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FUZZY BCI-SUBALGEBRAS WITH INTERVAL-VALUED MEMBERSHIP FUNCTIONS

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Abstract. The purpose of this paper is to define the notion of an interval-valued fuzzy BCI-subalgebra (briefly, an i-v fuzzy BCI-subalgebra) of a BCI-algebra. Necessary and sufficient conditions for an i-v fuzzy set to be an i-v fuzzy BCI-subalgebra are stated. A way to make a new i-v fuzzy BCI-subalgebra from old one is given. The images and inverse images of i-v fuzzy BCI-subalgebras are defined, and how the images or inverse images of i-v fuzzy BCI-subalgebras become i-v fuzzy BCI-subalgebras is studied.

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1. Introduction. The notion of BCK-algebras was proposed by Iami and Iséki in 1966. In the same year, Iséki [2] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras. Fuzzy sets were initiated by Zadeh [3]. In [4], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as an i-v fuzzy set. In [4], Zadeh also constructed a method of approximate inference using his i-v fuzzy sets. In [1], Biswas defined interval-valued fuzzy subgroups (i.e., i-v fuzzy subgroups) of Rosenfeld's nature, and investigated some elementary properties. In this paper, using the notion of interval-valued fuzzy set by Zadeh, we introduce the concept of an interval-valued fuzzy BCI-subalgebra (briefly, i-v fuzzy BCI-subalgebra) of a BCI-algebra, and study some of their properties. Using an i-v level set of an i-v fuzzy set, we state a characterization of an i-v fuzzy BCI-subalgebra. We prove that every BCI-subalgebra of a BCI-algebra X can be realized as an i-v level BCI-subalgebra of an i-v fuzzy BCI-subalgebra of X. In connection with the notion of homomorphism, we study how the images and inverse images of i-v fuzzy BCI-subalgebras become i-v fuzzy BCI-subalgebras.

2. Preliminaries. In this section, we include some elementary aspects that are necessary for this paper.

Recall that a *BCI-algebra* is an algebra $(X,*,0)$ of type $(2,0)$ satisfying the following axioms:

(I) $((x*y)*(x*z))*(z*y)=0$,

(II) $(x*(x*y))*y=0$,

(III) $x * x = 0$, and
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$,
for every $x, y, z \in X$.

Note that the equality $0 * (x * y) = (0 * x) * (0 * y)$ holds in a BCI-algebra. A non-empty subset S of a BCI-algebra X is called a *BCI-subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A mapping $f : X \to Y$ of BCI-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

We now review some fuzzy logic concepts. Let X be a set. A *fuzzy set* in X is a function $\mu : X \to [0,1]$. Let f be a mapping from a set X into a set Y. Let ν be a fuzzy set in Y. Then the *inverse image* of ν, denoted by $f^{-1}[\nu]$, is the fuzzy set in X defined by $f^{-1}[\nu](x) = \nu(f(x))$ for all $x \in X$. Conversely, let μ be a fuzzy set in X. The *image* of μ, written as $f[\mu]$, is a fuzzy set in Y defined by

$$f[\mu](y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \mu(z) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

for all $y \in Y$, where $f^{-1}(y) = \{x \mid f(x) = y\}$.

An *interval-valued fuzzy set* (briefly, *i-v fuzzy set*) A defined on X is given by

$$A = \{(x, [\mu_A^L(x), \mu_A^U(x)])\}, \quad \forall x \in X \text{ (briefly, denoted by } A = [\mu_A^L, \mu_A^U]), \tag{2.2}$$

where μ_A^L and μ_A^U are two fuzzy sets in X such that $\mu_A^L(x) \leq \mu_A^U(x)$ for all $x \in X$.

Let $\bar{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$, $\forall x \in X$ and let $D[0,1]$ denotes the family of all closed subintervals of $[0,1]$. If $\mu_A^L(x) = \mu_A^U(x) = c$, say, where $0 \leq c \leq 1$, then we have $\bar{\mu}_A(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to $D[0,1]$. Thus $\bar{\mu}_A(x) \in D[0,1]$, $\forall x \in X$, and therefore the i-v fuzzy set A is given by

$$A = \{(x, \bar{\mu}_A(x))\}, \quad \forall x \in X, \text{ where } \bar{\mu}_A : X \longrightarrow D[0,1]. \tag{2.3}$$

Now let us define what is known as *refined minimum* (briefly, rmin) of two elements in $D[0,1]$. We also define the symbols "$\geq$", "$\leq$", and "$=$" in case of two elements in $D[0,1]$. Consider two elements $D_1 := [a_1, b_1]$ and $D_2 := [a_2, b_2] \in D[0,1]$. Then

$$\begin{aligned} &\operatorname{rmin}(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]; \\ &D_1 \geq D_2 \quad \text{if and only if } a_1 \geq a_2,\ b_1 \geq b_2; \end{aligned} \tag{2.4}$$

and similarly we may have $D_1 \leq D_2$ and $D_1 = D_2$.

Definition 2.1. A fuzzy set μ in a BCI-algebra X is called a *fuzzy BCI-subalgebra* of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

3. Interval-valued fuzzy BCI-subalgebras. In what follows, let X denote a BCI-algebra unless otherwise specified. We begin with the following two propositions.

Proposition 3.1. *Let f be a homomorphism from a BCI-algebra X into a BCI-algebra Y. If ν is a fuzzy BCI-subalgebra of Y, then the inverse image $f^{-1}[\nu]$ of ν is a fuzzy BCI-subalgebra of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned} f^{-1}[\nu](x * y) &= \nu(f(x * y)) = \nu(f(x) * f(y)) \\ &\geq \min\{\nu(f(x)), \nu(f(y))\} \\ &= \min\{f^{-1}[\nu](x), f^{-1}[\nu](y)\}. \end{aligned} \tag{3.1}$$

Hence $f^{-1}[\nu]$ is a fuzzy BCI-subalgebra of X. □

Proposition 3.2. *Let $f : X \to Y$ be a homomorphism between BCI-algebras X and Y. For every fuzzy BCI-subalgebra μ of X, the image $f[\mu]$ of μ is a fuzzy BCI-subalgebra of Y.*

Proof. We first prove that

$$f^{-1}(y_1) * f^{-1}(y_2) \subseteq f^{-1}(y_1 * y_2) \tag{3.2}$$

for all $y_1, y_2 \in Y$. For, if $x \in f^{-1}(y_1) * f^{-1}(y_2)$, then $x = x_1 * x_2$ for some $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Since f is a homomorphism, it follows that $f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2$ so that $x \in f^{-1}(y_1 * y_2)$. Hence (3.2) holds. Now let $y_1, y_2 \in Y$ be arbitrarily given. Assume that $y_1 * y_2 \notin \mathrm{Im}(f)$. Then $f[\mu](y_1 * y_2) = 0$. But if $y_1 * y_2 \notin \mathrm{Im}(f)$, that is, $f^{-1}(y_1 * y_2) = \emptyset$, then $f^{-1}(y_1) = \emptyset$ or $f^{-1}(y_2) = \emptyset$ by (3.2). Thus $f[\mu](y_1) = 0$ or $f[\mu](y_2) = 0$, and so

$$f[\mu](y_1 * y_2) = 0 = \min\{f[\mu](y_1), f[\mu](y_2)\}. \tag{3.3}$$

Suppose that $f^{-1}(y_1 * y_2) \neq \emptyset$. Then we should consider the two cases:

$$f^{-1}(y_1) = \emptyset \quad \text{or} \quad f^{-1}(y_2) = \emptyset, \tag{3.4}$$

$$f^{-1}(y_1) \neq \emptyset \quad \text{and} \quad f^{-1}(y_2) \neq \emptyset. \tag{3.5}$$

For the case (3.4), we have $f[\mu](y_1) = 0$ or $f[\mu](y_2) = 0$, and so

$$f[\mu](y_1 * y_2) \geq 0 = \min\{f[\mu](y_1), f[\mu](y_2)\}. \tag{3.6}$$

Case (3.5) implies, from (3.2), that

$$\begin{aligned} f[\mu](y_1 * y_2) &= \sup_{z \in f^{-1}(y_1 * y_2)} \mu(z) \geq \sup_{z \in f^{-1}(y_1) * f^{-1}(y_2)} \mu(z) \\ &= \sup_{x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)} \mu(x_1 * x_2). \end{aligned} \tag{3.7}$$

Since μ is a fuzzy BCI-subalgebra of X, it follows from the definition of a fuzzy BCI-subalgebra that

$$\begin{aligned} f[\mu](y_1 * y_2) &\geq \sup_{x_1 \in f^{-1}(y_1),\, x_2 \in f^{-1}(y_2)} \min\{\mu(x_1), \mu(x_2)\} \\ &= \sup_{x_1 \in f^{-1}(y_1)} \left(\min\left\{ \sup_{x_2 \in f^{-1}(y_2)} \mu(x_1), \mu(x_2) \right\} \right) \\ &= \sup_{x_1 \in f^{-1}(y_1)} \left(\min\left\{ \mu(x_1), \sup_{x_2 \in f^{-1}(y_2)} \mu(x_2) \right\} \right) \\ &= \sup_{x_1 \in f^{-1}(y_1)} \left(\min\{\mu(x_1), f[\mu](y_2)\} \right) \\ &= \min\left\{ \sup_{x_1 \in f^{-1}(y_1)} \mu(x_1), f[\mu](y_2) \right\} \\ &= \min\{f[\mu](y_1), f[\mu](y_2)\}. \end{aligned} \tag{3.8}$$

Hence $f[\mu](y_1 * y_2) \geq \min\{f[\mu](y_1), f[\mu](y_2)\}$ for all $y_1, y_2 \in Y$. This completes the proof. □

Definition 3.3. An i-v fuzzy set A in X is called an *interval-valued fuzzy BCI-subalgebra* (briefly, *i-v fuzzy BCI-subalgebra*) of X if

$$\bar{\mu}_A(x * y) \geq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} \quad \forall x, y \in X. \tag{3.9}$$

Example 3.4. Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table:

Table 3.1.

$*$	0	a	b	c
0	0	c	0	a
a	a	0	a	c
b	b	c	0	a
c	c	a	c	0

let an i-v fuzzy set A defined on X be given by

$$\bar{\mu}_A(x) = \begin{cases} [0.2, 0.8] & \text{if } x \in \{0, b\}, \\ [0.1, 0.7] & \text{otherwise.} \end{cases} \tag{3.10}$$

It is easy to check that A is an i-v fuzzy BCI-subalgebra of X.

Lemma 3.5. *If A is an i-v fuzzy BCI-subalgebra of X, then $\bar{\mu}_A(0) \geq \bar{\mu}_A(x)$ for all $x \in X$.*

Proof. For every $x \in X$, we have

$$\begin{aligned}\bar{\mu}_A(0) &= \bar{\mu}_A(x * x) \geq \operatorname{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(x)\}\\ &= \operatorname{rmin}\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(x), \mu_A^U(x)]\}\\ &= [\mu_A^L(x), \mu_A^U(x)] = \bar{\mu}_A(x),\end{aligned} \tag{3.11}$$

this completes the proof. □

Theorem 3.6. *Let A be an i-v fuzzy BCI-subalgebra of X. If there is a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} \bar{\mu}_A(x_n) = [1,1], \tag{3.12}$$

then $\bar{\mu}_A(0) = [1,1]$.

Proof. Since $\bar{\mu}_A(0) \geq \bar{\mu}_A(x)$ for all $x \in X$, we have $\bar{\mu}_A(0) \geq \bar{\mu}_A(x_n)$ for every positive integer n. Note that

$$[1,1] \geq \bar{\mu}_A(0) \geq \lim_{n\to\infty} \bar{\mu}_A(x_n) = [1,1]. \tag{3.13}$$

Hence $\bar{\mu}_A(0) = [1,1]$. □

Theorem 3.7. *An i-v fuzzy set $A = [\mu_A^L, \mu_A^U]$ in X is an i-v fuzzy BCI-subalgebra of X if and only if μ_A^L and μ_A^U are fuzzy BCI-subalgebras of X.*

Proof. Suppose that μ_A^L and μ_A^U are fuzzy BCI-subalgebras of X. Let $x, y \in X$. Then

$$\begin{aligned}\bar{\mu}_A(x * y) &= [\mu_A^L(x * y), \mu_A^U(x * y)]\\ &\geq [\min\{\mu_A^L(x), \mu_A^L(y)\}, \min\{\mu_A^U(x), \mu_A^U(y)\}]\\ &= \operatorname{rmin}\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\}\\ &= \operatorname{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}.\end{aligned} \tag{3.14}$$

Hence A is an i-v fuzzy BCI-subalgebra of X.

Conversely, assume that A is an i-v fuzzy BCI-subalgebra of X. For any $x, y \in X$, we have

$$\begin{aligned}[\mu_A^L(x * y), \mu_A^U(x * y)] &= \bar{\mu}_A(x * y) \geq \operatorname{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}\\ &= \operatorname{rmin}\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\}\\ &= [\min\{\mu_A^L(x), \mu_A^L(y)\}, \min\{\mu_A^U(x), \mu_A^U(y)\}].\end{aligned} \tag{3.15}$$

It follows that $\mu_A^L(x * y) \geq \min\{\mu_A^L(x), \mu_A^L(y)\}$ and $\mu_A^U(x * y) \geq \min\{\mu_A^U(x), \mu_A^U(y)\}$. Hence μ_A^L and μ_A^U are fuzzy BCI-subalgebras of X. □

Theorem 3.8. *Let A be an i-v fuzzy set in X. Then A is an i-v fuzzy BCI-subalgebra of X if and only if the nonempty set*

$$U(A; [\delta_1, \delta_2]) := \{x \in X \mid \bar{\mu}_A(x) \geq [\delta_1, \delta_2]\} \tag{3.16}$$

is a BCI-subalgebra of X for every $[\delta_1, \delta_2] \in D[0,1]$.

We then call $\bar{U}(A;[\delta_1,\delta_2])$ the *i-v level BCI-subalgebra* of A.

Proof. Assume that A is an i-v fuzzy BCI-subalgebra of X and let $[\delta_1,\delta_2]\in D[0,1]$ be such that $x,y\in\bar{U}(A;[\delta_1,\delta_2])$. Then

$$\bar{\mu}_A(x*y)\geq \text{rmin}\{\bar{\mu}_A(x),\bar{\mu}_A(y)\}\geq \text{rmin}\{[\delta_1,\delta_2],[\delta_1,\delta_2]\}=[\delta_1,\delta_2], \tag{3.17}$$

and so $x*y\in\bar{U}(A;[\delta_1,\delta_2])$. Thus $\bar{U}(A;[\delta_1,\delta_2])$ is a BCI-subalgebra of X.

Conversely, assume that $\bar{U}(A;[\delta_1,\delta_2])$ $(\neq\emptyset)$ is a BCI-subalgebra of X for every $[\delta_1,\delta_2]\in D[0,1]$. Suppose there exist $x_0,y_0\in X$ such that

$$\bar{\mu}_A(x_0*y_0)<\text{rmin}\{\bar{\mu}_A(x_0),\bar{\mu}_A(y_0)\}. \tag{3.18}$$

Let $\bar{\mu}_A(x_0)=[\gamma_1,\gamma_2]$, $\bar{\mu}_A(y_0)=[\gamma_3,\gamma_4]$, and $\bar{\mu}_A(x_0*y_0)=[\delta_1,\delta_2]$. Then

$$[\delta_1,\delta_2]<\text{rmin}\{[\gamma_1,\gamma_2],[\gamma_3,\gamma_4]\}=[\min\{\gamma_1,\gamma_3\},\min\{\gamma_2,\gamma_4\}]. \tag{3.19}$$

Hence $\delta_1<\min\{\gamma_1,\gamma_3\}$ and $\delta_2<\min\{\gamma_2,\gamma_4\}$. Taking

$$[\lambda_1,\lambda_2]=\frac{1}{2}\left(\bar{\mu}_A(x_0*y_0)+\text{rmin}\{\bar{\mu}_A(x_0),\bar{\mu}_A(y_0)\}\right), \tag{3.20}$$

we obtain

$$\begin{aligned}[\lambda_1,\lambda_2]&=\frac{1}{2}\left([\delta_1,\delta_2]+[\min\{\gamma_1,\gamma_3\},\min\{\gamma_2,\gamma_4\}]\right)\\&=\left[\frac{1}{2}(\delta_1+\min\{\gamma_1,\gamma_3\}),\frac{1}{2}(\delta_2+\min\{\gamma_2,\gamma_4\})\right].\end{aligned} \tag{3.21}$$

It follows that

$$\begin{aligned}\min\{\gamma_1,\gamma_3\}>\lambda_1&=\frac{1}{2}(\delta_1+\min\{\gamma_1,\gamma_3\})>\delta_1,\\\min\{\gamma_2,\gamma_4\}>\lambda_2&=\frac{1}{2}(\delta_2+\min\{\gamma_2,\gamma_4\})>\delta_2\end{aligned} \tag{3.22}$$

so that $[\min\{\gamma_1,\gamma_3\},\min\{\gamma_2,\gamma_4\}]>[\lambda_1,\lambda_2]>[\delta_1,\delta_2]=\bar{\mu}_A(x_0*y_0)$. Therefore, $x_0*y_0\notin\bar{U}(A;[\lambda_1,\lambda_2])$. On the other hand,

$$\begin{aligned}\bar{\mu}_A(x_0)&=[\gamma_1,\gamma_2]\geq[\min\{\gamma_1,\gamma_3\},\min\{\gamma_2,\gamma_4\}]>[\lambda_1,\lambda_2],\\\bar{\mu}_A(y_0)&=[\gamma_3,\gamma_4]\geq[\min\{\gamma_1,\gamma_3\},\min\{\gamma_2,\gamma_4\}]>[\lambda_1,\lambda_2],\end{aligned} \tag{3.23}$$

and so $x_0,y_0\in\bar{U}(A;[\lambda_1,\lambda_2])$. It contradicts that $\bar{U}(A;[\lambda_1,\lambda_2])$ is a BCI-subalgebra of X. Hence $\bar{\mu}_A(x*y)\geq\text{rmin}\{\bar{\mu}_A(x),\bar{\mu}_A(y)\}$ for all $x,y\in X$. This completes the proof. □

Theorem 3.9. *Every BCI-subalgebra of X can be realized as an i-v level BCI-subalgebra of an i-v fuzzy BCI-subalgebra of X.*

Proof. Let Y be a BCI-subalgebra of X and let A be an i-v fuzzy set on X defined by

$$\bar{\mu}_A(x)=\begin{cases}[\alpha_1,\alpha_2] & \text{if } x\in Y,\\ [0,0] & \text{otherwise},\end{cases} \tag{3.24}$$

where $\alpha_1,\alpha_2\in(0,1]$ with $\alpha_1<\alpha_2$. It is clear that $\bar{U}(A;[\alpha_1,\alpha_2])=Y$. We show that A

is an i-v fuzzy BCI-subalgebra of X. Let $x, y \in X$. If $x, y \in Y$, then $x * y \in Y$ and so

$$\bar{\mu}_A(x * y) = [\alpha_1, \alpha_2] = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}. \tag{3.25}$$

If $x, y \notin Y$, then $\bar{\mu}_A(x) = [0,0] = \bar{\mu}_A(y)$ and thus

$$\bar{\mu}_A(x * y) \geq [0,0] = \text{rmin}\{[0,0], [0,0]\} = \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}. \tag{3.26}$$

If $x \in Y$ and $y \notin Y$, then $\bar{\mu}_A(x) = [\alpha_1, \alpha_2]$ and $\bar{\mu}_A(y) = [0,0]$. It follows that

$$\bar{\mu}_A(x * y) \geq [0,0] = \text{rmin}\{[\alpha_1, \alpha_2], [0,0]\} = \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}. \tag{3.27}$$

Similarly for the case $x \notin Y$ and $y \in Y$, we get $\bar{\mu}_A(x * y) \geq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$. Therefore A is an i-v fuzzy BCI-subalgebra of X, and the proof is complete. □

Theorem 3.10. *Let Y be a subset of X and let A be an i-v fuzzy set on X which is given in the proof of Theorem 3.9. If A is an i-v fuzzy BCI-subalgebra of X, then Y is a BCI-subalgebra of X.*

Proof. Assume that A is an i-v fuzzy BCI-subalgebra of X. Let $x, y \in Y$. Then $\bar{\mu}_A(x) = [\alpha_1, \alpha_2] = \bar{\mu}_A(y)$, and so

$$\bar{\mu}_A(x * y) \geq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]. \tag{3.28}$$

This implies that $x * y \in Y$. Hence Y is a BCI-subalgebra of X. □

Theorem 3.11. *If A is an i-v fuzzy BCI-subalgebra of X, then the set*

$$X_{\bar{\mu}_A} := \{x \in X \mid \bar{\mu}_A(x) = \bar{\mu}_A(0)\} \tag{3.29}$$

is a BCI-subalgebra of X.

Proof. Let $x, y \in X_{\bar{\mu}_A}$. Then $\bar{\mu}_A(x) = \bar{\mu}_A(0) = \bar{\mu}_A(y)$, and so

$$\bar{\mu}_A(x * y) \geq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} = \text{rmin}\{\bar{\mu}_A(0), \bar{\mu}_A(0)\} = \bar{\mu}_A(0). \tag{3.30}$$

Combining this and Lemma 3.5, we get $\bar{\mu}_A(x * y) = \bar{\mu}_A(0)$, that is, $x * y \in X_{\bar{\mu}_A}$. Hence $X_{\bar{\mu}_A}$ is a BCI-subalgebra of X. □

The following is a way to make a new i-v fuzzy BCI-subalgebra from old one.

Theorem 3.12. *For an i-v fuzzy BCI-subalgebra A of X, the i-v fuzzy set A^* in X defined by $\bar{\mu}_{A^*}(x) = \bar{\mu}_A(0 * x)$ for all $x \in X$ is an i-v fuzzy BCI-subalgebra of X.*

Proof. Since the equality $0 * (x * y) = (0 * x) * (0 * y)$ holds for all $x, y \in X$, we have

$$\begin{aligned} \bar{\mu}_{A^*}(x * y) &= \bar{\mu}_A(0 * (x * y)) = \bar{\mu}_A((0 * x) * (0 * y)) \\ &\geq \text{rmin}\{\bar{\mu}_A(0 * x), \bar{\mu}_A(0 * y)\} \\ &= \text{rmin}\{\bar{\mu}_{A^*}(x), \bar{\mu}_{A^*}(y)\} \end{aligned} \tag{3.31}$$

for all $x, y \in X$. Therefore A^* is an i-v fuzzy BCI-subalgebra of X. □

Definition 3.13 (Biswas [1]). Let f be a mapping from a set X into a set Y. Let B be an i-v fuzzy set in Y. Then the *inverse image* of B, denoted by $f^{-1}[B]$, is the i-v fuzzy set in X with the membership function given by $\bar{\mu}_{f^{-1}[B]}(x) = \bar{\mu}_B(f(x))$ for all $x \in X$.

Lemma 3.14 (Biswas [1]). *Let f be a mapping from a set X into a set Y. Let $m = [m^L, m^U]$ and $n = [n^L, n^U]$ be i-v fuzzy sets in X and Y, respectively. Then*

(i) $f^{-1}(n) = [f^{-1}(n^L), f^{-1}(n^U)]$,

(ii) $f(m) = [f(m^L), f(m^U)]$.

Theorem 3.15. *Let f be a homomorphism from a BCI-algebra X into a BCI-algebra Y. If B is an i-v fuzzy BCI-subalgebra of Y, then the inverse image $f^{-1}[B]$ of B is an i-v fuzzy BCI-subalgebra of X.*

Proof. Since $B = [\mu_B^L, \mu_B^U]$ is an i-v fuzzy BCI-subalgebra of Y, it follows from Theorem 3.7 that μ_B^L and μ_B^U are fuzzy BCI-subalgebras of Y. Using Proposition 3.1, we know that $f^{-1}[\mu_B^L]$ and $f^{-1}[\mu_B^U]$ are fuzzy BCI-subalgebras of X. Hence, by Lemma 3.14 and Theorem 3.7, we conclude that $f^{-1}[B] = [f^{-1}[\mu_B^L], f^{-1}[\mu_B^U]]$ is an i-v fuzzy BCI-subalgebra of X. □

Definition 3.16 (Biswas [1]). Let f be a mapping from a set X into a set Y. Let A be an i-v fuzzy set in X. Then the *image* of A, denoted by $f[A]$, is the i-v fuzzy set in Y with the membership function defined by

$$\bar{\mu}_{f[A]}(y) = \begin{cases} \operatorname{rsup}_{z \in f^{-1}(y)} \bar{\mu}_A(z) & \text{if } f^{-1}(y) \neq \varnothing, \ \forall y \in Y, \\ [0,0] & \text{otherwise}, \end{cases} \tag{3.32}$$

where $f^{-1}(y) = \{x \mid f(x) = y\}$.

Theorem 3.17. *Let f be a homomorphism from a BCI-algebra X into a BCI-algebra Y. If A is an i-v fuzzy BCI-subalgebra of X, then the image $f[A]$ of A is an i-v fuzzy BCI-subalgebra of Y.*

Proof. Assume that A is an i-v fuzzy BCI-subalgebra of X. Note that $A = [\mu_A^L, \mu_A^U]$ is an i-v fuzzy BCI-subalgebra of X if and only if μ_A^L and μ_A^U are fuzzy BCI-subalgebras of X. It follows from Proposition 3.2 that the images $f[\mu_A^L]$ and $f[\mu_A^U]$ are fuzzy BCI-subalgebras of Y. Combining Theorem 3.7 and Lemma 3.14, we conclude that $f[A] = [f[\mu_A^L], f[\mu_A^U]]$ is an i-v fuzzy BCI-subalgebra of Y. □

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