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THE PRIME FILTER THEOREM OF LATTICE IMPLICATION ALGEBRAS

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ABSTRACT. Using a special set $x^{-1}F$, we give an equivalent condition for a filter to be prime, and applying this result, we provide the prime filter theorem in lattice implication algebras.

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1. Introduction. In order to research the logical system whose propositional value is given in a lattice, Xu [3] proposed the concept of lattice implication algebras, and discussed some of their properties. Xu and Qin [4] introduced the notion of filters and implicative filters in a lattice implication algebra and investigated their properties. The present author [1] gave an equivalent condition of a filter and provided some equivalent conditions for a filter to be an implicative filter. Also, by using these results, an extension property for implicative filter was constructed. In [2], Liu and Xu defined the notion of prime filters and studied a decomposition theorem of lattice implication algebras.

In this paper, we first give an equivalent condition for a filter to be prime by using a special set $x^{-1}F$ and applying this result we provide the prime filter theorem in lattice implication algebras.

2. Preliminaries. First of all, we recall a few notions and properties.

By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution "$'$" and a binary operation "$\to$" satisfying the following axioms:

(I1) $x \to (y \to z) = y \to (x \to z)$,
(I2) $x \to x = 1$,
(I3) $x \to y = y' \to x'$,
(I4) $x \to y = y \to x = 1 \Rightarrow x = y$,
(I5) $(x \to y) \to y = (y \to x) \to x$,
(L1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(L2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,

for all $x, y, z \in L$.

In what follows the binary operation "$\to$" will be denoted by juxtaposition. We can define a partial ordering "$\le$" on a lattice implication algebra L by $x \le y$ if and only if $xy = 1$.

In a lattice implication algebra L, the following hold (see [3]):

(1) $0x = 1$, $1x = x$, and $x1 = 1$.

(2) $x' = x0$.
(3) $xy \le (yz)(xz)$.
(4) $x \vee y = (xy)y$.
(5) $x \le y$ implies $yz \le xz$ and $zx \le zy$.
(6) $x \le (xy)y$.

A subset F of a lattice implication algebra L is called a *filter* of L if it satisfies:
(F1) $1 \in F$,
(F2) $x \in F$ and $xy \in F$ imply $y \in F$,
for all $x, y \in L$.

Any filter F of a lattice implication algebra L has the property: if $x \le y$ and $x \in F$, then $y \in F$.

3. The prime filter theorem. In the rest of this paper, the letter L will be reserved, so far as is possible, for a lattice implication algebra.

Note that for a subset F of L,

$$\langle F\rangle = \{x \in L \mid a_1(a_2 \cdots (a_n x)\cdots) = 1;\ a_1, a_2, \ldots, a_n \in F\} \tag{3.1}$$

is the smallest filter containing F and is called the *filter generated* by F (see [4]).

For any nonnegative integer n, we define $n(x)y$ recursively as follows: $0(x)y = y$, $1(x)y = xy$, and $(n+1)(x)y = x(n(x)y)$ for all $x, y \in L$. Using (I1) and (1) we know that $y(n(x)y) = 1$, that is, $y \le n(x)y$ for all $x, y \in L$.

Proposition 3.1. *Let F be a filter of L and let $x \in L$. Then*

$$\langle F \cup \{x\}\rangle = \{y \in L \mid n(x)y \in F \text{ for some nonnegative integer } n\}. \tag{3.2}$$

Proof. Let $y \in \langle F \cup \{x\}\rangle$. Then

$$m(x)(a_1(a_2 \cdots (a_n y)\cdots)) = 1 \tag{3.3}$$

for some $a_1, a_2, \ldots, a_n \in F$ and some nonnegative integer m. Using (I1) repeatedly, we know that

$$a_1(a_2 \cdots (a_n(m(x)y))\cdots) = 1. \tag{3.4}$$

It follows from (F2) that $m(x)y \in F$ so that

$$\langle F \cup \{x\}\rangle \subseteq \{y \in L \mid n(x)y \in F \text{ for some nonnegative integer } n\}. \tag{3.5}$$

Conversely, assume that $n(x)y \in F$ for some nonnegative integer n. It follows from $F \subseteq \langle F \cup \{x\}\rangle$ that $x((n-1)(x)y) = n(x)y \in \langle F \cup \{x\}\rangle$. Since $x \in \langle F \cup \{x\}\rangle$, we have $(n-1)(x)y \in \langle F \cup \{x\}\rangle$ by (F2). Repeating this process we know that $y = 0(x)y \in \langle F \cup \{x\}\rangle$. Hence

$$\{y \in L \mid n(x)y \in F \text{ for some nonnegative integer } n\} \subseteq \langle F \cup \{x\}\rangle, \tag{3.6}$$

This completes the proof. □

Definition 3.2. For any nonempty subset F of L and $x \in L$, we define

$$x^{-1}F := \{y \in L \mid x \vee y \in F\}. \tag{3.7}$$

Note that if F is a filter of L, then $1 \in x^{-1}F$.

Proposition 3.3. *If F is a filter of L, then $x^{-1}F$ is a filter of L containing F.*

Proof. Let $y \in x^{-1}F$ and $yz \in x^{-1}F$. Then $x \vee y \in F$ and $x \vee (yz) \in F$. Now

$$(x \vee y)(x \vee z) = ((yx)x)((zx)x) \geq (zx)(yx) \geq yz \tag{3.8}$$

and $(x \vee y)(x \vee z) \geq x \vee z \geq x$. It follows that $x \vee (yz) \leq (x \vee y)(x \vee z)$ so that $(x \vee y)(x \vee z) \in F$. Using the fact that F is a filter and $x \vee y \in F$, we get $x \vee z \in F$, that is, $z \in x^{-1}F$. This shows that $x^{-1}F$ is a filter of L. Let $y \in F$. Since $y \leq x \vee y$, it follows that $x \vee y \in F$, that is, $y \in x^{-1}F$. Hence $F \subseteq x^{-1}F$, this completes the proof. □

Proposition 3.4. *Let F and G be filters of L. Then*
(i) $x^{-1}F = L$ *if and only if* $x \in F$,
(ii) $x \leq y$ *in* $L \Rightarrow x^{-1}F \subseteq y^{-1}F$,
(iii) $F \subseteq G \Rightarrow x^{-1}F \subseteq x^{-1}G$,
(iv) $x^{-1}(F \cap G) = x^{-1}F \cap x^{-1}G$ *and* $x^{-1}(F \cup G) = x^{-1}F \cup x^{-1}G$,
(v) $(x \vee y)^{-1}F = x^{-1}(y^{-1}F)$,
(vi) $(x \wedge y)^{-1}F \subseteq x^{-1}F \cap y^{-1}F$,
for all $x, y \in L$.

Proof. (i) If $x \in F$, then $x \vee y \in F$ for all $y \in L$, that is, $y \in x^{-1}F$. Hence $x^{-1}F = L$. Conversely, assume that $x^{-1}F = L$. Then $x \vee y \in F$ for all $y \in L$, in particular $x = x \vee x \in F$.

(ii) Assume that $x \leq y$ in L and let $z \in x^{-1}F$. Then $x \vee z \in F$ and $x \vee z \leq y \vee z$. It follows that $y \vee z \in F$, that is, $z \in y^{-1}F$.

(iii)–(vi) Clear. □

Definition 3.5 (see [2, Definition 4])**.** A proper filter P of L is said to be *prime* if for every $x, y \in L$, $x \vee y \in P$ implies $x \in P$ or $y \in P$.

Proposition 3.6. *Let P and F be filters of L such that $F \subseteq P$. If P is prime, then $x^{-1}F \subseteq P$ for all $x \in L \setminus P$.*

Proof. Let $z \in x^{-1}F$ for all $x \in L \setminus P$. Then $x \vee z \in F \subseteq P$. Since P is prime, it follows that $z \in P$ because $x \notin P$. Hence $x^{-1}F \subseteq P$. □

Proposition 3.7. *If P is a prime filter of L, then $L \setminus P$ is $\vee$-closed, that is, $x \vee y \in L \setminus P$ whenever $x \in L \setminus P$ and $y \in L \setminus P$.*

Proof. The proof is straightforward. □

The following theorem gives a characterization of prime filters.

Theorem 3.8. *A filter P of L is prime if and only if $x^{-1}P = P$ for all $x \in L \setminus P$.*

Proof. Suppose P is a prime filter of L and let $x \in L \setminus P$. The inclusion $P \subseteq x^{-1}P$ follows from Proposition 3.3. Let $y \in x^{-1}P$. Then $x \vee y \in P$ and so $y \in P$ because P is prime and $x \notin P$. This proves that $x^{-1}P = P$. Conversely, assume that $x^{-1}P = P$ for all $x \in L \setminus P$. Let $y \vee z \in P$ and $z \notin P$. It follows from the hypothesis that $z^{-1}P = P$ so that $y \in z^{-1}P = P$. This shows that P is prime. □

Proposition 3.9. *If F is a filter of L, then $F = x^{-1}F \cap \langle F \cup \{x\}\rangle$ for all $x \in L \setminus F$.*

Proof. Clearly, $F \subseteq x^{-1}F \cap \langle F \cup \{x\}\rangle$. Let $y \in x^{-1}F \cap \langle F \cup \{x\}\rangle$. Then $x \vee y \in F$ and $y \in \langle F \cup \{x\}\rangle$. It follows from Proposition 3.1 that there exists a nonnegative integer n such that $n(x)y \in F$. Now

$$n(x)y = x((n-1)(x)y) = (x \vee (n-1)(x)y)(n-1)(x)y. \tag{3.9}$$

Since $y \leq (n-1)(x)y$, therefore $x \vee y \leq x \vee (n-1)(x)y$ and so $x \vee (n-1)(x)y \in F$. From $n(x)y = (x \vee (n-1)(x)y)(n-1)(x)y \in F$ it follows that $(n-1)(x)y \in F$. Continuing this process, we get $y \in F$ and, consequently, $x^{-1}F \cap \langle F \cup \{x\}\rangle \subseteq F$. This completes the proof. □

Finally, we provide the prime filter theorem. This is a generalization of Liu and Xu's result [2, Theorem 4] because every lattice ideal is necessarily $\vee$-closed.

Theorem 3.10 (prime filter theorem)**.** *Let F be a filter of L and S a $\vee$-closed subset of L such that $F \cap S = \emptyset$. Then there exists a prime filter P of L such that $F \subseteq P$ and $P \cap S = \emptyset$.*

Proof. The existence of a filter P being the maximal element of the family of all filters that contain F and have empty intersection with S follows from an application of Zorn's lemma. We now prove that P is prime. Suppose P is not prime. By Theorem 3.8, there exists an element $x \in L \setminus P$ such that $x^{-1}P \neq P$. Now P is properly contained in both $x^{-1}P$ and $\langle P \cup \{x\}\rangle$; therefore the maximality of P implies that $x^{-1}P \cap S \neq \emptyset$ and $\langle P \cup \{x\}\rangle \cap S \neq \emptyset$. Let $y \in x^{-1}P \cap S$ and $z \in \langle P \cup \{x\}\rangle \cap S$. Then $y \in x^{-1}P$ and $z \in \langle P \cup \{x\}\rangle$ and hence $y \vee z \in x^{-1}P \cap \langle P \cup \{x\}\rangle = P$ by Proposition 3.9. Also $y \vee z \in S$ because S is $\vee$-closed. Consequently, $y \vee z \in P \cap S$ and so $P \cap S \neq \emptyset$, a contradiction. This completes the proof. □

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