R^k,\ \beta \in R^k,\ s \in R^k,\ \gamma \in R^k$, $\mu \in R^k$ and $\eta \in R$ not all zero and $w_i \in R^n$ $(1 \le i \le k)$ such that the following Fritz

John optimality conditions [3] are satisfied at $(x_0,y_0,\lambda^0,z_1^0,z_2^0,\dots,z_k^0)$,

$$\sum_{i=1}^{k}\alpha_i(\nabla_x f_i+w_i^0)+(\beta-\eta y_0)^T\sum_{i=1}^{k}\lambda_i(\nabla_{yx}f_i)=s, \tag{3.24}$$

$$\forall i\in\{1,2,\dots,k\},\quad w_i^0\in C_i, \tag{3.25}$$

$$\forall i\in\{1,2,\dots,k\},\quad x_0^T w_i^0=s(x\mid C_i), \tag{3.26}$$

$$\sum_{i=1}^{k}(\alpha_i-\eta\lambda_i^0)[\nabla_y f_i-z_i]+(\beta-\eta y_0)^T\sum_{i=1}^{k}\lambda_i^0[\nabla_{yy}f_i]=0, \tag{3.27}$$

$$\forall i\in\{1,2,\dots,k\},\quad (\beta-\eta y_0)^T[\nabla_y f_i-z_i]-\mu_i=0, \tag{3.28}$$

$$\forall i\in\{1,2,\dots,k\},\quad \alpha_i y_0-(\beta-\eta y_0)^T\quad \lambda^0\in N_{D_i}(z_i^0), \tag{3.29}$$

$$\beta^T\sum_{i=1}^{k}\lambda_i^0(\nabla_y f_i-z_i^0)=0, \tag{3.30}$$

$$\eta y_0^T\sum_{i=1}^{k}\lambda_i^0(\nabla_y f_i-z_i^0)=0, \tag{3.31}$$

$$s^T x_0=0, \tag{3.32}$$

$$\mu^T\lambda^0=0, \tag{3.33}$$

$$(\alpha,\beta,s,\mu,\eta)\geq 0, \tag{3.34}$$

$$(\alpha,\beta,s,\mu,\eta)\neq 0. \tag{3.35}$$

Since $\lambda^0>0$ and $\mu\geqq 0$, (3.33) implies $\mu=0$. Consequently (3.28) yields

$$(\beta-\eta y_0)^T(\nabla_y f_i-C_i w_i)=0. \tag{3.36}$$

Multiplying left-hand side of (3.27) by $(\beta-\eta y_0)^T$ and using (3.36), we have

$$(\beta-\eta y_0)^T\left\{\sum_{i=1}^{k}\lambda_i^0[\nabla_{yy}f_i]\right\}(\beta-\eta y_0)=0 \tag{3.37}$$

which, in view of (i), yields

$$\beta=\eta y_0. \tag{3.38}$$

From (3.27) and (3.38), we have

$$\sum_{i=1}^{k}(\alpha_i-\eta\lambda_i^0)[\nabla_y f_i-z_i^0]=0. \tag{3.39}$$

According to assumption (ii), equation (3.39) implies

$$\alpha_i=\eta\lambda_i^0,\quad i=1,2,\dots,k. \tag{3.40}$$

□

If $\eta=0$, then $\alpha_i=0$, $i=1,2,\dots,k$ and from (3.38), $\beta=0$. From (3.24), $s=0$. From (3.28), $\mu_i=0$, $i=1,2,\dots,k$. Thus, we obtain $(\alpha,\beta,\gamma,s,\mu,\eta)=0$ which contradicts condition (3.35). Hence $\eta > 0$. From (3.40) and $\lambda > 0$, we have $\alpha_i > 0$, $i = 1,2,\dots,k$. By (3.24), (3.38), and (3.40) we get

$$\sum_{i=1}^{k} \lambda_i^0 (\nabla_x f_i + w_i) = \frac{s}{\eta} \geq 0. \tag{3.41}$$

By (3.34), (3.38), and $\eta > 0$ we have

$$y_0 = \frac{\beta}{\eta} \geq 0. \tag{3.42}$$

From (3.32) and (3.41), it follows that

$$x_0^T \sum_{i=1}^{k} \lambda_i^0 (\nabla_x f_i + w_i) = 0. \tag{3.43}$$

From (3.25), (3.41), (3.42), and (3.43), we know that $(x_0, y_0, \lambda^0, z_1^0, x_2^0, \dots, z_k^0)$ is feasible for (VD).

Now from (3.29) and (3.38) we obtain

$$y_0^T z_i^0 = s(y_0 \mid D_i), \quad i = 1,2,\dots,k. \tag{3.44}$$

Using (3.26) and (3.44) we get

$$f_i(x_0, y_0) + s(x_0 \mid C_i) - y_0^T z_i^0 = f(x_0, y_0) + x_0^T z_i^0 - s(y_0 \mid D_i). \tag{3.45}$$

Thus, $(x_0, y_0, \lambda^0, w_1^0, w_1^0, w_2^0, \dots, w_k^0)$ is feasible for (VD) and the objective values of (VP) and (VD) are equal.

We claim that $(x_0, y_0, \lambda^0, w_1^0, w_2^0, \dots, w_k^0)$ is an efficient solution of (VD), for if it is not true, then there would exist $(u, v, \lambda_0, w_1, w_2, \dots, w_k)$ feasible for (VD) such that

$$\begin{aligned} &f_i(u,v) + u^T w_i - s(v \mid D_i) \geq f_i(x_0, y_0) + x_0^T w_i^0 - s(y_0 \mid D_i), \quad \forall i = 1,2,\dots,k; \\ &f_j(u,v) + u^T w_j - s(v \mid D_j) > f_j(x_0, y_0) + x_0^T w_j^0 - s(y_0 \mid D_j), \end{aligned} \tag{3.46}$$

for some $j \in \{1,2,\dots,k\}$. Using equalities (3.26) and (3.44), a contradiction to Theorem 3.1 is obtained.

If $(x_0, y_0, \lambda^0, w_1^0, w_2^0, \dots, w_k^0)$ is improperly efficient, then, for every scalar $M > 0$, there exists a feasible solution $(u, v, \lambda_0, w_1, w_2, \dots, w_k)$ in (VD) and an index i such that

$$\begin{aligned} &f_i(u,v) + u^T w_i - s(v \mid D_i) - f_i(x_0, y_0) + x_0^T w_i^0 - s(y_0 \mid D_i) \\ &\quad > M\{f_j(x_0, y_0) + x_0^T w_j^0 - s(y_0 \mid D_j)\} - f_j(u,v) + u^T w_j - s(v \mid D_j) \end{aligned} \tag{3.47}$$

for all j satisfying

$$f_j(x_0, y_0) + x_0^T w_j^0 - s(y_0 \mid D_j) > f_j(u,v) + u^T w_j - s(v \mid D_j) \tag{3.48}$$

whenever

$$f_i(u,v)+u^T w_i - s(v \mid D_j) > f_i(x_0,y_0)+x_0^T w_i^0 - s(y_0 \mid D_i). \tag{3.49}$$

Since $x_0^T w_i^0 = s(x \mid C_i)$ and $y_0^T z_i^0 = s(y_0 \mid D_i)$, $(i = 1,2,\dots,k)$, it implies that

$$f_i(u,v)+u^T w_i - s(v \mid D_i) - f_i(x_0,y_0)+s(x \mid C_i) - y_0^T z_i^0 \tag{3.50}$$

can be made arbitrarily large and hence for λ^0 with $\lambda_i^0 > 0$, we have

$$\sum_{i=1}^{k} \lambda_i^0 \{f_i(u,v)+u^T w_i - s(v \mid D_i)\} > \sum_{i=1}^{k} \lambda_i^0 \{f_i(x_0,y_0)+s(x \mid C_i) - y_0^T z_i^0\}, \tag{3.51}$$

which contradicts weak duality (Theorem 3.1).

In a similar manner to that of Theorem 3.3 we can prove the following.

Theorem 3.4 (strong duality). *Let $(x_0,y_0,\lambda^0,z_1^0,z_2^0,\dots,z_k^0)$ be a properly efficient solution for (VP) and fix $\lambda = \lambda^0$ in (VD); and the assumptions of Theorem 3.2 are fulfilled. Assume that (i) and (ii) of Theorem 3.3 hold. Then there exist $w_i^0 \in R^n$ $(1 \le i \le k)$ such that $(x_0,y_0,\lambda^0,w_1^0,w_2^0,\dots,w_k^0)$ is a properly efficient solution of (VD).*

4. Special cases. It is readily shown that $(x^T Ax)^{1/2} = s(x \mid C)$, where $C = \{Ay, y^T Ay \le 1\}$ and that this set C is compact and convex.

(i) If, for all $i \in \{1,2,\dots,k\}$, $C_i = 0$, and $D_i = 0$, then (VP) and (VD) reduce to programs studied by Weir and Mond [7].

(ii) If $(x^T B_i x)^{1/2} = s(x \mid C_i)$, where $C_i = \{B_i y, y^T B_i y \le 1\}$, $(x^T C_i x)^{1/2} = s(x \mid D_i)$, and $D_i = \{C_i y, y^T C_i y \le 1\}$, $i = 1,2,\dots,k$; then programs (VP) and (VD) become a pair of symmetric dual nondifferentiable programs considered by Mond, Husain, and Prasad [6].

(iii) If, in (FP) and (FD), $k = 1, (x^T B_i x)^{1/2} = s(x \mid C_i)$, where $C_i = \{B_i y, y^T B_i y \le 1\}$, $(x^T C_i x)^{1/2} = s(x \mid D_i)$, where $D_i = \{C_i y, y^T C_i y \le 1\}$, then we obtain the symmetric dual problems of Chandra, Craven, and Mond [1].

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