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FINITE-RANK INTERMEDIATE HANKEL OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. Let $L^2 = L^2(D, r\,dr\,d\theta/\pi)$ be the Lebesgue space on the open unit disc and let $L_a^2 = L^2 \cap \mathcal{H}ol(D)$ be the Bergman space. Let P be the orthogonal projection of L^2 onto L_a^2 and let Q be the orthogonal projection onto $\bar{L}_{a,0}^2 = \{g \in L^2;\ \bar{g} \in L_a^2,\ g(0) = 0\}$. Then $I - P \geq Q$. The big Hankel operator and the small Hankel operator on L_a^2 are defined as: for ϕ in L^∞, $H_\phi^{\text{big}}(f) = (I-P)(\phi f)$ and $H_\phi^{\text{small}}(f) = Q(\phi f)(f \in L_a^2)$. In this paper, the finite-rank intermediate Hankel operators between H_ϕ^{big} and H_ϕ^{small} are studied. We are working on the more general space, that is, the weighted Bergman space.

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1. Introduction. Let D be the open unit disc in $\mathbb{C}$ and let $d\mu$ be the finite positive Borel measure on D. Let $L^2 = L^2(\mu) = L^2(D, d\mu)$ and $\mathcal{H}ol(D)$ be the set of all holomorphic functions on D. The weighted Bergman space $L_a^2 = L_a^2(\mu)$ is the intersection of L^2 and $\mathcal{H}ol(D)$. In general, L_a^2 is not closed. In [6, Theorem 8], when $(\operatorname{supp}\mu) \cap D$ is a uniqueness set for $\mathcal{H}ol(D)$, the first author and M. Yamada gave a necessary and sufficient condition for that L_a^2 is closed. Throughout this paper, we assume that L_a^2 is closed. When $d\mu = r\,dr\,d\theta/\pi$, L_a^2 is the usual Bergman space.

For μ such that $L_a^2(\mu)$ is closed, when $\mathcal{M}$ is the closed subspace of $L^2(\mu)$ and $z\mathcal{M} \subseteq \mathcal{M}$, $\mathcal{M}$ is called an invariant subspace. Suppose that $\mathcal{M} \supseteq zL_a^2$. $P^{\mathcal{M}}$ denotes the orthogonal projection from L^2 onto $\mathcal{M}$. For ϕ in $L^\infty = L^\infty(\mu) = L^\infty(D, d\mu)$, the intermediate Hankel operator $H_\phi^{\mathcal{M}}$ is defined by

$$H_\phi^{\mathcal{M}} f = (I - P^{\mathcal{M}})(\phi f) \quad (f \in L_a^2). \tag{1.1}$$

When $\mathcal{M} = L_a^2$, $H_\phi^{\mathcal{M}}$ is called a big Hankel operator H_ϕ^{big} and when $\mathcal{M} = (\bar{z}\bar{L}_a^2)^\perp$, $H_\phi^{\mathcal{M}}$ is called a small Hankel operator H_ϕ^{small}. Note that $H_\phi^{\mathcal{M}}$ is called a little Hankel operator when $\mathcal{M} = (\bar{L}_a^2)^\perp$.

For arbitrary symbol ϕ in L^∞, in the case of $d\mu = r\,dr\,d\theta/\pi$, both H_ϕ^{big} and H_ϕ^{small} were studied when they are compact operators or Schatten class operators (see [12]). However it seems to have not been studied when they are finite-rank operators. When $\bar{\phi}$ is in L_a^2, it is known (see [12, page 155]) that if H_ϕ^{big} is a finite-rank operator, then $H_\phi^{\text{big}} = 0$ and if $\bar{\phi}$ is a polynomial, then H_ϕ^{small} is a finite-rank operator. In this paper, for arbitrary symbol ϕ in L^∞ we show that if H_ϕ^{big} is a finite-rank operator, then $H_\phi^{\text{big}} = 0$, and we study when H_ϕ^{small} is a finite-rank operator. In fact, we study such problems for the intermediate Hankel operators $H_\phi^{\mathcal{M}}$ on the weighted Bergman space $L_a^2(\mu)$.

In [2, 7, 9, 10], intermediate Hankel operators were studied in special weights, $d\mu = (\alpha+1)(1-r^2)^\alpha r\,dr\,d\theta/\pi$ for $-1 < \alpha < \infty$. In particular, Strouse [9] studied finite-rank intermediate Hankel operators.

Let $d\mu = d\sigma(r)\,d\theta$ be a Borel measure on D, where $d\sigma(r)$ is a positive measure on $[0,1)$ with $d\sigma([0,1)) = 1/2\pi$ and $d\theta$ is the Lebesgue measure on ∂D. $L_a^2(\mu)$ is closed if $d\sigma([t,1)) > 0$ for any $t > 0$ (see [6]). For this type measures, it is possible to study more precisely the intermediate Hankel operators. In fact, L^2 has the following orthogonal decomposition:

$$L^2 = \sum_{j=-\infty}^{\infty} \oplus \mathcal{L}^2 e^{ij\theta}, \tag{1.2}$$

where $\mathcal{L}^2 = L^2(d\sigma) = L^2([0,1), d\sigma)$. Set

$$\mathbf{H}^2 = \sum_{j=0}^{\infty} \oplus \mathcal{L}^2 e^{ij\theta}, \tag{1.3}$$

then $L_a^2 \subset \mathbf{H}^2 \subset (\bar{z}\bar{L}_a^2)^\perp$ and $L^2 = \mathbf{H}^2 \oplus e^{-i\theta}\bar{\mathbf{H}}^2$. If $\mathcal{M} = \mathbf{H}^2$, it is easy comparatively to determine finite-rank Hankel operators $H_\phi^{\mathcal{M}}$ and we can do it completely in Section 5. We can expect that $H_\phi^{\mathcal{M}}$ is close to H_ϕ^{big} in case $\mathcal{M} \subseteq \mathbf{H}^2$ (see Section 5) and $H_\phi^{\mathcal{M}}$ is close to H_ϕ^{small} in case $\mathcal{M} \supseteq \mathbf{H}^2$ (see Section 6).

In Section 2, we describe an invariant subspace in L_a^2 whose codimension is of finite. Moreover we show that there does not exist an invariant subspace which contains L_a^2 properly and in which L_a^2 is of finite codimension. We also give a lot of examples of invariant subspaces which contain L_a^2 and in which Hankel operators are studied in this paper. In Section 3, we describe finite-rank intermediate Hankel operators for arbitrary measure μ such that $L_a^2(\mu)$ is closed. Moreover, we show that there does not exist any nonzero finite-rank Hankel operators H_ϕ^{big} and there exists a nonzero finite-rank Hankel operator H_ϕ^{small}. In fact, we give two necessary and sufficient conditions for that if $H_\phi^{\mathcal{M}}$ is of finite rank $\le \ell$, then $H_\phi^{\mathcal{M}} = 0$. In Sections 3, 4, and 5, we use the Fourier coefficients $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$ of $\mathcal{M}$ and so we assume $d\mu = d\sigma(r)\,d\theta$. Using the Fourier coefficients of ϕ and $\mathcal{M}$, we give a necessary and sufficient condition for that $H_\phi^{\mathcal{M}}$ is of finite rank $\le \ell$. Assuming that ϕ is a harmonic function, we can get a better necessary and sufficient condition. When $\mathcal{M} \subseteq \mathbf{H}^2$, using the Fourier coefficients $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$, we give a necessary condition and a sufficient condition for that if $H_\phi^{\mathcal{M}}$ is of finite rank $\le \ell$, then $H_\phi^{\mathcal{M}} = 0$. Two conditions are very similar but are a little different. Applications are given to examples in Section 2.

2. Invariant subspaces. In this section, we assume that $d\mu = d\sigma(r)\,d\theta$ and $d\sigma([t,1)) > 0$ for any $t > 0$, except Propositions 2.1 and 2.2. For our purpose, the invariant subspace $\mathcal{M}$ must contain zL_a^2 but $\ker H_\phi^{\mathcal{M}}$ is an invariant subspace in L_a^2. If $H_\phi^{\mathcal{M}}$ is of finite rank, then the codimension of $\ker H_\phi^{\mathcal{M}}$ in L_a^2 is finite. In order to study finite-rank intermediate Hankel operators, we need the generalization of a result of Axler and Bourdon [1] which determines finite codimensional invariant subspaces in L_a^2 when $d\mu = r\,dr\,d\theta/\pi$. In Propositions 2.1 and 2.2, the measure μ is an arbitrary finite positive Borel measure such that L_a^2 is closed and $(\operatorname{supp}\mu) \cap D$ is a uniqueness set for $\mathcal{H}ol(D)$. Since $\mathbf{H}^2 \cap L^\infty$ is an extended weak-$*$ Dirichlet algebra in L^∞, Proposition 2.3 is a corollary of [4, Theorem 1]. We will give several examples of invariant subspaces which contain zL_a^2.

Proposition 2.1. *Suppose $\mathcal{M}$ is an invariant subspace in L_a^2 and ℓ is a positive integer. The codimension of $\mathcal{M}$ in L_a^2 is ℓ, if and only if $\mathcal{M} = qL_a^2$, where $q = \prod_{j=1}^{\ell}(z-a_j)$ and $a_j \in D$ $(1 \le j \le \ell)$.*

Proof. The proof is almost parallel to that in [1, Theorem 1]. We will give a sketch of it. Suppose $\mathcal{M}^\perp = L_a^2 \ominus \mathcal{M}$ and $\dim \mathcal{M}^\perp = \ell$. Put

$$S_z f = P(zf) \quad (f \in \mathcal{M}^\perp), \tag{2.1}$$

where P is an orthogonal projection. Since $\ell < \infty$, there exists an analytic polynomial b such that $b(S_z) = S_{b(z)} = 0$ and the degree of b is less than or equal to ℓ. Hence $b\mathcal{M}^\perp \subseteq \mathcal{M}$ and so $bL_a^2 \subseteq \mathcal{M}$. We show that the zeros of b are only in D and the degree of $b = \ell$. Then $\mathcal{M} = bL_a^2$. It is clear that the degree of $b = \ell$. In this direction, we did not need the condition such that $(\operatorname{supp}\mu) \cap D$ is a uniqueness set.

If $a \notin D$, $(z-a)L_a^2$ is dense in L_a^2. Assuming $a \ge 1$ and so $a = 1$ without a loss of generality, if $\varepsilon > 0$, then $(z-1)L_a^2 = (z-1)\{z-(1+\varepsilon)\}^{-1}L_a^2$. For any $f \in L_a^2$, it is easy to see that

$$\int_D \left| \frac{z-1}{z-(1+\varepsilon)} f - f \right|^2 d\mu \longrightarrow 0 \quad (\varepsilon \longrightarrow 0). \tag{2.2}$$

This implies that $(z-1)L_a^2$ is dense in L_a^2. Thus all zeros of b must be in D. The "if" part is clear because any point $a \in D$ gives a bounded evaluation functional. Here we used the condition such that $(\operatorname{supp}\mu) \cap D$ is a uniqueness set (see [6, (1) of Theorem 8]). □

Proposition 2.2. *Suppose that $(z-a)^{-1}$ does not belong to L^2 for each $a \in D$. If $\mathcal{M}$ is an invariant subspace which contains L_a^2 properly, then the codimension of L_a^2 in $\mathcal{M}$ is infinite.*

Proof. If $\dim \mathcal{M} \ominus L_a^2 = \ell < \infty$, by the proof of Proposition 2.1, there exists a polynomial $b = \prod_{j=1}^{\ell}(z-a_j)$ such that $b\mathcal{M} \subseteq L_a^2$ and $a_j \in D$ $(1 \le j \le \ell)$. Hence there exists a function ϕ in $\mathcal{M}$ such that $\phi \notin L_a^2$ and $g = b\phi \in L_a^2$. If $g(a_k) \ne 0$ for some k, then $g/(z-a_k) = \phi \prod_{j \ne k}(z-a_j)$ cannot belong to L^2 because $(z-a_k)^{-1} \notin L^2$. Hence $g(a_j) = 0$ for any j. By [6, the proof in (1) of Theorem 8], $g \in bL_a^2$ and so $\phi = g/b$ belongs to L_a^2. This contradiction implies that $\dim \mathcal{M} \ominus L_a^2 = \infty$.

For an invariant subspace $\mathcal{M}$, set

$$\mathcal{M}_j = \left\{ f_j \in \mathcal{L}^2;\ f \in \mathcal{M},\ f(z) = \sum_{j=-\infty}^{\infty} f_j(r) e^{ij\theta} \right\}. \tag{2.3}$$

Then $\mathcal{M}_j$ is a subspace in $\mathcal{L}^2$, $r\mathcal{M}_j \subseteq \mathcal{M}_{j+1}$ and hence $\dim \mathcal{M}_{j+1} \ge \dim \mathcal{M}_j$. We call $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$ the Fourier coefficients of $\mathcal{M}$. $\mathcal{M}_j e^{ij\theta}$ may not belong to $\mathcal{M}$. If $\mathcal{M}_j e^{ij\theta}$ belongs to $\mathcal{M}$ for any j, then $\mathcal{M}$ has the following decomposition:

$$\mathcal{M} = \sum_{j=-\infty}^{\infty} \oplus \mathcal{M}_j e^{ij\theta}. \tag{2.4}$$

This decomposition is called the Fourier decomposition of $\mathcal{M}$. In general, $\mathcal{M}$ does not have the Fourier decomposition but we can get an extension $\tilde{\mathcal{M}}$ of $\mathcal{M}$ which has the following Fourier decomposition:

$$\tilde{\mathcal{M}} = \sum_{j=-\infty}^{\infty} \oplus(\text{closure of } \mathcal{M}_j)e^{ij\theta}. \tag{2.5}$$

□

Proposition 2.3. *If $\mathcal{M}$ is an invariant subspace which contains L_a^2 and $e^{i\theta}\mathcal{M} \subseteq \mathcal{M}$, then $\mathcal{M} = \chi_E \bar{q}\mathbf{H}^2 \oplus \chi_{E^c} L^2$, where χ_E is a characteristic function in $\mathcal{L}^2$ and q is a unimodular function in $\mathbf{H}^2$. Hence $\mathcal{M} \supseteq \mathbf{H}^2$. If $\bigcap_{j=0}^{\infty} e^{ij\theta}\mathcal{M} = \{0\}$, then $\mathcal{M} = \bar{q}\mathbf{H}^2$.*

Proof. Suppose $S_0 = \mathcal{M} \ominus e^{i\theta}\mathcal{M}$, then $\mathcal{M} = (\sum_{j=0}^{\infty} \oplus S_0 e^{ij\theta}) \oplus \mathcal{M}_{-\infty}$, where $\mathcal{M}_{-\infty} = \bigcap_{j=0}^{\infty} e^{ij\theta}\mathcal{M}$, and $rS_0 \subset S_0$ because $r\mathcal{M}_j \subseteq \mathcal{M}_{j+1}$. It is well known that $\mathcal{M}_{-\infty} = \chi_G L^2$ for a characteristic function χ_F of some measurable subset in D. Put $E = G^c$ then there exists a function f in S_0 such that

$$|f| > 0 \quad \text{on } E \qquad \text{and} \qquad f = 0 \quad \text{on } F. \tag{2.6}$$

Since f is orthogonal to $fe^{ij\theta}$ for all $j \geq 0$. $|f|^2$ belongs to $\mathcal{L}^1 = L^1(d\sigma) = L^1([0,1), d\sigma)$ and so $|f|$ belongs to $\mathcal{L}^2$. Hence χ_E belongs to $\mathcal{L}^2$. Set

$$F(re^{i\theta}) = \begin{cases} \dfrac{f(re^{i\theta})}{|f(re^{i\theta})|} & \text{if } f \neq 0, \\ 1 & \text{if } f = 0, \end{cases} \tag{2.7}$$

then F is a unimodular function in L^2. Since $rS_0 \subseteq S_0$, we can show that $\chi_E F$ belongs to S_0 and so $S_0 = \chi_E F \mathcal{L}^2$. Hence $\mathcal{M} \ominus \mathcal{M}_\infty = \chi_E F \mathbf{H}^2$. Since $1 \in \mathcal{M}$, $\chi_E \bar{F} \in \mathbf{H}^2$ and $q = \bar{F} \in \mathbf{H}^2$, □

Example 2.4. (i) For $0 < \beta < 1$, put

$$T_\beta = \overline{\text{span}}\{z^n \bar{z}^m;\ \beta n \geq m \geq 0\}. \tag{2.8}$$

Then T_β is an invariant subspace and $T_\beta \supseteq L_a^2$. Put $T_\beta = L_a^2$ for $\beta = 0$ and $T_\beta = \mathbf{H}^2$ for $\beta = 1$. In general, $L_a^2 \subseteq T_\beta \subseteq \mathbf{H}^2$ and T_β $(0 \leq \beta < 1)$ has the following Fourier decomposition:

$$T_\beta = \sum_{j=0}^{\infty} \oplus (T_\beta)_j e^{ij\theta}, \tag{2.9}$$

where $(T_\beta)_j = \overline{\text{span}}\{r^j p_j(r^2);\ \ p_j \text{ is a polynomial of degree at most } \beta j/(1-\beta)\}$. Janson and Rochberg [2] studied $H_\phi^{\mathcal{M}}$ when $\mathcal{M} = (\bar{T}_\beta)^\perp$. Then $(\bar{T}_\beta)^\perp = e^{i\theta}\mathbf{H}^2 \oplus \sum_{j=0}^{\infty} \oplus \{\mathcal{L}^2 \ominus (\bar{T}_\beta)_j\} e^{-ij\theta}$.

(ii) For $k \geq 0$, put $E^k = \overline{\text{span}}\{z^m \bar{z}^n;\ m = 0, 1, \ldots, k;\ n = m, m+1, \ldots\}$. $\bar{E}^k$ is an invariant subspace and $L_a^2 \subseteq \bar{E}^k \subseteq \mathbf{H}^2$. $\bar{E}^k$ has the following Fourier decomposition:

$$\bar{E}^k = \sum_{j=0}^{\infty} \oplus (\bar{E}^k)_j e^{ij\theta}, \tag{2.10}$$

where $(\bar{E}^k)_j = \text{span}\{r^j, \ldots, r^{j+2k}\}$. Strouse [9] studied $H_\phi^{\mathcal{M}}$ when $\mathcal{M} = (E^k)^\perp$. Then $(E^k)^\perp = e^{i\theta}\mathbf{H}^2 \oplus \sum_{j=0}^{\infty} \oplus \{\mathcal{L}^2 \ominus (E^k)_j\} e^{-ij\theta}$.

(iii) Fix a polynomial p of degree k, that is, $p = \sum_{j=0}^{k} a_j z^j$. Put

$$\begin{aligned} Y(p) &= \overline{\operatorname{span}}\{z^n,\ z^m \bar{p};\ n \ge 0,\ m \ge 0\}, \\ Y^k &= \overline{\operatorname{span}}\{z^\ell \bar{z}^j;\ \ell \ge 0,\ 0 \le j \le k\}. \end{aligned} \tag{2.11}$$

Both $Y(p)$ and Y^k are invariant subspaces and $L_a^2 \subseteq Y(p) \subseteq Y^k$, and Y^k has the following Fourier decomposition:

$$Y^k = \sum_{j=-k}^{\infty} \oplus (Y^k)_j e^{ij\theta}, \tag{2.12}$$

where $Y_0^k = \operatorname{span}\{1, r^2, \ldots, r^{2k}\}$ and $(Y^k)_j = r^j(Y_0^k)$ for $j \ge 0$, and $(Y^k)_{-j} = \operatorname{span}\{r^{2\ell - j}; j \le \ell \le k\}$ for $1 \le j \le k$. $(Y(p))_j \subseteq (Y^k)_j$ for any j but $Y(p)$ does not have a Fourier decomposition. If $a_j \ne 0$ for $1 \le j \le k$, $(Y(p))_j = (Y^k)_j$ for any j and so $\tilde{Y}(p) = Y^k$. Peng, Rochberg, and Wu [7] and Wang and Wu [10] studied $H_\phi^{\mathcal{M}}$ when $\mathcal{M} = (\bar{Y}^k)^\perp$. In general, we can define $Y(g)$ for any function g in L^2. Usually, $Y(g)$ does not have the Fourier decomposition.

(iv) For a unimodular function q in $\mathbf{H}^2$, put $\mathcal{M} = \bar{q}\mathbf{H}^2$. Then $\mathcal{M}$ is an invariant subspace which contains $\mathbf{H}^2$. In general, $\bar{q}\mathbf{H}^2$ may not have the Fourier decomposition but for $q = e^{i\ell\theta}$, for some $\ell \ge 0$,

$$\mathcal{M} = \sum_{j=-\ell}^{\infty} \oplus \mathcal{L}^2 e^{ij\theta}. \tag{2.13}$$

There are a lot of invariant subspaces between $\mathbf{H}^2$ and $e^{-i\ell\theta}\mathbf{H}^2$ even if $\ell = 1$.

(v) For arbitrary closed subspaces S in $\mathcal{L}^2$, put $\mathcal{M} = \mathbf{H}^2 \oplus S e^{-i\theta}$. Then $\mathcal{M}$ is an invariant subspace between $\mathbf{H}^2$ and $e^{-i\theta}\mathbf{H}^2$.

3. Kronecker's theorem. In this section, the measure μ is an arbitrary finite positive Borel measure such that L_a^2 is closed. We will write

$$\mathcal{M}^\infty = \mathcal{M} \cap L^\infty \tag{3.1}$$

and, for each positive integer ℓ,

$$\mathcal{M}^{\infty,\ell} = \left\{\phi \in L^\infty;\ \phi(z) = g(z) \prod_{j=1}^{\ell} (z - a_j)^{-1} \text{ a.e. } \mu \text{ on } D, g \in \mathcal{M}^\infty \text{ and } a_1, \ldots, a_\ell \in D\right\}. \tag{3.2}$$

Then $\mathcal{M}^\infty \subseteq \mathcal{M}^{\infty,1} \subseteq \mathcal{M}^{\infty,2} \subseteq \cdots$.

Kronecker (cf. [11, page 210]) described finite-rank Hankel operators on the Hardy space. Theorem 3.1 describes finite-rank intermediate Hankel operators on the (weighted) Bergman space. However the situation is very different from that of Kronecker because $\mathcal{M}^\infty = \mathcal{M}^{\infty,\ell}$ may happen for some $\ell > 0$. See Corollaries 3.3 and 3.4.

Theorem 3.1. *Suppose $\mathcal{M}$ is an invariant subspace which contains zL_a^2, and ϕ is a function in L^∞. $H_\phi^{\mathcal{M}}$ is of finite rank $\le \ell$ if and only if ϕ belongs to $\mathcal{M}^{\infty,\ell}$.*

Proof. Note that $\ker H_\phi^{\mathcal{M}} = \{f \in L_a^2;\ \phi f \in \mathcal{M}\}$. Since $\mathcal{M}$ is an invariant subspace, $\ker H_\phi^{\mathcal{M}}$ is also an invariant subspace. Proposition 2.1 implies the theorem. □

Theorem 3.2. *Suppose $\mathcal{M}$ is an invariant subspace which contains L_a^2, and ϕ is a function in L^∞. Then the following are equivalent:*

(1) *If $H_\phi^{\mathcal{M}}$ is of finite rank, then $H_\phi^{\mathcal{M}} = 0$.*

(2) *$\mathcal{M}^\infty = \mathcal{M}^{\infty,\ell}$ for any $\ell > 0$.*

(3) *If $g \in \mathcal{M}^\infty$, $a \in D$ and $(g(z)-g(a))/(z-a) \in L^\infty$, then $(g(z)-g(a))/(z-a)$ belongs to $\mathcal{M}^\infty$.*

(4) *If $\mathcal{M}'$ is an invariant subspace and $(\mathcal{M}')^\infty \supsetneq \mathcal{M}^\infty$, then there does not exist a nonzero polynomial b such that $b(\mathcal{M}')^\infty \subseteq \mathcal{M}^\infty$.*

Proof. By Theorem 3.1, (1)⇔(2) is clear.

(1)⇒(3). If there exists $g \in \mathcal{M}^\infty$ such that $(g-g(a))/(z-a) \in L^\infty$ does not belong to $\mathcal{M}^\infty$, put $\phi = (g-g(a))/(z-a)$, then $H_\phi^{\mathcal{M}}$ is of rank 1 and $H_\phi^{\mathcal{M}} \neq 0$.

(3)⇒(4). If (4) is not true, there exists ψ such that $\psi \notin \mathcal{M}^\infty$, $\psi \in (\mathcal{M}')^\infty$ and $b\psi \in \mathcal{M}^\infty$ for some polynomial: $b = \prod_{j=1}^{\ell}(z-a_j)$ and $a_j \in D (1 \le j \le \ell < \infty)$. We may assume that $\phi = \psi\prod_{j=1}^{\ell-1}(z-a_j) \notin \mathcal{M}^\infty$ and $g = (z-a_\ell)\phi \in \mathcal{M}^\infty$. Then

$$\frac{g-g(a_\ell)}{z-a_\ell} = \phi \in L^\infty, \quad \phi \notin \mathcal{M}^\infty. \tag{3.3}$$

(4)⇒(1). By Theorem 3.1, if $H_\phi^{\mathcal{M}}$ is of finite rank $\le \ell$, then $\phi \in \mathcal{M}^{\infty,\ell}$. If $\phi \notin \mathcal{M}^\infty$, suppose $\mathcal{M}'$ is an invariant subspace generated by ϕ and $\mathcal{M}$, then $(\mathcal{M}')^\infty \supsetneq \mathcal{M}^\infty$ but there does not exist a nonzero polynomial b such that $b(\mathcal{M}')^\infty \subseteq \mathcal{M}^\infty$. Since $\phi \in \mathcal{M}'$, this contradicts that $\phi \in \mathcal{M}^{\infty,\ell}$. □

Corollary 3.3. *Suppose* $(\operatorname{supp}\mu) \cap D$ *is a uniqueness set for $\mathcal{H}ol(D)$. If H_ϕ^{big} is of finite rank, then $H_\phi^{\text{big}} = 0$.*

Proof. Theorem 3.2(3) implies the corollary. In fact, if $g \in L_a^2 \cap L^\infty$, then $g(z) - g(a) \in (z-a)L_a^2$ by [6, the proof in (1) of Theorem 5.4]. Thus $(g(z)-g(a))/(z-a)$ belongs to $L_a^2 \cap L^\infty$. □

Corollary 3.4. *Suppose $d\mu = r\,dr\,d\theta/\pi$. Let D_0 be an open subset of D and $\mathcal{M} = \{f \in L^2;\ f$ is analytic on $D_0\}$. Then $\mathcal{M}$ is an invariant subspace and if $H_\phi^{\mathcal{M}}$ is of finite rank then $H_\phi^{\mathcal{M}} = 0$.*

Proof. It is easy to see that $\mathcal{M}^\infty$ satisfies Theorem 3.2(3). □

Corollary 3.5. *Suppose that if $H_\phi^{\mathcal{M}}$ is of finite rank then $H_\phi^{\mathcal{M}} = 0$. If $\mathcal{M}'$ is an invariant subspace which contains $\mathcal{M}$ properly, then the codimension of $\mathcal{M}$ in $\mathcal{M}'$ is infinite or $(\mathcal{M}')^\infty = \mathcal{M}^\infty$.*

Proof. If $\dim \mathcal{M}'/\mathcal{M} < \infty$, as in the proof of Proposition 2.2, then there exists a nonzero polynomial b such that $b\mathcal{M}' \subseteq \mathcal{M}$. Hence $b(\mathcal{M}')^\infty \subseteq \mathcal{M}^\infty$. If $(\mathcal{M}') \neq \mathcal{M}^\infty$, by Theorem 3.2, this contradicts that if $H_\phi^{\mathcal{M}}$ is of finite rank, then $H_\phi^{\mathcal{M}} = 0$. □

4. General case.

In this section, we assume that $d\mu = d\sigma(r)\,d\theta$ and $d\sigma([t,1)) > 0$ for any $t > 0$. Hence we can define the Fourier coefficients $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$ of $\mathcal{M}$. We assume $\mathcal{M} = \tilde{\mathcal{M}}$, that is, $\mathcal{M}$ has the Fourier decomposition.

Theorem 4.1. *Suppose $\mathcal{M}$ is an invariant subspace which contains zL_a^2 and $\phi = \sum_{j=-\infty}^{\infty} \phi_j(r)e^{ij\theta}$ is a function in L^∞. Then $H_\phi^{\mathcal{M}}$ is of finite rank $\le \ell$ if and only if there exist complex numbers $b_0, \dots, b_\ell$ such that $b_\ell = 1$ and, for any integer n,*

$$\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in \mathcal{M}_n. \tag{4.1}$$

If ℓ is the minimum number of complex numbers $b_1, \dots, b_\ell$ such that $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in \mathcal{M}_n$ for all n, then $H_\phi^{\mathcal{M}}$ is of rank ℓ.

Proof. If $H_\phi^{\mathcal{M}}$ is of rank $\le \ell$, by Theorem 3.1 there exists a polynomial $b = \sum_{j=0}^{\ell} b_j z^j$ such that $b\phi \in \mathcal{M}$. Then

$$\left(\sum_{j=-\infty}^{\infty} \phi_j(r)e^{ij\theta}\right)\left(\sum_{j=0}^{\ell} b_j r^j e^{ij\theta}\right) = \sum_{n=-\infty}^{\infty}\left(\sum_{j=0}^{\ell} \phi_{n-j}(r) b_j r^j\right)e^{in\theta} \tag{4.2}$$

and so $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in \mathcal{M}_n$ for any n. The converse and the second statement are clear by Theorem 3.2. □

Corollary 4.2. *Let $\phi = \phi_t(r)e^{it\theta}$ for some integer t in Theorem 4.1. Then $H_\phi^{\mathcal{M}}$ is of finite rank $\le \ell$ if and only if there exist complex numbers $b_0, \dots, b_\ell$ such that $b_\ell = 1$ and for $t \le n \le \ell + t$, $b_{n-t} r^{n-t} \phi_t(r) \in \mathcal{M}_n$.*

Proof. Since $\phi_j(r) = 0$ for $j \ne t$, if $n < t$ or $n > \ell + t$, then $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = 0$. For $t \le n \le \ell + t$, $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = b_{n-t} r^{n-t} \phi_t(r)$, thus the corollary follows. □

Corollary 4.3. *Let $\phi = \sum_{j=1}^{\infty} a_j z^j + \sum_{j=0}^{\infty} a_{-j} \bar{z}^j$ in Theorem 4.1. Then $H_\phi^{\mathcal{M}}$ is of rank $\le \ell$ if and only if there exist complex numbers $b_0, \dots, b_\ell$ such that $b_\ell = 1$ and for any nonpositive integer $n \sum_{j=0}^{\ell} b_j a_{n-j} r^{2j-n} \in \mathcal{M}_n$ and, for $0 < n < \ell$, $\sum_{j=n}^{\ell} b_j a_{n-j} r^{2j-n} \in \mathcal{M}_n$.*

Proof. If $n \ge \ell$ and $n \ne 0$, then

$$\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = \sum_{j=0}^{\ell} b_j a_{n-j} r^{j+n-j} = \left(\sum_{j=0}^{\ell} b_j a_{n-j}\right) r^n \tag{4.3}$$

and hence $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in \mathcal{M}_n$ because $zL_a^2 \subseteq \mathcal{M}$. Now Theorem 4.1 implies the corollary.

Theorem 4.1 does not give an exact relation between the rank of $H_\phi^{\mathcal{M}}$ and the number ℓ of complex numbers $b_0, \dots, b_\ell$ such that $b_\ell = 1$. However, we can show the following: if $H_\phi^{\mathcal{M}}$ is of rank ℓ, then there exist complex numbers $b_0, \dots, b_\ell$ such that $b_\ell = 1$, $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in \mathcal{M}_n$ for any n and $b = \sum_{j=0}^{\ell} b_j z^j$ has just ℓ zeros in D. That is, if $\ell = 1$, then $|b_0| < 1$.

By Theorem 4.1, $H_\phi^{\mathcal{M}} = 0$ if and only if $\phi_n \in \mathcal{M}_n$ for any n (i.e., $\phi \in \mathcal{M}$). Moreover, $H_\phi^{\mathcal{M}}$ is of rank ≤ 1 if and only if there exist complex numbers $(b_0, b_1) \ne (0,0)$ such that $b_1 = 1$ and $b_0\phi_n + b_1 r \phi_{n-1} \in \mathcal{M}_n$ for any n. □

5. Big Hankel operator and $\mathcal{M} \subseteq H^2$. In this section, we assume that $d\mu = d\sigma(r)\,d\theta$ and $d\sigma([t,1)) > 0$ for any $t > 0$. Hence we can define the Fourier coefficients $\{\mathcal{M}_j\}_{j=-k}^{\infty}$ of $\mathcal{M}$ and we assume $\mathcal{M} = \tilde{\mathcal{M}}$. In this case, $H_\phi^{\mathcal{M}}$ is close to H_ϕ^{big}. Recall examples in Section 2, that is, T_β, $\bar{E}^k, Y(p)$, and Y^k.

Corollary 5.1. *Suppose $\mathcal{M}$ is an invariant subspace between zL_a^2 and $\mathbf{H}^2$, and $\phi = \sum_{j=1}^{\infty} a_j z^j + \sum_{j=0}^{\infty} a_{-j}\bar{z}^j$. Then $H_\phi^{\mathcal{M}}$ is of finite rank $\le \ell$ if and only if $a_{-n} = 0$ for $n > \ell$ and there exists complex numbers $b_0, \dots, b_\ell$ such that $b_\ell = 1$ and $\sum_{j=n}^{\ell} b_j a_{n-j} r^{2j-n} \in \mathcal{M}_n$ for $0 \le n \le \ell$ and $\sum_{j=0}^{\ell} b_j a_{n-j} r^{2j-n} = 0$ for $-\ell < n < 0$.*

Proof. Since $\mathcal{M} \subseteq \mathbf{H}^2$, by Corollary 4.3 $H_\phi^{\mathcal{M}}$ is of finite rank $\le \ell$ if and only if there exist complex numbers $b_0, \dots, b_\ell$ such that $b_\ell = 1$ and $\sum_{j=0}^{\ell} b_j a_{n-j} r^{2j-n} = 0$ for $n < 0$ and $\sum_{j=n}^{\ell} b_j a_{n-j} r^{2j-n} \in \mathcal{M}_n$ for $0 \le n \le \ell$. If $\sum_{j=0}^{\ell} b_j a_{n-j} r^{2j-n} = 0$ for $n < 0$, then $b_j a_{n-j} = 0$ for $0 \le j \le \ell$ and $n < 0$. Hence for each j $(0 \le j \le \ell)$, $b_j a_{-t} = 0$ if $t > j$. Thus $a_{-t} = 0$ if $t > \ell$. □

Proposition 5.2. *Suppose $\mathcal{M}$ is an invariant subspace between zL_a^2 and $e^{-ik\theta}\mathbf{H}^2$ where $k \ge 0$, and $\phi = \sum_{j=0}^{\infty} \phi_{-j}(r) e^{-ij\theta}$ is a function in L^∞. Then $H_\phi^{\mathcal{M}}$ is of finite rank $\le \ell$ if and only if*

$$\phi(z) = \frac{\sum_{j=-k}^{\ell} \psi_j(r) e^{ij\theta}}{\sum_{j=0}^{\ell} b_j r^j e^{ij\theta}}, \tag{5.1}$$

where $\psi_n = \sum_{j=0}^{\ell} b_j r^j \phi_{n-j} \in \mathcal{M}_n$, for $-k \le n \le \ell$, and $(b_0, \dots, b_\ell) \in \mathbb{C}^\ell$.

Proof. Note that $\mathcal{M} \subseteq e^{-ik\theta}\mathbf{H}^2$ and $\phi_j(r) = 0$ for $j > 0$. If $H_\phi^{\mathcal{M}}$ is of finite rank $\le \ell$, then, by Theorem 4.1,

$$\left(\sum_{j=0}^{\ell} b_j r^j e^{ij\theta}\right)\left(\sum_{j=0}^{\infty} \phi_{-j}(r) e^{-ij\theta}\right) = \sum_{n=-k}^{\ell} \psi_n(r) e^{in\theta} \tag{5.2}$$

and $\psi_n = \sum_{j=0}^{\ell} b_j r^j \phi_{n-j} \in \mathcal{M}_n$ for $-k \le n \le \ell$. The converse is also a result of Theorem 3.1. □

Corollary 5.3. *Suppose $\mathcal{M}$ is an invariant subspace in Proposition 5.2. If $\phi = \phi_+ + \phi_- = \sum_{j=1}^{\infty} a_j z^j + \sum_{j=0}^{\infty} a_{-j}\bar{z}^j$ and $\phi_- \in L^\infty$, then $H_\phi^{\mathcal{M}}$ is of finite rank $\le \ell$ if and only if*

$$\phi(z) = \phi_+ + \frac{\sum_{j=-k}^{\ell} \psi_j(r) e^{ij\theta}}{\sum_{j=0}^{\ell} b_j r^j e^{ij\theta}}, \tag{5.3}$$

where $\psi_n = \sum_{j=0}^{\ell} b_j a_{n-j} r^{j+|n-j|} \in \mathcal{M}_n$, for $-k \le n \le \ell$, and $(b_0, \dots, b_\ell) \in \mathbb{C}^\ell$. If $(b_0, \dots, b_\ell) = (0, \dots, 0)$, then $\psi_n = 0$ and so $\phi = \phi_+$.

Theorem 5.4. *Suppose $\mathcal{M}$ is an invariant subspace between zL_a^2 and $e^{-ik\theta}\mathbf{H}^2$ where $k \ge 0$, and $\phi = \sum_{j=1}^{\infty} \phi_{-j}(r) e^{ij\theta}$ is a function in L^∞.*

(1) *If $\mathcal{M}_j \cap r^{j+1}\mathcal{L}^2 = \{0\}$ for any $j \ge 0$, then there does not exist any finite rank $H_\phi^{\mathcal{M}}$ except $H_\phi^{\mathcal{M}} = 0$.*

(2) *If there does not exist any finite rank $H_\phi^{\mathcal{M}}$ except $H_\phi^{\mathcal{M}} = 0$, then $\mathcal{M}_{-(k-j)} \cap r^{j+1}\mathcal{L}^\infty = \{0\}$ for any $j \ge 0$.*

Proof. (1) If $H_\phi^{\mathcal{M}}$ is of finite rank ℓ, by Proposition 5.2,

$$\psi_n = \sum_{j=n}^{\ell} b_j r^j \phi_{n-j} \in \mathcal{M}_n, \tag{5.4}$$

for $0 \le n \le \ell$ because $\phi_{n-j}(r) = 0$ for $0 \le j \le n-1$. We may assume $b_\ell = 1$. As $n = \ell - 1$, $r^\ell \phi_{-1}(r) \in \mathcal{M}_{\ell-1}$. Since $\mathcal{M}_{\ell-1} \cap r^\ell \mathcal{L}^2 = \{0\}$, $\phi_{-1}(r) = 0$. As $n = \ell - 2$,

$$b_{\ell-1} r^{\ell-1} \phi_{-1}(r) + r^\ell \phi_{-2}(r) \in \mathcal{M}_{\ell-2}. \tag{5.5}$$

Since $\mathcal{M}_{\ell-2} \cap r^{\ell-1}\mathcal{L}^2 = \{0\}$ and $\phi_{-1}(r) = 0$, $\phi_{-2}(r) = 0$. we can get $\phi_{-j}(r) = 0$ for $j \le \ell$. In Proposition 5.2, $\psi_n = 0$ for $0 \le n \le \ell$ and so $\phi \equiv 0$.

(2) If $r^{j+1} g \in \mathcal{M}_{-(k-j)} \cap r^{j+1}\mathcal{L}^\infty$, then put $\phi = g e^{-i(k+1)\theta}$. If $g \ne 0$ then $\phi \notin \mathcal{M}$ and

$$z^{j+1}\phi = r^{j+1} g e^{-i(k-j)\theta} \in \mathcal{M}_{-(k-j)} e^{-i(k-j)\theta}. \tag{5.6}$$

Since $\mathcal{M}$ has the Fourier decomposition, $\mathcal{M}_j e^{ij\theta} \subseteq \mathcal{M}$ and so $z^{j+1}\phi \in \mathcal{M}$. Theorem 3.1 gives a contradiction. □

We will apply results in this section to Example 2.4 in Section 2.

Example 5.5. (i) Suppose $\mathcal{M} = T_\beta$ $(0 \le \beta < 1)$.

(1) When $\phi = \sum_{j=1}^\infty \phi_{-j}(r) e^{-ij\theta}$ is a function in L^∞, there does not exist any finite rank $H_\phi^{\mathcal{M}}$ except $H_\phi^{\mathcal{M}} = 0$ if and only if $\beta = 0$.

(2) When $\phi = \sum_{j=0}^\infty a_j z^j + \sum_{j=1}^\infty a_{-j}\bar{z}^j$ is a function in L^∞, there does not exist any finite rank $H_\phi^{\mathcal{M}}$ except $H_\phi^{\mathcal{M}} = 0$ if and only if $\beta = 0$.

Proof. Recall that $T_\beta = \sum_{j=0}^\infty \oplus (T_\beta)_j e^{ij\theta}$ and $(T_\beta)_j = \operatorname{span}\{r^j p_j(r^2);\ p_j$ is a polynomial of degree at most $\beta j/1-\beta\}$.

(1) If $\beta = 0$, then $(T_\beta)_j \cap r^{j+1}\mathcal{L}^2 = \{0\}$ for any $j \ge 0$ and if $\beta \ne 0$, then $(T_\beta)_j \cap r^{j+1}\mathcal{L}^\infty \ne \{0\}$ for enough large j. Theorem 5.4 implies (1).

(2) If $\beta \ne 0$, then there exists n such that $1-\beta \le \beta(n-1)$. Hence $(T_\beta)_{n-1} \ni r^{n+1}$. Suppose $\phi = \bar{z}$, then $z^n\phi = r^{n+1} e^{i(n-1)\theta}$ and so $z^n\phi \in (T_\beta)_{n-1} e^{i(n-1)\theta} \subset T_\beta$. By Theorem 3.1, $H_\phi^{\mathcal{M}}$ is of rank $\le n$ and $H_\phi^{\mathcal{M}} \ne 0$. □

(ii) Suppose $\mathcal{M} = \bar{E}^m$ $(0 \le m < \infty)$.

(1) When $\phi = \sum_{j=1}^\infty \phi_{-j}(r) e^{-ij\theta}$, there does not exist any finite rank $H_\phi^{\mathcal{M}}$ except $H_\phi^{\mathcal{M}} = 0$ if and only if $m = 0$.

(2) When $\phi = \sum_{j=0}^\infty a_j z^j + \sum_{j=1}^\infty a_{-j}\bar{z}^j$ is a function in L^∞, there does not exist any finite rank $H_\phi^{\mathcal{M}}$ except $H_\phi^{\mathcal{M}} = 0$ if and only if $m = 0$ or 1.

Proof. We recall that $(\bar{E})^m = \sum_{j=0}^\infty \oplus (\bar{E}^m)_j e^{ij\theta}$ and $(\bar{E}^m)_j = \operatorname{span}\{r^j, \dots, r^{j+2m}\}$.

(1) If $m = 0$, then $(\bar{E}^m)_j \cap r^{j+1}\mathcal{L}^2 = \{0\}$ for any $j \ge 0$ and if $m \ne 0$, then $(\bar{E}^m)_j \cap r^{j+1}\mathcal{L}^\infty \ne \{0\}$ for any $j \ge 0$. Theorem 5.4 implies (1).

(2) If $m = 0$, by (1) there does not exist any finite rank $H_\phi^{\mathcal{M}}$ except $H_\phi^{\mathcal{M}} = 0$. If $m = 1$, then $(E^m)_n = \operatorname{span}\{r^n, r^{n+2}\}$ for $n \ge 0$. When H_ϕ^M is of finite rank ℓ, by Corollary 5.1, $a_{-n} = 0$ for $n > \ell$ and if $0 \le n \le \ell$,

$$\sum_{j=n}^{\ell} b_j a_{n-j} r^{2j-n} = c r^n + d r^{n+2} \tag{5.7}$$

for complex constants c, d. Hence, for $0 \le n \le \ell$,

$$b_j a_{n-j} = 0 \quad \text{for } n+2 \le j \le \ell. \tag{5.8}$$

Since $b_\ell = 1$, $a_{n-\ell} = 0$ for $0 \le n \le \ell$ and so $a_{-j} = 0$ for $0 \le j \le \ell$. When $m \ge 2$, if $\phi = \bar{z}$, then $z\phi = r^2 \in (\bar{E}^m)_0 = \operatorname{span}\{1, r^2, \dots, r^{2m}\}$ and $z\phi \in \bar{E}^m$ because $(\bar{E}^m)_0 \subset \bar{E}^m$. However $H_\phi^{\mathcal{M}} \ne 0$. □

(iii) Suppose $\mathcal{M} = Y^k$.

(1) When $\phi = \sum_{j=1}^{\infty} \phi_{-j}(r) e^{-ij\theta}$, there does not exist any finite rank $H_\phi^{\mathcal{M}}$ except $H_\phi^{\mathcal{M}} = 0$ if and only if $k = 0$.

(2) When $\phi = \phi_+ + \phi_- = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=1}^{\infty} a_{-j} \bar{z}^j$ and ϕ_+ are functions in L^∞, there does not exist any finite rank $H_\phi^{\mathcal{M}}$ except $H_\phi^{\mathcal{M}} = 0$ if and only if $k = 0$.

Proof. Since $H_\phi^{\mathcal{M}} = H_{\phi_-}^{\mathcal{M}}$, it is sufficient to prove (1). We recall that $Y^k = \sum_{j=-k}^{\infty} \oplus (Y^k)_j e^{ij\theta}$, where $Y_0^k = \operatorname{span}\{1, r^2, \dots, r^{2k}\}$ and $(Y^k)_j = r^j (Y^k)_0$ for $j \ge 0$, and $(Y^k)_{-j} = \operatorname{span}\{r^{2\ell - j},\ j \le \ell \le k\}$ for $1 \le j \le k$. If $k = 0$, then $Y^k = L_a^2$. If $k \ge 1$, $(Y^k)_{-k} = \operatorname{span}\{r^k\}$. Theorem 5.4(2) implies that there exists a nonzero finite rank $H_\phi^{\mathcal{M}}$. □

6. Small Hankel operator and $\mathcal{M} \supseteq \mathbf{H}^2$. In this section, we assume that $d\mu = d\sigma(r)\, d\theta$ and $d\sigma([t,1)) > 0$ for any $t > 0$. Hence we can define the Fourier coefficients $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$ of $\mathcal{M}$. In this case, $H_\phi^{\mathcal{M}}$ is close to H_ϕ^{small} and far from H_ϕ^{big}. Note that if $\mathcal{M}'$ is an invariant subspace and $\mathcal{M}' \subseteq e^{i\theta}\mathbf{H}^2$, then $\mathcal{M} = (\bar{\mathcal{M}}')^\perp$ is an invariant subspace and $\mathcal{M} \supseteq e^{i\theta}\mathbf{H}^2$.

Proposition 6.1. *Suppose $\mathcal{M}$ is an invariant subspace which contains $e^{ik\theta}\mathbf{H}^2$ for some nonnegative integer k. If $\mathcal{M} \ne L^2$, there exists at least a nonzero finite rank $H_\phi^{\mathcal{M}}$.*

Proof. If $\bar{z}^n \in \mathcal{M}$ for all $n \ge 1$, then $z^\ell \bar{z}^n \in \mathcal{M}$ for all $\ell \ge 1$ because $z\mathcal{M} \subseteq \mathcal{M}$. Let $\mathscr{E}$ be the closed linear span of $\{z^\ell \bar{z}^n;\ n \ge 1,\ \ell \ge 0\}$, then $\mathscr{E} \subseteq \mathcal{M}$ and $g\mathscr{E} \subseteq \mathscr{E}$ for arbitrary polynomial g of z and $\bar{z}$. It is well known that $\mathscr{E} = L^2$. This contradiction implies that there exists at least n such that $\bar{z}^n \notin \mathcal{M}$ and $n \ge 1$. If $\phi = \bar{z}^n$, then $z^{n+k}\phi \in \mathcal{M}$. Then $H_\phi^{\mathcal{M}} \ne 0$ but $H_\phi^{\mathcal{M}}$ is of finite rank $\le n+k$, by Theorem 3.1. □

Proposition 6.2. *Suppose $\mathcal{M}$ is an invariant subspace which contains $e^{ik\theta}\mathbf{H}^2$ for some nonnegative integer k. The following statements are valid.*

(1) If $\phi = \sum_{j=-\infty}^{\infty} \phi_j(r) e^{ij\theta}$ is a function in L^∞, then there exists a function ϕ' in L^2 such that $\phi' = \sum_{j=0}^{k-1} \phi_j(r) e^{ij\theta} + \sum_{j=1}^{\infty} \phi_{-j}(r) e^{-ij\theta}$ and $H_{\phi'}^{\mathcal{M}} = H_\phi^{\mathcal{M}}$.

(2) If $\phi = \sum_{j=k}^{\infty} \phi_j(r) e^{ij\theta}$ is a function in L^∞, then $H_\phi^{\mathcal{M}} = 0$.

(3) If $\phi = \sum_{j=-\ell}^{\infty} \phi_j(r) e^{ij\theta}$ is a function in L^∞, then $H_\phi^{\mathcal{M}}$ is of rank $\le \ell + k < \infty$. Conversely, if one of (1) or (2) is valid, then $\mathcal{M}$ contains $e^{ik\theta}\mathbf{H}^2$.

Proof. Both (1) and (2) are clear because $\mathcal{M} \supseteq e^{ik\theta}\mathbf{H}^2$. (3) is a result of Theorem 3.1. The converse is also clear. □

We will consider Example 2.4 in Section 2.

Example 6.3. (ii) Suppose $\mathcal{M} = (E^k)^\perp (0 \le k < \infty)$ and $\phi = \sum_{j=-\infty}^{\infty} \phi_j(r) e^{ij\theta}$ is a function in L^∞.

(1) $H_\phi^{\mathcal{M}} = 0$ if and only if

$$\int_0^1 \phi_{-j}(r) r^{j+2t} d\sigma = 0 \quad (j \geq 0,\ 0 \leq t \leq k). \tag{6.1}$$

(2) $H_\phi^{\mathcal{M}}$ is of rank ≤ 1 if and only if there exist complex numbers $(b_0, b_1) \neq (0,0)$ such that

$$b_0 \int_0^1 \phi_{-j}(r) r^{j+2t} d\sigma = -b_1 \int_0^1 \phi_{-j-1}(r) r^{j+2t+1} d\sigma \tag{6.2}$$

for $j \geq 0,\ 0 \leq t \leq k$.

(3) Suppose $d\sigma = r\,dr/2\pi$. When $\phi = \sum_{j=0}^\infty a_j z^j + \sum_{j=1}^\infty a_{-j} \bar{z}^j$, if $H_\phi^{\mathcal{M}}$ is of rank ≤ 1, then $H_\phi^{\mathcal{M}} = 0$.

Proof. From the remark in the last part of Section 4, (1) and (2) follows. (3) By (2), $H_\phi^{\mathcal{M}}$ is of rank ≤ 1 if and only if there exist complex numbers $(b_0, b_1) \neq (0,0)$ such that

$$b_0 a_{-j} \frac{1}{2j+2t+1} = -b_1 a_{-j-1} \frac{1}{2j+2t+3} \tag{6.3}$$

for $j \geq 0,\ 0 \leq t \leq k$. When $k \neq 0$, for each j, as $t = 0$,

$$\begin{aligned} b_0 a_{-j} \frac{1}{2j+1} &= -b_1 a_{-j-1} \frac{1}{2j+3}, \\ b_0 a_{-j} \frac{1}{2j+3} &= -b_1 a_{-j-1} \frac{1}{2j+5}. \end{aligned} \tag{6.4}$$

This implies that $a_{-j} = a_{-j-1} = 0$, for $j \geq 0$, and so $\phi = \sum_{j=1}^\infty a_j z^j$. When $k = 0$, Corollary 3.3 implies (3) □

(iv) Suppose $\mathcal{M} = \bar{q}\mathbf{H}^2$ for some unimodular function q in $\mathbf{H}^2$ and ϕ is a function in L^∞. $H_\phi^{\mathcal{M}}$ is of finite rank ℓ if and only if

$$\phi = \bar{q} \sum_{j=-\ell}^{\infty} \psi_j(r) e^{ij\theta}, \tag{6.5}$$

where $\psi_{-\ell}(r) \neq 0$.

Proof. If $\phi = \bar{q} \sum_{j=-\ell}^\infty \psi_j(r) e^{ij\theta}$, then $z^\ell \phi \in \mathcal{M}$ and so, by Theorem 3.1, $H_\phi^{\mathcal{M}}$ is of finite rank $\leq \ell$. Since $\psi_{-\ell}(r) \neq 0$, $b\phi \notin \mathcal{M}$ for any polynomial b of degree $\leq \ell - 1$ and so $H_\phi^{\mathcal{M}}$ is of finite rank ℓ. The converse is clear. □

(v) Suppose $\mathcal{M} = \mathbf{H}^2 \oplus S e^{-i\theta}$ and S is a closed subspace in $\mathcal{L}^2$. Let $\phi = \sum_{j=-\infty}^\infty \phi_j(r) e^{ij\theta}$ be a function in L^∞. By Theorems 3.1 and 4.1, $H_\phi^{\mathcal{M}}$ is of finite rank $\leq \ell$ if and only if $\phi_j(r) = 0$ for $j \leq -(\ell+2)$ and there exist complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$,

$$\begin{gathered} \sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = 0 \quad \text{for } -(\ell+1) \leq n < -1, \\ \sum_{j=0}^{\ell} b_j r^j \phi_{-1-j}(r) \in S. \end{gathered} \tag{6.6}$$

7. Restricted shift operator and $\mathcal{M} \subseteq L_a^2$. In this section, we assume $\mu = r\,dr\,d\theta/\pi$ for simplicity. Let $\mathcal{M}$ be an invariant subspace in L_a^2 and $\mathcal{K} = L_a^2 \ominus \mathcal{M}$. For ϕ in $L_a^\infty = L_a^2 \cap L^\infty$,

$$S_\phi^{\mathcal{K}} f = (I - P^{\mathcal{K}})(\phi f) \quad (f \in \mathcal{K}), \tag{7.1}$$

where $P^{\mathcal{K}}$ is the orthogonal projection from L_a^2 to $\mathcal{K}$. $S_z^{\mathcal{K}}$ is called a restricted shift operator. For any ϕ in L_a^∞, $S_\phi^{\mathcal{K}}$ commutes with $S_z^{\mathcal{K}}$. We do not know whether if the bounded linear operator T on $\mathcal{K}$ commutes with $S_z^{\mathcal{K}}$, then $T = S_\phi^{\mathcal{K}}$ for some ϕ in L_a^∞. If $TS_z^{\mathcal{K}} = S_z^{\mathcal{K}} T$ and $\phi = TP^{\mathcal{K}}1$ is bounded, then it is easy to see that $T = S_\phi^{\mathcal{K}}$ (cf. [5, page 784]). In the Hardy space instead of the Bergman space, Sarason [8] showed that this is true without any condition and $\|T\| = \|\phi\|_\infty$.

We can define the Hankel operator $H_\phi^{\mathcal{M}}$ as in the introduction. However $H_\phi^{\mathcal{M}}$ is not an intermediate Hankel operator. It is not so difficult to see the following: when $\mathcal{K} = L_a^2 \ominus \mathcal{M}$ and ϕ in L_a^∞,

$$\|H_\phi^{\mathcal{M}}\| = \|S_\phi^{\mathcal{K}}\|. \tag{7.2}$$

This is known for the Hardy space. In fact, for f in L_a^2,

$$H_\phi^{\mathcal{M}} f = (I - P^{\mathcal{M}})\phi f = P^{\mathcal{K}} \phi P^{\mathcal{K}} f \tag{7.3}$$

and so $H_\phi^{\mathcal{M}} f = S_\phi^{\mathcal{K}} P^{\mathcal{K}} f$ for f in L_a^2. Hence $H_\phi^{\mathcal{M}}$ is of finite rank n if and only if $S_\phi^{\mathcal{K}}$ is of finite rank n. It is easy to see that $S_\phi^{\mathcal{K}}$ is of finite rank $\ell \le n$ if and only if there exists an analytic polynomial p of degree $\ell \le n$ such that $p(\phi) \in \mathcal{M}^\infty$. When ϕ is in L^∞, Theorems 3.1 and 4.1 are true for $H_\phi^{\mathcal{M}}$.

Suppose ϕ is a function in L_a^∞.

(1) $L_a^2 \supseteq \ker H_\phi^{\mathcal{M}} \supseteq \mathcal{M}$.

(2) When the common zero set $Z(\mathcal{M})$ of $\mathcal{M}$ in D is empty, if $H_\phi^{\mathcal{M}}$ is of finite rank then $H_\phi^{\mathcal{M}} = 0$. This is a result of (1) and Proposition 2.1.

(3) If $Z(\mathcal{M})$ is not empty, there exists a nonzero finite rank $H_\phi^{\mathcal{M}}$.

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