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TAUTNESS AND APPLICATIONS OF THE ALEXANDER-SPANIER COHOMOLOGY OF K-TYPES

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Abstract. The aim of the present work is centered around the tautness property for the two K-types of Alexander-Spanier cohomology given by the authors. A version of the continuity property is proved, and some applications are given.

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1. Introduction. It is well known that in the Alexander-Spanier cohomology theory [17, 18] or in the isomorphic theory of Čech [9], if the coefficient group G is topological then either the theory does not take into account the topology on G [9, 18], or considers only the case when G is compact to obtain a compact cohomology [4, 1]. Continuous cohomology naturally arises when the coefficient group of a cohomology theory is topological [2, 3, 11]. The partially continuous Alexander-Spanier cohomology theory [14] can be considered as a variant of the continuous cohomology of a space with two topologies in the sense of Bott-Haefliger [15]; also it is isomorphic to the continuous cohomology of a simplicial space defined by Brown-Szczarba [2].

The idea of K-groups [5, 6], where K is a locally-finite simplicial complex, is used to introduce the K-types of Alexander-Spanier cohomology with coefficients in a pair (G,G') of topological abelian groups [7, 8]; namely, K-Alexander-Spanier and partially continuous K-Alexander-Spanier cohomologies $\bar{H}^*_K, \tilde{H}^*_K$. It is proved that these K-types satisfied the seven Eilenberg-Steenrod axioms [9]; the excision axiom for the second K-type is verified for compact Hausdorff spaces when (G,G') are absolutely retract. Therefore, the uniqueness theorem of the cohomology theory on the category of compact polyhedral pairs [9] asserts that our Alexander-Spanier K-types over a pair of absolute retract coefficient abelian groups are naturally isomorphic.

In the present work, we prove that the K-Alexander-Spanier cohomology of a closed subset in a paracompact space is isomorphic to the direct limit of the K-Alexander-Spanier cohomology of its neighborhoods, and that the partially continuous K-Alexander-Spanier cohomology of a neighborhood retract closed subspace of a Hausdorff space is isomorphic to the direct limit of the partially continuous K-Alexander-Spanier cohomology of its neighborhoods. Also a version of the continuity property is proved. Moreover, we study some applications of the K-type cohomologies.

2. Alexander-Spanier cohomology of K-types. Here we mention the notations which we used throughout [7, 8].

For an object (X,A) of the category Q of the pairs of topological spaces and their continuous maps, denote by $\Omega(X,A)[\tilde{\Omega}(X,A)]$ the set of the pairs $\bar{\alpha} = (\alpha,\alpha')$, where

α is an open covering of X and α' is a subcollection of α covering $A[\alpha' = \alpha \cap A]$; it is directed with respect to the refinement relation $\bar{\alpha} < \bar{\beta}$, that is, $\alpha < \beta$ and $\alpha' < \beta'$ [9]. Denote by $C^{q(\tau)}(\tilde{X})$ the group of functions $\varphi^{\tau} : \tilde{X}^{q(\tau)+1} \to G$, where τ is a simplex in K, $q(\tau) = q + \dim\tau$, $q \geq 0$, and $\tilde{X}$ denotes either a space X or $\alpha \in \Omega(X)$. Let $C^{q(\tau)}(\tilde{X})$ be the subgroup of the direct product $\prod_{\tau\in K} C^{q(\tau)}(\tilde{X})$ consisting of such $\varphi = \{\varphi^{\tau}\}$ for which the condition (k) is satisfied, which states that there is a cofinite subset $\check{\tau}(\varphi)$ of K, that is, $K - \check{\tau}(\varphi)$ is finite such that $(\varphi^{\tau})^{-1}(G') = \tilde{X}^{q(\tau)+1}$, $\forall \tau \in \check{\tau}(\varphi)$. The coboundary $\delta^q : C^q(\tilde{X}) \to C^{q+1}(\tilde{X})$ is given by

$$(\delta^q\varphi)^{\tau} = \sum_{i=1}^{q(\tau)+1} (-1)^i \varphi^{\tau} p_i^{(q(\tau)+1)} + (-1)^{q(\tau)+1} \sum_{\sigma\in \mathrm{st}(\tau)} [\sigma : \tau]\varphi^{\sigma}, \tag{2.1}$$

where $\mathrm{st}(\tau) = \{\sigma \in K : \tau \text{ is } (\dim\sigma - 1)\text{-face of } \sigma\}$, $p_i^{(\tau)} : X^{\tau+1} \to X^{\tau}$ is the projection defined by: if $\hat{t}_i$ is the τ-tuple consisting of $t = (x_0,\dots,x_{\tau}) \in X^{\tau+1}$ with x_i omitted, then $p_1^{(\tau)}(t) = \hat{t}_i$, $0 \leq i \leq \tau$. The cohomology groups of the cochain complex $C^{\neq}(X) = \{C^q(X), \delta^q\}$ is, in general, uninteresting, as shown in the following theorem [8].

Theorem 2.1. *If $\dim K = 0$, then $H^q(C^{\neq}(X)) \cong G^{*K}$ (the subgroup of $G^K = \prod_{\tau\in K} G^{\tau}$, $G^{\tau} = G$, consisting of those elements having all but a finite number of their τ-coordinates in G'), and $H^q(C^{\neq}(X)) = 0$, when $q \neq 0$.*

To pass to more interesting cohomology groups, the topology of the space X will be used to define that $\varphi \in C^q(X)$ is said to be K-locally zero on $M \subseteq X$ if there is $\alpha \in \Omega_X(M)$ (the set of external covering of M by open subsets of X) such that φ vanishes on $\alpha \cap M$, that is, each φ^{τ} vanishes on $(\alpha\cap M)^{q(\tau)+1}$, where $\alpha^{\tau} = \cup\{u_{\alpha}^{\tau} : u_{\alpha} \in \alpha\}$. The subgroups of $C^q(X)$ consisting of those elements which are K-locally zero on X, A, respectively, are denoted by $C_0^q(X), C^q(X,A)$. The K-Alexander-Spanier cohomology of (X,A) over (G,G'), denoted by $\bar{H}_K^*(X,A)$, is the cohomology of the quotient cochain complex $\bar{C}_K^{\neq}(X,A) = C^{\neq}(X,A)/C_0^{\neq}(X)$. If $f : (X,A) \to (Y,B)$ is in Q, $\bar{\beta} \in \Omega(Y,B)$ and $\bar{\alpha} = f^{-1}(\bar{\beta})$, then f defines a cochain map $\bar{f}^{\neq} : \bar{C}_K^{\neq}(Y,B) \to \bar{C}_K^{\neq}(X,A)$, where $\check{\tau}(f^q\varphi) = \check{\tau}(\varphi)$ for each $\varphi \in C^q(Y)$. In turn, $\bar{f}^{\neq}$ induces the homomorphism $\bar{f}^* : \bar{H}_K^*(Y,B) \to \bar{H}_K^*(X,A)$.

On the other hand, for $\bar{\alpha} \in \Omega(X,A)$, denote by $C_{\bar{\alpha}}^q$ the subgroup of $C_{\alpha}^q = C^q(\alpha)$ consisting of those φ that vanish on $\alpha' \cap A$. Then we obtain a direct system $\{C_{\bar{\alpha}}^{\neq}\}_{\Omega(X,A)}$ such that any map $f \in Q$ constitutes a map $F : \{C_{\bar{\beta}}^{\neq}\}_{\Omega(Y,B)} \to \{C_{\bar{\alpha}}^{\neq}\}_{\Omega(X,A)}$ [9]; its limit is F^{∞}.

Theorem 2.2. *The K-Alexander-Spanier cohomology functor $\{\bar{H}_K^*, \bar{f}^*\}$ is naturally isomorphic to the functor $\{\varinjlim\{H^*(C_{\bar{\alpha}}^{\neq})\}_{\Omega(X,A)}, F^{\infty *}\}$ [7].*

In the previous part, the topology on (G,G) plays no role; to pass to the second cohomology of K-type we characterize an element $\varphi \in C^q(X)$ to be K-partially continuous if it is continuous on some $\alpha \in \Omega(X)$, that is, $\varphi^{\tau} \mid \alpha^{q(\tau)+1}$ are continuous functions. Let $L^q(X)$ be the group of all such elements, and $M_K^{\neq}(X) = L^{\neq}(X)/C_0^{\neq}(X)$. The subgroup of C_{α}^q, where $\alpha \in \Omega(X)$, consisting of the K-continuous elements φ, that is, φ^{τ} are continuous, is denoted by M_{α}^q. Let $i : A \hookrightarrow X$, define $M_K^{\neq}(X,A)$ to be the mapping cone of $i^{\neq} : M_K^{\neq}(X) \to M_K^{\neq}(A)$, (see [13, 18]), assuming that $M_K^q(X,A) = M_K^q(X) \oplus M_K^{q-1}(A)$, and the coboundary is $\triangle^q(\varphi,\psi) = (-\delta^q\varphi, i^q\varphi + \delta^{q-1}\psi)$. The cohomology of $M_K^{\neq}(X,A)$ is the partially continuous K-Alexander-Spanier cohomology of (X,A) over the topological pair (G,G') of coefficient groups; it is denoted by $\tilde{H}_K^*(X,A)$.

On the other hand, if $\bar{\alpha} \in \tilde{\Omega}(X,A)$, then i defines a cochain map $i_{\alpha}^{\neq} : M_{\alpha}^{\neq} \to M_{\alpha'}^{\neq}$; its mapping cone is denoted by $M_{\bar{\alpha}}^{\neq}$.

Theorem 2.3. *For a pair $(X,A) \in Q$ with A closed, $M_K^{\neq}(X,A)$ is naturally isomorphic to $\varinjlim\{M_{\bar{\alpha}}^{\neq}\}_{\tilde{\Omega}(X,A)}$* [7].

Theorem 2.4. *For a discrete space, and $q \geq 0$, $\tilde{H}_K^q(X) \cong \bar{H}_K^q(X)$.*

Proof. Since $X^{q(\tau)+1}$ admits a discrete topology, it follows that each τ-coordinate φ^τ of $\varphi \in C_K^q(X)$ is continuous [16]. Then φ is K-partially continuous with respect to any $\alpha \in \Omega(X)$. Therefore, $L^q(X) = C_K^q(X)$ and $M_K^{\neq}(X) = \bar{C}_K^{\neq}(X)$. $\square$

3. Tautness and continuity properties.

This article is devoted to study the tautness property for both Alexander-Spanier cohomology of K-types. One of its applications is the continuity property.

The star of a subset A in a space X with respect to $\alpha \in \Omega(X)$ is

$$\mathrm{st}(A,\alpha) = \cup\{U_\alpha \in \alpha : U_d \cap A \neq \varnothing\}. \tag{3.1}$$

The star of α is

$$\alpha^* = \{\mathrm{st}(U_\alpha,\alpha) : u_\alpha \in \alpha\}. \tag{3.2}$$

Definition 3.1. Let $\alpha,\beta \in \Omega(X)$, then β is a star-refinement of α, written $\alpha <^* \beta$ if $\alpha < \beta^*$.

Denote by $\mathcal{N}(A)$ the collections of neighborhoods $\{N\}$ of A in X; it is directed downward by inclusion. If $N_1 < N_2$, then the inclusion $\pi_{N_1N_2} : N_2 \hookrightarrow N_1$ induces the homomorphisms $\bar{\pi}_{N_1N_2}^* : \bar{H}_K^q(N_1) \to \bar{H}_K^q(N_2)$. Also $i_N : A \hookrightarrow N$ induces $\bar{i}_N^* : \bar{H}_K^q(N) \to \bar{H}_K^q(A)$, and they define a homomorphism

$$I^\infty : \varinjlim\{\bar{H}_K^q(N), \bar{\pi}_{N_1N_2}^*\}_{\mathcal{N}(A)} \longrightarrow \bar{H}_K^q(A). \tag{3.3}$$

Theorem 3.2 (Tautness). *A closed subspace of a paracompact space is a taut subspace relative to the K-Alexander-Spanier cohomology, that is, I^∞ is an isomorphism for each q and any pair (G,G') of coefficient groups.*

Proof. (1) I^∞ is an epimorphism. Let $h \in \bar{H}_K^q(A)$ with representative $\bar{\varphi} \in \bar{C}_K^q(A)$, written as $h = [\bar{\varphi}]$. Let $\varphi \in C^q(A)$ such that $\varphi \in \bar{\varphi}$. Then there is $\alpha = \{u_\alpha = v_\alpha \cap A : v_\alpha \subseteq X \text{ is open}\} \in \Omega(A)$ such that

$$(\delta^q\varphi) \mid \alpha^{q(\tau)+2} = 0. \tag{3.4}$$

Since A is closed, it follows that $\beta = \{v_\alpha\} \cup \{X - A\} \in \Omega(X)$. The paracompactness of X is equivalent to the existence of such $\gamma \in \Omega(X)$ that $\beta <^* \gamma$, and a neighborhood N of A and an extension $f : N \to A$ (not necessarily continuous) of the identity map id_A of A, that is, $f i_N = \mathrm{id}_A$, such that $f(u_\gamma \cap N) \subseteq \mathrm{st}(u_\gamma,\gamma)$ for each $u_\gamma \in \gamma$ [18]. One can show that f defines a cochain map $f^{\neq} : C^{\neq}(A) \to C^{\neq}(N)$ by $(f^q\varphi)^\tau = \varphi^\tau f^{(q(\tau)+1)}$ with

$\check{\tau}(f^q\varphi) = \check{\tau}(\varphi)$, where $f^{(\tau)} : N^\tau \to A^\tau$ given by $f(x_0,\dots,x_{\tau-1}) = (f(x_0),\dots,f(x_{\tau-1}))$. The relation $\beta < \gamma^*$ yields that for each $u_\gamma \in \gamma$ there is $u_\beta \in \beta$ such that $f(u_\gamma \cap N) \subseteq \operatorname{st}(u_\gamma,\gamma) \subseteq u_\beta$. Because $f(N) = A$, then $f(u_\gamma \cap N) \subseteq u_\beta \cap A \subseteq u_\alpha$ for some $u_\alpha \in \alpha$. By using (3.4), we get $(\delta^q f^q\varphi)^\tau \mid (\gamma\cap N)^{q(\tau)+2} = 0$, that is, $\delta^q(f^q\varphi) \in C_0^{q+1}(N)$. Then $f^q\varphi$ represents a cocycle $\overline{f^q\varphi} \in \bar{C}_K^q(N)$ which, in turn, defines $h_N \in \bar{H}_K^q(N)$, that is, $h_N = [\overline{f^q\varphi}]$. Let $t \in A^{q(\tau)+1}$, then

$$\left(i_N^q(f^q\varphi)\right)^\tau(t) = \varphi^\tau f^{(q(\tau)+1)} i_N^{(q(\tau)+1)}(t) = \varphi^\tau(t), \tag{3.5}$$

and therefore, $\bar{i}_N^* h_N = [\overline{(f i_N)^q\varphi}] = [\bar{\varphi}] = h$.

(2) I^∞ is a monomorphism. Let $h_1 \in \bar{H}_K^q(N_1)$, $\bar{\varphi}_1 \in \bar{C}_K^q(N_1)$ and $\varphi_1 \in C^q(N_1)$ such that $\varphi_1 \in \bar{\varphi}_1$, $\bar{\varphi}_1 \in h_1$, and $[h_1] \in \operatorname{Ker} I^\infty$.

First, one can consider that the neighborhood N_1 of A is a paracompact subset of X. For, if N_1 is not so, then there is a paracompact subset M_1 of X such that $M_1 < N_1$ (e.g., take $M_1 = X$) [10]. The inclusion $\pi_{M_1N_1}$ induces an epimorphism $\bar{\pi}^{\#}_{M_1N_1}$ [8], let $\bar{\pi}^q_{M_1N_1}\bar{\psi}_1 = \bar{\varphi}_1$. Thus the cohomology class of $\bar{H}_K^q(M_1)$ represented by $\bar{\psi}_1$ is $[h_1]$, which shows that N_1 can be taken paracompact.

Now, $\bar{\varphi}_1 \in \operatorname{Ker}\delta^q$, or equivalently, there is $\alpha = \{u_\alpha = v_\alpha \cap N_1 : v_\alpha \subseteq X \text{ is open}\} \in \Omega(N_1)$ such that

$$\left(\delta^q\varphi_1\right)^\tau \mid \alpha^{q(\tau)+2} = 0. \tag{3.6}$$

On the other hand, the assumption $\bar{i}_{N_1}^* h_1 = 0$ asserts that there exists $\bar{\varphi} \in \bar{C}_K^{q-1}(A)$ such that $i_{N_1}^q\varphi_1 - \delta^{q-1}\varphi \in C_0^q(A)$, where $\varphi \in \bar{\varphi}$. This means that there is $\beta = \{u_\beta = \omega_\beta \cap A : \omega_\beta \subseteq X \text{ is open}\} \in \Omega(A)$ such that

$$\left(i_{N_1}^q\varphi_1\right)^\tau = \left(\delta^{q-1}\varphi\right)^\tau \quad \text{on } \beta^{q(\tau)+1}. \tag{3.7}$$

Assume that $\beta_1 = \{u_{\beta_1} = \omega_\beta \cap N_1\} \cup \{N_1 - A\}$. The paracompactness of N_1 asserts the existence of $\gamma_1,\gamma_2 \in \Omega(N_1)$ for which $\alpha <^* \gamma_1$ and $\beta_1 <^* \gamma_2$. The directedness of $\Omega(N_1)$ implies that there is $\gamma \in \Omega(N_1)$ for which $\gamma_1,\gamma_2 < \gamma$; and so for each $u_\gamma \in \gamma$ there are $u_{\gamma_i} \in \gamma_i$, $i = 1,2$ and $u_\alpha \in \alpha$, $u_{\beta_1} \in \beta_1$ such that

$$u_\gamma \subset u_{\gamma_i} \subseteq \operatorname{st}\left(u_{\gamma_i},\gamma_i\right) \subseteq u_\alpha \cap u_{\beta_1}. \tag{3.8}$$

Then

$$\operatorname{st}\left(u_\gamma,\gamma\right) \subseteq u_\alpha \cap u_{\beta_1}, \tag{3.9}$$

that is, $\alpha,\beta_1 <^* \gamma$. According to [18, Lemma 6.6.1], there is a neighborhood N_2 of N_1 and $f : N_2 \to A$ (not necessarily continuous) such that $f i_{N_2} = \mathrm{id}_A$, and $u_{\beta_1} \in \beta_1$ such that

$$f\left(u_\gamma \cap N_2\right) \subseteq \operatorname{st}\left(u_\gamma,\gamma\right) \subseteq u_{\beta_1} \subseteq u_{\beta_1} \cap A = u_\beta. \tag{3.10}$$

Then, by (3.7), we get

$$\left(\delta^{q-1} f^{q-1}\varphi\right)^\tau = \left(f^q i_{N_1}^q\varphi_1\right) \quad \text{on } \left(\gamma \cap N_2\right)^{q(\tau)+1}. \tag{3.11}$$

Define $D^q : C^{q+1}(N_1) \to C^q(N_2)$ by

$$\text{if } t = \left(x_0,\dots,x_{q(\tau)}\right) \in N_2^{q(\tau)+1} \text{ and } \psi_1 \in C^{q+1}\left(N_1\right) \tag{3.12}$$

then

$$(D^q\psi_1)^\tau(t) = \sum_{r=0}^{q(\tau)} (-1)^\gamma \psi_1^\tau(y_0,\dots,y_\tau,z_\tau,\dots,z_{q(\tau)}), \tag{3.13}$$

where

$$y_j = \pi_{N_1N_2}(x_j), \qquad z_j = (i_{N_1}f)(x_j) = f(x_j), \tag{3.14}$$

and $\check{\tau}(D^q\psi_1) = \check{\tau}(\psi_1)$. By a similar calculation as given in [7], we get

$$(\delta^{q-1}D^{q-1}\varphi_1)^\tau = (f^q i_{N_1}^q\varphi_1)^\tau - (\pi_{N_1N_2}^q\varphi_1)^\tau - (D^q\delta^q\varphi_1)^\tau. \tag{3.15}$$

By (3.9), (3.10) for each $u_\gamma \in \gamma$, there is $u_\alpha \in \alpha$ such that

$$(u_\gamma \cap N_2) \cup f(u_\gamma \cap N_2) \subseteq u_\alpha. \tag{3.16}$$

Then, by (3.6), (3.11), and (3.15) consequently, we have

$$(\delta^{q-1}D^{q-1}\varphi_1)^\tau = (f^q i_{N_1}^q\varphi_1)^\tau - (\pi_{N_1N_2}^q\varphi_1)^\tau \quad \text{on } (\gamma\cap N_2)^{q(\tau)+1}, \tag{3.17}$$

and so

$$(\pi_{N_1N_2}^q\varphi_1)^\tau = (\delta^{q-1}(f^{q-1}\varphi - D^{q-1}\varphi_1))^\tau \quad \text{on } (\gamma\cap N_2)^{q(\tau)+1}. \tag{3.18}$$

Therefore

$$\psi_2 = f^{q-1}\varphi - D^{q-1}\varphi_1 \in C^{q-1}(N_2) \tag{3.19}$$

such that

$$(\pi_{N_1N_2}^q\varphi_1)^\tau = (\delta^{q-1}\psi_2)^\tau \quad \text{on } (\gamma\cap N_2)^{q(\tau)+1}, \tag{3.20}$$

that is, $\bar{\pi}_{N_1N_2}h_1 = 0$ which completes the proof. □

Corollary 3.3. *Any one-point subset of a paracompact is a taut subspace relative to $\bar{H}_K^*$.*

The next part is devoted to studying the tautness property for $\tilde{H}_K^*$, which is also valid for $\bar{H}_K^*$. The idea and results of α-β-contiguous maps, introduced in [7] plays an essential role in this study.

The inclusions $\pi_{N_1N_2}: N_2 \hookrightarrow N_1$, corresponding to the relations $N_1 < N_2$ in $\mathcal{N}(A)$, define the direct system $\{\tilde{H}_K^q(N), \tilde{\pi}_{N_1N_2}^*\}$. Also the inclusion $i_N : A \hookrightarrow N$, where $N \in \mathcal{N}(A)$, defines a map of direct systems [9]:

$$I_N : \{H^q(M_\alpha^{\mp}), \tilde{\pi}_{\alpha\beta}^*\}_{\Omega(N)} \longrightarrow \{H^q(M_{\tilde{\alpha}}^{\mp}), \tilde{\pi}_{\tilde{\alpha}\tilde{\beta}}^*\}_{\Omega(A)}, \tag{3.21}$$

where $\alpha \in \Omega(N)$, $\tilde{\alpha} = i_N^{-1}(\alpha) = \alpha \cap A$. On the other hand, $\{\tilde{i}_N^*\}$ defines a homomorphism

$$\tilde{I}^\infty : \varinjlim\{\tilde{H}_K^q(N), \tilde{\pi}_{N_1N_2}^*\}_{\mathcal{N}(A)} \longrightarrow \tilde{H}_K^q A. \tag{3.22}$$

Theorem 3.4 (Tautness). *If A is a closed subset in a Hausdorff space X such that A is a neighborhood retract, then A is a taut subspace relative to the cohomology $\tilde{H}_K^*$.*

Proof. (1) $\tilde{I}^\infty$ is an epimorphism. Let $h \in \tilde{H}^q_K(A)$, without loss of generality, the neighborhood retractness of A in X yields that A has an open neighborhood U(in X) such that $U \subseteq N$ and a retraction $\tau_1 : U \to A$ (if U_1 is an open neighborhood of A of which A is retract but $U_1 \nsubseteq N$, take $U = U_1 \cap \operatorname{Int} N$). Let $i_U : A \hookrightarrow U$ then, $\tilde{I}^\infty[\tilde{\tau}_1^*(h)] = \tilde{i}_U^*(\tilde{\tau}_1^* h) = \widetilde{\mathrm{id}}_A^*(h) = h$.

(2) $\tilde{I}^\infty$ is a monomorphism. Let $[h] \in \operatorname{Ker} \tilde{I}^\infty$. It is sufficient to construct $V \in \mathcal{N}(A)$ satisfying $N < V$ and $\tilde{\pi}^*_{NV} h = 0$. Since the cohomology functor commutes with the direct limit [18]. Theorem 2.3 asserts that one may assume that h belongs to $\varinjlim\{H^q(M^{\neq}_\alpha), \tilde{\pi}^*_{\alpha\beta}\}_{\Omega(N)}$ with representative $h_\alpha \in H^q(M^{\neq}_\alpha)$, where

$$\alpha = \{u_\alpha = \omega_\alpha \cap N : \omega_\alpha \subseteq X \text{ is open}\} \in \Omega(N). \tag{3.23}$$

Let $\alpha_1 = \{\omega_\alpha\} \cup \{X - A\}$, $\tilde{\alpha} = \alpha_1 \cap A$,

$$\beta = \{u_\beta = \tau_1^{-1}(u_{\tilde{\alpha}}) \cap (u_\alpha \cap U) : \phi \neq u_{\tilde{\alpha}} \in \tilde{\alpha}\}, \tag{3.24}$$

$V = \cup u_\beta$, $\tau = \tau_1 \mid V : V \hookrightarrow A$, and $\alpha' = \alpha_1 \cap V$. Then $\tilde{\alpha} \in \Omega(A)$, $\alpha' = \alpha \cap V \in \Omega(V)$, $u_{\tilde{\alpha}} \subseteq u_\beta$ for each $u_{\tilde{\alpha}} \neq \phi$, β is a family of open subsets in U and so open in X, V is an open neighborhood of A such that $V \subseteq U$, and $\beta \in \Omega(V)$. Since $u_\beta = u_\beta \cap u_\alpha \subseteq V \cap u_\alpha = u_{\alpha'}$, it follows that $\alpha' < \beta$. Also $\alpha' \cap A = \alpha \cap A = \tilde{\alpha}$ and $j^{-1}\beta = \tilde{\alpha}$, where $j : A \hookrightarrow V$. If $\ell : V \hookrightarrow N$, and $[\varphi] \in H^q(M^{\neq}_\alpha)$, then

$$\tilde{j}^*_\beta \tilde{\pi}^*_{\alpha'\beta} \tilde{\ell}^*_\alpha[\varphi] = \tilde{j}^*_\beta[\{(\varphi^\tau \mid \alpha'^{q(\tau)+1}) \mid \beta^{q(\tau)+1}\}] = [\{\varphi^\tau \mid \tilde{\alpha}^{q(\tau)+1}\}], \tag{3.25}$$

that is,

$$\tilde{j}^*_\beta \tilde{\pi}^*_{\alpha'\beta} \tilde{\ell}^*_\alpha = \tilde{i}^*_{N,\alpha}, \tag{3.26}$$

where $\tilde{i}^{\neq}_{N,\alpha} : M^{\neq}_\alpha \to M^{\neq}_{\tilde{\alpha}}$ is induced by $i_N : A \hookrightarrow N$.

On the other hand, $(j\tau)u_\beta \subseteq u_\beta$ and so $j\tau$, $\mathrm{id}_V : V \to V$ are $\beta - \beta$-contiguous [7].

It follows that $(\widetilde{\mathrm{id}}_V)^q_{\beta-\beta}$, $(\widetilde{j\tau})^q_{\beta-\beta} : M^q_\beta \to M^q_\beta$ are cochain homotopic [7]. Then $(\widetilde{\mathrm{id}}_V)^*_{\beta-\beta} = (\widetilde{j\tau})^*_{\beta-\beta} = \tilde{\tau}^*_{\tilde{\alpha}-\beta} \tilde{j}^*_\beta$, which yields that $\tilde{j}^*_\beta$ is a monomorphism. Because $\tilde{i}^*_{N,\alpha} h_\alpha = 0$, equation (3.26) yields that $\tilde{\pi}^*_{\alpha'\beta} \tilde{\ell}^*_\alpha h_\alpha = 0$. Since $\tilde{\ell}^*_\alpha h_\alpha, \tilde{\pi}^*_{\alpha'\beta}(\tilde{\ell}_\alpha h_\alpha)$ represent the zero element of $\varinjlim\{H^q(M^{\neq}_\alpha), \tilde{\pi}^*_{\alpha\beta}\}_{\Omega(N)}$, it follows that $\tilde{\pi}^*_{NV} h = [\tilde{\ell}^*_\alpha h_\alpha] = 0$. □

The rest of this article is centered around a special case of the continuity property for $\bar{H}^*_K$. As an application of the continuity property the cohomology groups satisfy a much stronger form of the excision axiom.

The following results can be deduced from those given in [9].

Lemma 3.5. *Let X be the intersection of a nested system $\{X_\alpha, \pi_{\beta\alpha}\}_\Lambda$, then*

(i) *X and $\varprojlim\{X_\alpha, \pi_{\beta\alpha}\}_\Lambda$ are homeomorphic.*

(ii) *If the nested system consists of compact Hausdorff spaces then X is a closed subset of each X_α.*

(iii) *If N is an open neighborhood of X in X_α (for some $\alpha \in \Lambda$), then there is $\beta > \alpha$ in Λ such that $X_\beta \subseteq N$.*

The inclusions $i_\alpha : X \hookrightarrow X_\alpha$ *define a map*

$$I : \{\bar{H}_K^q(X_\alpha), \bar{\pi}_{\alpha\beta}^*\}_\Lambda \longrightarrow \bar{H}_K^q(X), \tag{3.27}$$

its direct limit is denoted by $\bar{I}^\infty$*.*

Theorem 3.6 (weak continuity). *If X is the intersection of a nested system $\{X_\alpha, \pi_{\beta\alpha}\}_\Lambda$ of compact Hausdorff spaces, then $\bar{I}^\infty$ is an isomorphism.*

Proof. Since each X_α is a paracompact Hausdorff space [10] and X_α is closed in X (Lemma 3.5), it follows, by Theorem 3.2, that X is a taut subspace in X_α relative to $\bar{H}_K^*$.

(1) $\bar{I}^\infty$ is an epimorphism. Let $h \in \bar{H}_K^q(X)$, then, according to Theorem 3.2, there exists an open neighborhood N of X in X_α and $h_N \in \bar{H}_K^q(N)$, such that $\bar{i}_N^*(h_N) = h$. By Lemma 3.5, there is $\beta > \alpha$ in Λ such that $X_\beta \subseteq N$. Let $i_\beta : X \hookrightarrow X_\beta$, $j_\beta : X_\beta \hookrightarrow N$. Because $\bar{i}_\beta^*(\bar{j}_\beta^* h_N) = (\overline{j_\beta i_\beta})^* h_N = \bar{i}_N^* h_N = h$, then $\bar{I}^\infty[\bar{j}_\beta^* h_N] = h$.

(2) $\bar{I}^\infty$ is a monomorphism. Let $[h_\alpha] \in \operatorname{Ker} \bar{I}^\infty$, that is, $\bar{i}_\alpha^* h_\alpha = 0$. The tautness of X in X_α yields, by Theorem 3.2, an open neighborhood N of X in X_α such that h_N is the unique element for which $\bar{i}_N'^* h_N = 0$, where $i_N' : X \hookrightarrow N$. Because $\bar{i}_N'^*(\bar{i}_N^* h_\alpha) = \bar{i}_\alpha^* h_\alpha = 0$, then $\bar{i}_N^* h_\alpha = 0$. Let $\beta > \alpha$ in Λ such that $X_\beta \subseteq N$, then $\bar{\pi}_{\alpha\beta}^* h_\alpha = (\overline{i_N i_\beta})^* h_\alpha = \bar{j}_\beta^*(\bar{i}_N^* h_\alpha) = 0$, that is, $[h_\alpha] = 0$. □

4. Applications. One of the good applications of the Alexander-Spanier cohomology of K-types is the study of the 0-dimensional cohomology groups and their relation with the connectedness of the space [7]. In this article, two applications are given. In a next work, we hope to give more applications. As a first application, we define the partially continuous K-Alexander-Spanier cohomology of an excision map and calculate its value for some dimensions.

Let $\tilde{f}^{\neq} : M_K^{\neq}(Y,B) \to M_K^{\neq}(X,A)$ be the cochain map induced by the map f in Q. Define $M_K^{\neq}(f)$ to be the mapping cone of $\tilde{f}^{\neq}$ by

$$\begin{aligned} M_K^q(f) &= M_K^q(Y,B) \oplus M_K^{q-1}(X,A) \\ &= M_K^q(Y) \oplus M_K^{q-1}(B) \oplus M_K^{q-1}(X) \oplus M_K^{q-2}(A), \end{aligned} \tag{4.1}$$

and the coboundary is

$$\begin{aligned} &\tilde{\triangle}^q(\varphi_2,\psi_2,\varphi_1,\psi_1) \\ &\quad = \left(-\tilde{\triangle}^q(\varphi_2,\psi_2), \triangle^q(\varphi_1,\psi_1) + \tilde{f}^q(\varphi_2,\psi_2)\right) \\ &\quad = \left(\delta^q\varphi_2, -\tilde{i}^q\varphi_2 - \delta^{q-1}\psi_2, -\delta^{q-1}\varphi_1 + \tilde{f}^q\varphi_2, \tilde{i}^{q-1}\varphi_1 + \delta^{q-2}\psi_1 + \widetilde{f\mid A}\right)^{q-1}\psi_2. \end{aligned} \tag{4.2}$$

Then there is a short exact sequence

$$0 \longrightarrow \overset{+}{M}{}_K^{\neq}(X,A) \xrightarrow{\lambda^{\neq}} M_K^{\neq}(f) \xrightarrow{\chi^{\neq}} \bar{M}_K^{\neq}(Y,B) \longrightarrow \underline{O}_2, \tag{4.3}$$

where $\lambda^{\neq}, \chi^{\neq}$ are injection, projection, respectively; $\overset{+}{M}{}^{\neq}(X,A)$ is the complex $M_K^{\neq}(X,A)$ with the dimensions all raised by one, and $\bar{M}^{\neq}(Y,B)$ is the complex $M^{\neq}(Y,B)$ with the sign of the coboundary changed [12]. Note that $H^q(\bar{M}_K^{\neq}(Y,B)) = \tilde{H}_K^q(Y,B)$. Let V be an open subset of X such that $\bar{V} \subseteq \operatorname{Int} A$, $B = X - V$, and $C = A - V$. Put the excision map $e : (B,C) \hookrightarrow (X,A)$ in (4.3) instead of f, and then apply the cohomology functor to get the long exact sequence

$$\cdots \longrightarrow \tilde{H}_K^q(e) \xrightarrow{\tilde{\chi}^*} \tilde{H}_K^q(X,A) \xrightarrow{\tilde{e}^*} \tilde{H}_K^q(B,C) \xrightarrow{\tilde{\lambda}^*} \tilde{H}_K^{q+1}(e) \longrightarrow \cdots. \tag{4.4}$$

Thus the groups $\tilde{H}_K^q(e), \tilde{H}_K^{q+1}(e)$ measure how much the cohomological groups deviate from the excision axiom.

Theorem 4.1. *If $\dim K = 0$, $e : (B,C) \hookrightarrow (X,A)$ is an excision map, where A is closed and (G,G') any pair of topological abelian groups, then $\tilde{H}_K^q(e) = 0$ when $q = 0$ or $q = 1$.*

Proof. (1) Case $q = 0$. We have

$$M_K^0(e) = M_K^0(X,A) = M_K^0(X) = L_K^0(X). \tag{4.5}$$

Let $\varphi \in M_K^0(e)$ such that $\tilde{\triangle}_\varphi = 0$, then $\tilde{i}^0\varphi = 0$, $\tilde{e}\varphi = 0$. Then $\varphi = 0$ [7], which means that $\operatorname{Ker}\tilde{\triangle}^0 = 0$.

(2) Case $q = 1$. We have

$$M_K^1(e) = M_K'(X) \oplus L^0(A) \oplus L^0(B). \tag{4.6}$$

It is sufficient to show that $\operatorname{Ker}\tilde{\triangle}^1 \subseteq \operatorname{Im}\tilde{\triangle}^0$. Let $(\varphi_2,\psi_2,\varphi_1,0) \in \operatorname{Ker}\tilde{\triangle}^1$, then

$$\delta^1\varphi = 0, \qquad \tilde{i}'\varphi_2 = -\delta^0\psi_2, \tag{4.7}$$

$$\tilde{e}^1\varphi_2 = \delta^0\varphi_1, \tag{4.8}$$

$$\tilde{e}_1^0(-\psi_2) = \tilde{j}\varphi_1, \tag{4.9}$$

where $i : A \hookrightarrow X$, $j : C \hookrightarrow B$ and $e_1 = e \mid C$.

By (4.9), there exists $\varphi \in M_K^0(X)$ [7] such that

$$\tilde{i}^0\varphi = -\psi_2, \qquad \tilde{e}^0\varphi = \varphi_1. \tag{4.10}$$

By (4.8), (4.9), and (4.10), we get

$$\tilde{i}^1(\delta^0\varphi - \varphi_2) = 0, \qquad \tilde{e}^1(\delta^0\varphi - \varphi_2) = 0. \tag{4.11}$$

Then $\delta^0\varphi = \varphi_2$ [7], which together with (4.11) yield $(\varphi,0,0,0) \in M_K^0(e)$ such that $\tilde{\triangle}^0(\varphi,0,0,0) = (\varphi_2,\psi_2,\varphi_1,0)$. □

Combining the sequence (4.4) and the above theorem, we get the following result.

Corollary 4.2. *Under the assumptions of Theorem 4.1, the map $\tilde{e}^{*0} : \tilde{H}_K^0(X,A) \to \tilde{H}_K^0(B,C)$ is an isomorphism but $\tilde{e}^{*1}$ is a monomorphism.*

Next we give a second application to the work introduced in this paper.

Let $\eta : (G,G') \to (F,F')$ be a homeomorphism of pairs of (discrete) abelian groups, which is an epimorphism, $(L,L') = \operatorname{Ker}\eta$ and $\lambda : (L,L') \hookrightarrow (G,G')$. Then for each $\bar{\alpha} \in \Omega(X,A)$, the maps η, λ define naturally a short exact sequence

$$0 \longrightarrow C^q(\bar{\alpha},L,L') \longrightarrow C^q(\bar{\alpha};G,G') \longrightarrow C^q(\bar{\alpha};F,F') \longrightarrow 0; \tag{4.12}$$

its cohomology is a long exact sequence [12] denoted by $S_{\bar{\alpha}}$. One can show that $\{S_{\bar{\alpha}}\}_{\Omega(X,A)}$ is a direct system, its direct limit [7, 8] is

$$\begin{aligned} \cdots \longrightarrow \bar{H}_K^{q-1}(X,A;F,F') \longrightarrow \bar{H}_K^{q}(X,A';L,L') \longrightarrow \bar{H}_K^{q}(X,A;G,G') \\ \longrightarrow \bar{H}_K^{q}(X,A,F,F') \longrightarrow \bar{H}_K^{q+1}(X,A;L,L') \longrightarrow \cdots . \end{aligned} \tag{4.13}$$

Now instead of F take the factor group G/G' and so F' will be the null subgroup of G/G'. Then the above sequence yields the following result.

Theorem 4.3. *Consider (X,A) has a trivial $(q-1)$-dimensional space K-Alexander-Spanier cohomology group with finite cochains, and a trivial $(q+1)$-dimensional K-Alexander-Spanier cohomology with infinite cochains, taken over the coefficient groups G/G' and G', respectively. Then the group $\bar{H}_K^q(X,A;G,G')$ defined over an arbitrary pair (G,G') of coefficient groups is the extension of the cohomology group $\bar{H}_K^q(X,A;G')$ with infinite cochains over G' by the group $\bar{H}_K^q(X,A,G/G')$ with finite cochains over G/G'.*

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